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THE STRENGTH OF MATERIALS

THE STRENGTH OF MATERIALS

A TREATISE ON THE THEORY OF STRESS CALCULATIONS FOR ENGINEERS

BY

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TO
MY SON

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PREFACE TO FIRST EDITION

In all books on stress calculation, save the very elementary, the discussions of the earlier parts of the subject seem to be inadequate to the needs of the student who meets it for the first time. On the other hand, a book which deals fully with the elementary principles probably does not go sufficiently far for the student who is reading for an honours degree. My object here has been to produce a book with which the honours candidate can begin and complete his studies. To this end I have dealt with the elementary conceptions at some length, for it is at the beginning that difficulties of understanding arise. As the more advanced parts of the book are reached, I have reduced the elaboration of detail, on the assumption that a man who is good enough to read these parts at all, will be capable of following the work with less help from the author.

Again, it seems to me that, in most existing books, the examples too often are only exercises in algebraic substitution. I have endeavoured to remedy this, somewhat, by including examples which make some demand on the thinking powers of the student.

The practical engineer, if he require a textbook for reference, wants it to be as complete as possible. For this reason I have included many items not usually found in textbooks on Strength of Materials, hoping that this will increase the value of the book for those whose lot is to deal with the more unusual type of stress problem which occasionally arises in the drawing office. Sometimes considerations of space have militated against a full treatment of some branch of the subject; in these instances I have included many useful results among the examples at the ends of the chapters, These harder examples thus serve two purposes: they afford profitable exercises for the advanced student, and they constitute a scattered collection of formulæ which will be useful to those who have neither the time nor inclination to prove them. If a reader cannot find a particular problem worked out in the text, he should search the examples for a formula that meets his requirements. This will be facilitated by reference to the exhaustive index at the end of the book.

For the most part it has seemed better, for ease of arrangement, to develop each part of the theory as far as it is to be carried at all in this book. But the student will be well advised, for a first reading, to take only certain selected passages. For the benefit of those who may not be happily placed in regard to assistance, I give, at the end of the contents, what I consider to be a good programme of reading.

I have deliberately omitted all matter dealing with the experimental branches of the subject, for this is now so highly developed as to demand a separate textbook in itself. For the same reasons I have not touched on the metallurgical aspects of the strength of materials, with the exception of a few remarks on fatigue in the final chapter.

During the last few years a vast amount of new work has been done, considerably increasing the powers of the stress-calculator; I have included here the most useful results of modern research.

I owe a very large debt of gratitude to the many gentlemen who have given me assistance in the laborious and tedious work of checking the manuscript and proofs. The manuscript of the first seventeen chapters was read by Mr. G. Upjohn, B.A., of Trinity College, Cambridge, to whom I am also indebted for many valuable suggestions on matters of detail. I must also express my thanks to Mr. J. W. Blomfield, also of Trinity College, Cambridge, for reading some of the MS. of these early chapters. The manuscript of the last sixteen chapters was mainly read by Mr. H. A. Webb, M.A., of Trinity College, Cambridge, who assisted me by many helpful suggestions and criticisms, and by allowing me to include certain pieces of work which were done by him in the first place. At the same time I acknowledge with gratitude the careful checking of the details, of much of this part of the book, by Mr. W. A. Green, B.Sc., A.M.I.N.A. The very thankless task of proof-reading was generously undertaken by Instructor Lieut.-Commander A. E. Hall, O.B.E., A.R.C.S., R.N., whose labours have been untiring.

I am also indebted to the Editor of *The Engineer* for permission to reproduce the diagram which forms Figure 390.

Finally, I should like to express my very great appreciation of the great care and trouble which the publishers have taken in the production of the book.

JOHN CASE.

PREFACE TO THIRD EDITION

The necessity for a third edition of this book has provided an opportunity to include certain additional matter which should prove useful to students and designers, and to make certain alterations and corrections. Experience has led me to revise my ideas concerning the yield point of metals, both as a physical phenomenon and in relation to combined stresses, with the result that Chapter VI has been very largely re-written. I hope that the modified description of the graphical determination of B.M. diagrams and the application of the funicular polygon to deflections will be found useful. The appendix, as can be seen from the Table of Contents, includes the mathematical analysis of several problems which, for various reasons, could not be included in earlier editions, and a section dealing with stress concentrations, which are becoming increasingly important. I should like to take this opportunity of expressing my thanks to those readers who from time to time have written to point out errors and make suggestions, and to Instructor Lieut.-Commander J. Fleming, R.N., for his labours in checking the MS. and proofs of corrections and additions.

June, 1938. JOHN CASE.

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The following programme of reading is suggested for the benefit of those working with but little assistance:

§§ 1–14; 34, 36, 37, 46; 47–52; 54–56; 84 and 85 as far as proofs of the formulæ for the stresses; 88, 89; 94–95, 97–98, 100; Ex. 1 and 2, p. 120; 101, 102, 104–106; 117–122; 138–145; 158, 159 as far as equation (1), p. 218; 161, 162–165; 212–225, 227–229.

For a second reading, fill in the gaps left above, omitting Chapter XVII; read Chapters XX, XXII and XXIV.

The remainder of the book is only suitable for advanced students.

CHAPTER I

DIRECT STRESSES

1. The Purpose of the Theory of Stresses.—When an engineer embarks upon the design of any machine or structure, it is his bounden duty to use every means in his power to ensure that the material realization of his design shall not break or collapse. It is the theory of stress calculation, aided and verified by experiment, which enables the designer to estimate the strength of his machine before it is built. Unfortunately, several obstacles combine to prevent the exact calculation of strength in many cases: first an imperfect knowledge of the forces at work, secondly the failure of mathematical processes to deal with some particular problem, thirdly an incomplete understanding of the physical properties of the materials employed, and, fourthly, the difference between the properties of the materials of the practical world and those assumed as a basis of all our theory. For instance, in the case of an aeroplane, our knowledge of the distribution of air pressure over the surface of the wings is very limited; at present mathematical theory does not enable us to calculate exactly the strength of the crankshaft of a motor car. However, in spite of these limitations and difficulties, the engineer of to-day can, if he wish, form a very good idea of the strength of any design he may create, and, in very many instances, calculate it with great accuracy. The fact of being able to do this has a definite social value, apart from considerations of safety, for it saves the unnecessary expenditure of material which occurs when parts of a machine are made stronger than they need be.

2. Definitions of Load and Stress.—Any member of a machine or structure has usually to withstand the action of certain external applied forces; these forces constitute the "load" on the member. For example the load on the piston rod of a steam engine is the force due to the steam pressure on the piston; the load on a railway bridge is the weight of a train passing over it and the weight of the bridge itself; the load on the propeller shaft of a ship consists of the twisting moment applied by the engine, the thrust exerted by the water on the propeller, and its own weight.

The simplest kind of load we can have is a direct pull or push, or, in more technical language, a direct tension or compression. As an instance of the former we might take the lifting rope of a crane, and of the latter the piston rod of an engine during the out-stroke. In each case there is a force applied at one end of the member which must be balanced by an equal and opposite force at the other end : the pull applied by the winding engine to the crane rope is balanced by the weight of the load at the other end; the steam pressure on the piston acting on one end of the piston rod is balanced by the thrust of the connecting rod at the other end and the inertia forces.

Now this balance must be maintained throughout the length of the member.

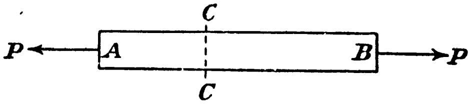

FIG. 1.

Suppose we have a rod *AB* (Fig. 1) and apply to each end equal and opposite tensions *P*; imagine a chalk mark made at *C*, then it is clear that the parts *AC* and *CB* must be pulling each other with a force equal to *P*. The part *AC* is in equilibrium under the action of the external force *P* at *A* and an equal force applied by the part *CB*. Similarly the part *CB* is in equilibrium under the action of *P* at *B* and the equal force at *C* applied by the part *AC*. This argument will hold good wherever we make the mark *C*. Thus, across every imaginary section of the rod perpendicular to the axis there is a resultant force *P*. These internal forces, whatever their nature, which are applied by one part of a body to the neighbouring parts, are collectively referred to as "*stresses*." If the rod is in tension the stress is called a ***tensile stress*** and is then usually considered positive. Referring again to Fig. 1, if the applied forces are thrusts, the action between any two neighbouring lengths of the rod will be a push : in other words the rod is in compression; the stresses are called *compressive stresses* and are usually reckoned negative. Tensile and compressive stresses are together referred to as *direct stresses*.

The whole action of one part of the body on the other, over a plane section, is sometimes referred to as the "total stress" on that section; we can thus define the total stress as the internal forces which, when applied to the section, will balance the external forces on one side of the section.

3. Measurement of Stress.—When we wish to give the stress a numerical value it is desirable, for purposes of comparison, to refer to the stress in relation to the area of the cross section, since it is to be expected, and can easily be verified by experiment, that the total tension which a rod can bear without breaking will be proportional to the area of the cross section. The total force acting on a section, divided by the area of that section, is called the *stress intensity* or, more often, simply the stress.*

---

* In future when we use the word "stress" without qualification it must be understood to mean "intensity of stress."

Thus, if P = total force or *total* stress acting on a cross section,
S = the area of that section,
p = the intensity of stress on that section,

then

$$\mathrm{p} = \frac{\mathrm{P}}{\mathrm{S}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The stress on any section of a body may or may not be the same all over, and we shall frequently meet instances where it will be necessary to speak of the stress at a point, when each unit of area is not transmitting the same amount of stress. In such cases we adopt the following conception: let δS be a small area enclosing a point A in which we are interested, and let δP be the total stress action, or force, across the area δS, between the parts of the body on either side of it; then in general the ratio $\delta P/\delta S$ will tend to a finite limit as δS is indefinitely decreased; the value of this limit is taken as the stress intensity at the point A.

In England and America it is usual to measure the load in pounds or tons weight, and the areas in square inches, so that stresses are expressed as lbs./in.^{2} or tons/in.^{2}

There are many instances, less obvious than the above, where the stress in the material is purely tensile, such as rotating rings: these will be considered later.

4. Strain.—When a body is subjected to the action of forces it is found that a certain deformation takes place: the shape and dimensions of the body are altered. This deformation is referred to as "strain." It is important that the student should clearly grasp the essential difference between stress and strain: the former partakes of the nature of a force, and is the cause of strain, which is purely geometrical in its manifestation.

5. Measurement of Strain in Tension and Compression.—If a rod of length l extend an amount δl, the longitudinal strain of the rod is reckoned as the increase of length *per unit length*. That is, if e denote the strain,

$$\mathrm{e} = \frac{\delta l}{l} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Since all the strains with which our theory deals are very small, so that δl is very small compared with l, we can, when more convenient, take

$$e = \frac{\delta l}{l + \delta l}.$$

If the rod be under compression the strain is specified in the same way, except that δl measures the contraction of length, and will be reckoned negative.

It must be noted carefully that strain is *not* an increase or decrease of length, but a ratio, that is a non-dimensional quantity.

6. Hooke's Law forms the basis of the whole of the mathematical theory of elasticity: it states that when the load increases or decreases the strain increases or decreases by a proportional amount, and that when the load is removed altogether the strain is reduced to zero. This law is obeyed by the majority of solid bodies within certain limits, the most notable exceptions being cast metals.

7. Young's Modulus.—From Hooke's Law we have the relation

$$\text{stress} = \text{strain} \times \text{a constant}.$$

The constant here mentioned is found to be the same for a given material whatever be the size or shape of the body made of this material; it is called Young's Modulus of Elasticity and is usually denoted by *E* in this country. Since strain is a non-dimensional quantity it follows that Young's Modulus has the same dimensions as stress, and is therefore measured in the same units, viz. lbs./in.^{2}, etc.

We can now write

$$E = \frac{\text{direct stress}}{\text{corresponding strain}} = \frac{p}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1), (2) and (3) we have

$$\delta l = el = \frac{lp}{E} = \frac{lP^*}{ES} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which expresses the extension or contraction of a rod in terms of its dimensions, the total load, and Young's Modulus.

For most materials Young's Modulus has the same value in tension and compression.

Example 1.—The piston of a steam-engine is 16″ diameter, and the piston rod 2·25″ diameter. The steam pressure is 150 lbs./in.^{2} Find the stress in the piston rod and elongation of a length of 30″, taking $E = 30 \times 10^6$ lbs./in.^{2}, when the piston is on the in-stroke.

The net area of the piston $= \frac{\pi}{4}(256 - 5{\cdot}06)\ \text{in.}^2 = 197\ \text{in.}^2$

The total load on the piston rod $= 197\ \text{in.}^2 \times 150\ \text{lbs./in.}^2$
$= 29{,}500\ \text{lbs.}$

The area of the piston rod $= 3{\cdot}98\ \text{in.}^2$

∴ the stress in the piston rod $= \dfrac{29{,}500\ \text{lbs.}}{3{\cdot}98\ \text{in.}^2}$
$= 7{,}400\ \text{lbs./in.}^2$

By formula (4) the elongation $= \dfrac{30'' \times 7{,}400\ \text{lbs./in.}^2}{30 \times 10^6\ \text{lbs./in.}^2} = 0{\cdot}0074''.$

Example 2.—The wire working a signal is 2,000 ft. long and $\frac{3}{16}''$ diameter. Assuming a pull on the wire of 400 lbs., find the movement which must be given to the signal box end of the wire if the movement at the signal end is to be 7″. Take $E = 30 \times 10^6$ lbs./in.^{2}

The area of the cross section of the wire $= 0{\cdot}0276$ in.^{2}

The extension of the wire, by equation (4),

$$= \frac{2{,}000\ \text{ft.} \times 400\ \text{lbs.}}{30 \times 10^6\ \text{lbs./in.}^2 \times 0{\cdot}0276\ \text{in.}^2} = \frac{800{,}000 \times 12}{0{\cdot}828 \times 10^6}\ \text{in.}$$

$= 11{\cdot}6''.$

∴ the signal box end of the wire must move 18·6″.

---

[*] If we take the second expression for *e* above, we get $\delta l = e(l + \delta l) = el + e\delta l$, and the last term is of the second order of small quantities and is negligible.

Example 3.—A circular rod of steel ⅜″ diameter is placed in a testing machine and it is found that when the tension is 1·1 ton, the total extension on a 12″ length is 0·01″. Find the value of E.

$$\text{Area of cross section} = 0{\cdot}1105 \text{ in.}^2$$

$$\text{The stress} = \frac{1{\cdot}1 \text{ tons}}{0{\cdot}1105 \text{ in.}^2}$$

$$\text{The strain} = \frac{0{\cdot}01 \text{ in.}}{12 \text{ in.}}$$

$$\therefore E = \frac{\text{stress}}{\text{strain}} = \frac{1{\cdot}1 \text{ tons} \times 12 \text{ in.}}{0{\cdot}1105 \text{ in.}^2 \times 0{\cdot}01 \text{ in.}} = 11{,}940 \text{ tons./in.}^2$$

Example 4.—Fig. 2 shows the dimensions of a big-end bolt for the connecting rod of a large marine engine.

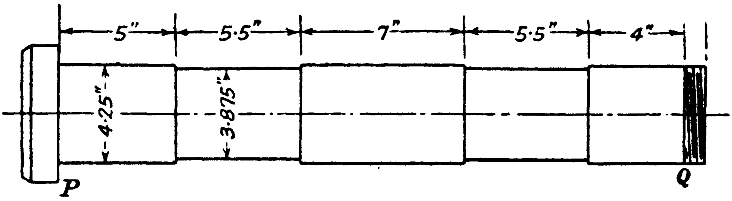

FIG. 2.

The pull on the ends is $P = 45$ tons. Taking $E = 13{,}500$ tons/in.², find the extension of the length PQ.

S_1 = Area of cross section, 4¼″ diam. = 14·19 in.²
l_1 = Total length of this section = 16·00 in.
S_2 = Area of cross section, 3⅞″ diam. = 11·79 in.²
l_2 = Total length of this section = 11·00 in.

$$\text{Then the elongation of } PQ = \frac{l_1P}{ES_1} + \frac{l_2P}{ES_2} = \frac{P}{E}\left(\frac{l_1}{S_1} + \frac{l_2}{S_2}\right)$$

$$= \frac{45 \text{ tons}}{13{,}500 \text{ tons/in.}^2}\left(\frac{16''}{14{\cdot}19 \text{ in.}^2} + \frac{11''}{11{\cdot}79 \text{ in.}^2}\right)$$

$$= \frac{45}{13{,}500}\left(\frac{16}{14{\cdot}19} + \frac{11}{11{\cdot}79}\right) \text{ in.} = 0{\cdot}00687 \text{ in.}$$

8. Stress-Strain Diagrams.—These are usually drawn by plotting a graph, using strains as abscissæ and the corresponding stresses as ordinates, and the diagrams most frequently employed are those obtained by tests on straight rods under tension. The shape of the curve obtained depends greatly on the material under test and the method of testing, and for full information on testing the reader is referred to works on the testing of materials and articles in the technical press.

In Fig. 3, (i) shows a typical stress-strain curve obtained by the use of an extensometer, with the ordinary methods of attaching it and of holding the specimen, for mild steel or wrought iron. From O to A the graph is a straight line, the material obeying Hooke's law: A is the *limit of proportionality*. From A to B there is a range of imperfect elasticity, and, if the load be removed, some of the strain will remain as *permanent set*, i.e. will not disappear. At *B* "*yield*" occurs, the material stretching considerably without increase of load.

At this stage we are entering a new phase in the relations between the changes in internal structure and the external agency which is causing those changes. Up to a point, the structure of the metal adapts itself to the increasing applied forces, but at " yield " this process of adaptation brings about a lack of capacity for any further adaptation. The quantitative change of increasing stress has brought about a qualitative change in the resisting powers of the metal, just as surely as the gradual supply of heat to a liquid at a critical point breaks up molecular cohesion and so causes evaporation. Viewed in this light, it is only to be expected that careful experiment would reveal a discontinuity in the behaviour of the metal. This, in fact, has been found to be true, and Fig. 3 (ii) shows the earlier stages of the graph when special care is taken to ensure axial loading and to avoid local stress concentrations.*

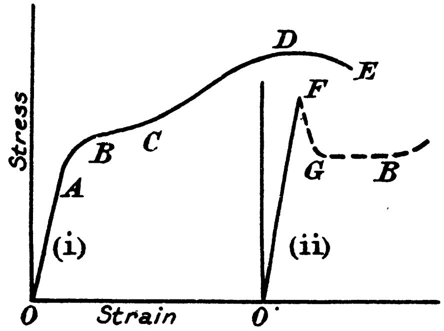

FIG. 3.

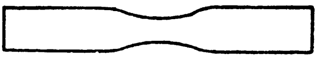

FIG. 4.

The portion *OF* is perfectly straight, and at *F* occurs the " higher yield point." This point is shown by a sudden change in the appearance of the polished surface of the specimen, characteristic of yield: what are known as Lüder's lines become noticeable. If, now, the load be removed and again applied gradually until this marking begins to spread, it is found that creep will begin at a stress considerably below the higher yield point. This stress is the " lower yield point." It is usually not possible to obtain any points on the graph between *F* and *B*. In less careful tests the existence of the higher yield point is obscured by the presence of local stress concentrations, and the " yield point " found corresponds with the lower yield point in a careful test. For practical purposes the lower yield point is generally the more important, since absolute uniformity of stress distribution and the absence of local stress concentrations can rarely, if ever, be ensured.†

When yield has taken place, the metal has become plastic, and

* See Report of British Association on Stresses in Overstrained Materials (1931).

† But see a paper by Coker and Haigh, *An Experimental Investigation of Cracking in Mild Steel Plates and Welded Seams*, Inst. of Naval Architects, 1935.

stretches rapidly with increase of load, a visible neck forms (Fig. 4), and fracture occurs. The second phase of adjustment between internal structure and external environment has collapsed. It must be noted that, in plotting these diagrams, the stress is calculated on the original area of cross section, so that the graph may bend over, as *DE*. The stress plotted in this way is the "nominal" stress. Also, it must be remembered that the properties revealed are only "statistical" qualities; the properties of single crystals are constant.

With materials other than mild steels and wrought irons the graph will follow the same general shape, except that the limit of proportionality may not be clearly marked or may be zero, and the same remark applies to the yield point.

9. Magnitudes of Stresses and Strains.—The following figures will give the reader an idea of the magnitudes with which he will have to deal in the study of stresses. In general these run to two extremes: Young's Modulus is a large quantity when measured in any of the common units; strain, within the limits of linear elasticity or even just below the yield point, is a very small quantity, whilst stresses are of more ordinary magnitude. For instance, for mild steel, Young's modulus is about 13,500 tons per square inch, the strain will be about 0·00112 with a stress of about 15 tons per square inch, the yield point being perhaps about 17 or 18 tons per square inch. If a mild steel bar of one square inch section, and twelve inches long, be pulled with a force of 15 tons the elongation will be 0·0132 inch, about. In practice the strain will never exceed 1/1000.

As further illustration it may be mentioned that the ultimate strength of ordinary mild steel is about 30 tons/in.^{2}, that of piano wire about 120 tons/in.^{2}, brass about 15 tons/in.^{2}, timber about 2 to 8 tons/in.^{2}, all in tension.

10. Limitations and Scope of Mathematical Theory.—In the following pages we develop the theory of the strength of solid bodies under various conditions of loading, and it is important that the reader should have clear in his mind the assumptions that are tacitly made hereafter, and the consequent limitations imposed on the application of the theoretical results.

In the first place the material is assumed to be homogeneous and *isotropic*. The former of these terms is sufficiently well understood to need no explanation here; the second implies that the elastic properties of the material are the same in all directions. No actual materials within our experience are isotropic, but most metals closely approach isotropy. Timber on the other hand is far from isotropic: the elastic properties in directions radial or tangential to the annual rings differ between themselves and from the properties in the longitudinal direction of growth.

Secondly the material is assumed to obey Hooke's Law. This is almost exactly true within the working limits for certain metals, such as wrought iron and steel, but many other materials, such as cast iron, stone, etc., do not obey the law for any measurable strain.

The strains are supposed to be so small that their squares and products can be neglected, and it is unlikely that this leads to any serious error. It is also assumed that the strains are within the limits of perfect elasticity, i.e. that the strain disappears on removal of the load; in practice the limits are in reality extremely narrow, although the "sets" which occur below the limit of linear elasticity are so small as to be negligible.

It is further assumed that the loads are applied gradually from an initial state of zero stress.

In spite of these assumptions the theory of stresses is able to yield extremely valuable results. For instance a theory of the stresses in beams or girders is given below, all the above assumptions and certain others as well being made; when these results are applied to beams made of wood, a material which has none of the assumed properties, the results agree remarkably well with experiment.* Theory is able to predict such phenomena as the increase of stress at the sharp corners of keyways in shafts, or due to the presence of flaws; it can indicate when certain members are likely to fail by buckling, and many other striking results. So that, although we know that our assumptions are more or less untrue, their use is amply justified by the results.

One fact, however, must always be remembered, that the formulæ developed will not indicate at all what will happen when the yield point is reached, and the second elastic phase is entered.

In attempting to solve new problems, their mathematical complexity frequently drives us to add considerably to the list of assumptions, and any new results which may be obtained must always be verified by experiment before they can be taken as established.

Most usually the problem before the engineer is this: he knows the nature and magnitude of the loads which a particular member must be designed to stand, he knows the material of which it is to be made; what must be the dimensions in order that it may be strong enough? According to the working conditions and the material, a certain maximum limit, obtained as the result of experiments, such as referred to in § 8, will be imposed upon the stress; the theoretical methods given below enable the engineer to calculate his dimensions so that this stress is not exceeded. Thus the subject of the strength of structures and machines has two aspects: the mathematical, which enables us to calculate the stresses under given circumstances, and the experimental which shows us the limits which must not be exceeded by the stresses, and by which new theories and formulæ are initiated and verified.

Other considerations, such as cost, ease of manufacture, etc., may influence the design, but that is not our concern here.

11. Factor of Safety and Working Stress.—The greatest estimated stress in any part of a machine or structure is called the working stress on that part. This will be a certain fraction of the ultimate breaking strength of the material, and the value of the ratio $\frac{\text{ultimate strength}}{\text{working stress}}$

---

* The explanation is that the wooden beam exhibits statistically the qualities of a homogeneous isotropic material, provided the dimensions are large in comparison with the spacing of the fibres.

is called the factor of safety; its selection depends on the judgment of the designer if not laid down by law. At the same time of course the working stress must be kept below the elastic limit.

The factor of safety varies greatly according to circumstances, such as the nature of the stresses, whether these are constant or fluctuating, liability to corrosion, possible effects of bad workmanship, non-uniformity of material, the probable accuracy of the calculated loads and of the method of calculation, and so on. The factor of safety will usually not be less than three and may be as high as twelve.

12. Fluctuating Stresses.—In practically every machine at least some of the parts are subjected to stresses which vary in a periodic manner between certain limits, for instance the piston rod of a reciprocating engine is put into a state of tension and compression alternately in every revolution of the crankshaft. In some cases the stress varies between equal, or nearly equal, tensile and compressive values, in others it is increased from zero to a maximum tension or compression, and then decreased to zero, and the process is repeated periodically. In the former instance we usually speak of alternating or reversed stresses. The question which concerns us, in both cases, is: will it take the same load to rupture the material under alternating or repeated stresses as under a stress gradually applied once for all? The answer is a decided negative, so that it at once becomes important to know what alternating stress will break a material.

We shall call the stress which, by one gradual application, breaks a material, the *ultimate statical strength* of the material. It is found that frequently repeated stresses of an intensity far less than the ultimate statical strength will suffice to bring about rupture, if the number of repetitions is large enough, and that if the stresses are reversed stresses, an even smaller stress will break the material for a given number of cycles. These phenomena are usually termed *fatigue*, and the range of stress which, after an indefinitely great number of cycles, will not cause fracture is called the *fatigue range.* Thus, if it were found that for a certain material a stress of 25 tons per square inch was the largest stress which could be repeated from zero an indefinitely large number of times without causing rupture, we should say that the fatigue range for that kind of fluctuating stress was 25 tons per square inch; again if it appeared that the stress could be alternated between the limits $\pm$ 10 tons per square inch, we should say the fatigue range was 20 tons per square inch for reversed stresses. We cannot, in an experiment, subject a material to an infinite number of stress cycles, but we can obtain a curve such as Fig. 5, and deduce from it the limiting value of the range to which the curve is asymptotic, and take this as the true value of the fatigue range for an infinite number of cycles. At the same time it is not yet certain that there is any range which will ensure the material infinite life under repeated or alternating stresses. It is common practice to specify fatigue ranges for six million cycles, although they are sometimes stated for one million, but this is certainly too few.

It is found that the range of stress is more important than the actual values of the stress, for instance Wöhler found for a specimen of Krupp axle steel, having a statical ultimate strength of 52 tons per sq. in., that the limiting stress which could be repeated from zero was 26·5 tons per sq. in., and that the limiting alternating stress was ± 14·05 tons per sq. in.; the range in the first case was thus 26·5, and in the second 28·1, tons per sq. in.

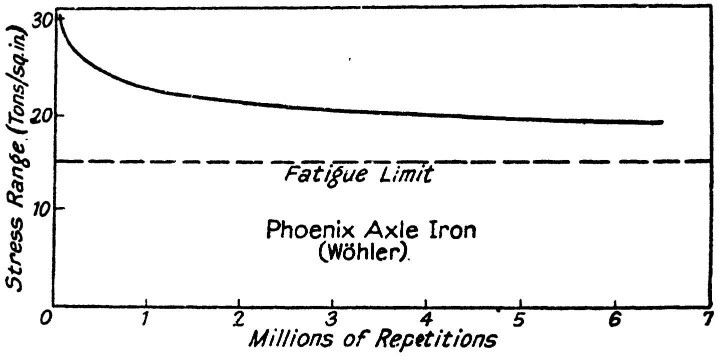

FIG. 5.

13. Principle of St. Venant.—In order to reduce a practical problem to one which is capable of relatively simple mathematical solution we frequently have to assume an ideal distribution of load which does not obtain in practice. For example in considering the stresses in a uniform rod under tension we assume that the load is distributed uniformly over a cross section of the rod, whereas in reality it may all be applied to the surface, which would be the case if the rod were being pulled by a hollow socket into which it was brazed or soldered. In such cases we find comfort in a principle stated by St. Venant: * "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." As an illustration of the meaning of this we will consider the case of a rod *AB* (Fig. 6) soldered into two sockets *R* and *S*, by means of which a

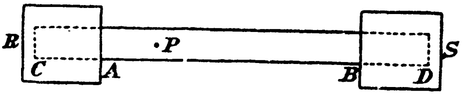

FIG. 6.

tensile load is applied to the rod. In this case forces are applied to the curved surface of the rod between the planes *A* and *C*, and the stress at a point *P* between *A* and *B* will not be *exactly* the same as if the load were evenly distributed across the plane *A*. But when the distance *AP* exceeds two or three times the greatest dimension of the cross section of the rod the effect on the strain at P will not be appreciable. For, consider the part of the bar between A and C and suppose a uniformly distributed load is applied to the cross section A, having a resultant equal and opposite to the resultant of the forces applied to the curved surface AC: the piece of rod AC will be in equilibrium, the applied systems of forces are together equivalent to zero force and zero couple, and we can conclude that the strains at P will be quite inappreciable when AP exceeds a few times the greatest diameter of the cross section.*

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* A. E. H. Love, *Treatise on the Mathematical Theory of Elasticity*, 2nd Ed. p. 129.

Hence the argument amounts to this: the net result of applying a distribution of force to the curved surface of AC (to the left, say), and a uniformly distributed force to the cross section at A (to the right, say), the two distributions having equal and opposite resultants, is that the strains at a point P will be inappreciable. Therefore the difference between the stress at P due to either distribution by itself can be disregarded. The distribution over the curved surface is what really happens, the uniform distribution is one with which we can deal in calculation.

14. Initial Stresses.—It frequently happens that, before any load is applied to some part of a machine or structure, it is already in a state of stress. For instance the bolts holding down the head of the cylinder of a steam engine are put into tension by tightening up the nuts; the same remark applies to the bolts in a flanged coupling of a steam pipe, or the big-end bolts of the connecting rod. In some of these cases we have to take into account the relative rigidity of the bodies in question, in others we need not. In the case of the cylinder head, this will deform so little under the pressure of the nuts compared with the deformation of the bolts that we can treat it as rigid, and the total intensity of stress in the bolt, when steam is in the cylinder, will be the sum of the initial stress and the stress due to the steam pressure. On the other hand, in dealing with the flanged joint we must consider the elasticity of the packing: any tension which is applied to the joint will be taken up partly by extra tension in the bolts and partly by reduced compression in the packing. The following example should show how such cases are to be treated.

Example.—Two pieces A and B, shown in Fig. 7, fit freely into the ends of a straight tube and are drawn together by a bolt and nut. The section of the tubular distance piece has the same area as the bolt, and they are of the same material. The nut is initially screwed up so that the tension in the bolt is 5 tons. The pieces A and B are then subjected to forces

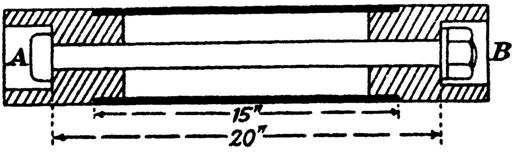

FIG. 7.

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* This case has been investigated mathematically by L. N. G. Filon, *Phil. Trans.*, Vol. 198, A., pp. 147–233.

of 3 tons tending to pull them apart. Calculate the resulting tension in the bolt. (Intercoll. Exam., Cambridge, 1908.)

Let S in.^{2} = cross sectional area of bolt or tube.
,, E tons/in.^{2} = Young's modulus.

In screwing up the bolt the tube contracts $\dfrac{15 \times 5}{ES}$ in. . (i)

It was remarked in § 5, p. 3, that we can, without sensible error, reckon the stretches as fractions of the strained length, so that although the length of the bolt between the nuts is altered by tightening up the latter, we may take 20 in. as the length on which to estimate the strains due to the extra load which is applied after tightening the nuts.

When the external load is applied let the tension in the bolt become T tons.

The extra extension of the bolt will be $\dfrac{20(T-5)}{ES}$ in.

The thrust in the tube will be $T - 3$ tons, so that the contraction of the tube will be $\dfrac{15(T-3)}{ES}$ in.

The decrease of the contraction from its previous value (i) is therefore

$$\frac{75}{ES} - \frac{15(T-3)}{ES}.$$

This must equal the extra extension of the bolt. Hence

$$\frac{75}{ES} - \frac{15(T-3)}{ES} = \frac{20(T-5)}{ES}$$

or

$$75 - 15(T-3) = 20(T-5)$$

which gives

$$T = 6\tfrac{2}{7} \text{ tons}.$$

15. Rods of Varying Section, and Distributed Axial Loads.—

Most cases of this nature are best dealt with graphically, and the method should be sufficiently clear from the examples worked out below. In some cases, e.g. Example 1 below, we do not have to resort to graphical processes.

Example 1.—A straight steel rod of circular section, and 24″ long, rotates about an axis through one end perpendicular to its length. Take the weight of steel as $\rho = 0{\cdot}283$ pound per cubic inch, and E as 30×10^6 lbs./in.^{2}, and calculate what speed of rotation will produce a maximum stress of 5 tons/in.^{2} in the rod, and what will then be the elongation of the rod.

Let ω = the angular velocity of the rod in radians/sec.
S = the area of the cross section.
The weight of the rod per unit length $= S\rho$.
Then the centrifugal force on an element of length dx', at a distance x' from the axis of rotation, $= \dfrac{\rho S x' \omega^2}{g} dx'$.
l = the length of the rod = 24 in.

Hence the tension T on a section distant $x (x < x')$ from the axis of rotation is given by

$$T = \int_x^l \frac{\rho S x' \omega^2}{g} dx' = \frac{\rho S \omega^2}{2g}(l^2 - x^2)$$

which is a maximum when $x = 0$.

Therefore the maximum stress

$$=\frac{T}{S}=\frac{\rho\omega^2l^2}{2g}=\frac{0{\cdot}283\text{ lbs./in.}^3\times 576\text{ in.}^2\times\omega^2\text{ (secs.)}^{-2}}{2\times(32{\cdot}2\times 12)\text{ ins./sec.}^2}=0{\cdot}211\omega^2\text{ lbs./in.}^2$$

If this = 5 tons/in.² we must have

$$\omega^2=\frac{5\times 2240}{0{\cdot}211}=53{,}100.$$

$$\omega=231\text{ radians/sec.}$$
$$=2{,}200\text{ r.p.m.}$$

Now, from above, the stress at distance x from the axis of rotation is

$$\frac{\rho\omega^2(l^2-x^2)}{2g}$$

The extension of an element dx will be

$$\frac{\rho\omega^2(l^2-x^2)dx}{2gE}$$

Hence the total extension will be

$$\frac{\rho\omega^2}{2gE}\int_0^l(l^2-x^2)dx$$

$$=\frac{1}{3}\cdot\frac{\rho\omega^2l^3}{gE}$$

$$=\frac{0{\cdot}283\text{ lbs./in.}^3\times 53{,}100\text{ (secs.)}^{-2}\times 13{,}824\text{ in.}^3}{3\times(12\times 32{\cdot}2)\text{ ins./sec.}^2\times 30{\cdot}10^6\text{ lbs./in.}^2}$$

$$=0{\cdot}006''\text{ nearly.}$$

Example 2.—An aeroplane propeller rotates at 1,650 r.p.m. and is 8′ 4″ diameter. It is made of walnut weighing 0·024 pound per cubic inch, for which $E = 1{\cdot}4.\ 10^6$ lbs./in.² The area of the cross section (S) is given below for different distances (x) from the axis of rotation, measured in inches. Draw a curve showing the variation of stress along the blade and estimate the total extension between the tip and a section 5″ from the axis.

x (in.) .	5	10	15	20	25	30	35	40	45	47·5	50
S (in.²) .	28	21·5	16·5	13·6	11·4	9·1	6·7	4·3	2·4	1·5	0

The centrifugal force per unit length is given by

$$X=\frac{\rho S\omega^2x}{g}$$

where ρ is the weight per unit volume of the material, ω the angular velocity, x the distance from the axis, and g is measured in inch units since x is in inch units. Hence

$$X=\frac{0{\cdot}024}{32{\cdot}2\times 12}\times\left(\frac{1650\times 2\pi}{60}\right)^2Sx=1{\cdot}85\,Sx.$$

The total force on any section will be $1{\cdot}85\int_x^{50}Sxdx$. Hence we must plot a curve of Sx and integrate it. The load on a section distant 10″, say, from the axis will be the area of this curve between $x = 10$ and $x = 50$, corrected for scales and multiplied by 1·85. Dividing the load by the area of the section we obtain the stress p. The extension is then given by $\frac{1}{E}\int_x^l p.dx$. Consequently we must plot a curve of p and integrate it.

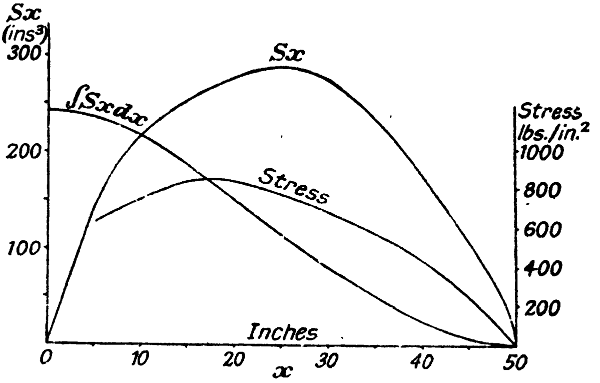

FIG. 8.

The curves are given in Fig. 8, and the rest of the calculations below.

x (in.)	5	10	15	20	25	30	35	40	45	47·5	50
Sx (in.^{3})	140	215	248	272	285	273	235	172	108	71·2	0
$\int_x^{50} Sx.dx$	8,660	8,300	7,400	6,120	4,720	3,300	2,040	1,040	320	100	0
Total Force (lbs.)	17,550	15,900	13,700	11,350	8,750	6,220	3,780	1,925	592	185	0
Stress lbs./in.^{2}	626	740	830	835	768	683	562	448	282	123	0

From the area of the stress curve we find

$$\int_5^{50} pdx = 27,320 \ \frac{\text{lbs.}}{\text{in.}}$$

Hence the total extension of that part of the blade

$$= \frac{27,320 \text{ lbs./in.}}{1\cdot4 \times 10^6 \text{ lbs./in.}^2} = 0\cdot0195''.$$

16. Composite Bars in Tension or Compression.—By a composite bar we mean a bar composed of two or more materials, such as steel rods surrounded by concrete, or a steel tube filled with wood, or rods of two different metals, etc. The construction of the bar is supposed to be such that all the constituent members must extend or contract equally.

As an illustration, suppose we have three rods of different materials, the areas of their cross sections being constant and equal to S_1, S_2, S_3, and Young's Modulus for the three materials being E_1, E_2, E_3, respectively.

Let the total load on the composite bar be P, which may be either a tension or a compression.

Let P_1, P_2, P_3 be the portions of the load taken by the three rods.

Let l be the length of the bar.

Then if all the rods have to stretch or contract equally we must have

$$\frac{lP_1}{E_1S_1} = \frac{lP_2}{E_2S_2} = \frac{lP_3}{E_3S_3}.$$

We must also have

$$P_1 + P_2 + P_3 = P.$$

Solving these equations for P_1, P_2 and P_3 we find :—

$$P_1 = \frac{E_1S_1}{E_1S_1 + E_2S_2 + E_3S_3}P,$$

with similar expressions for P_2 and P_3.

Hence the load is divided among the members in the ratio of the values of ES, and the same will hold for any number of constituent bars. The stress in each bar is then found by dividing the load on it by its cross sectional area. Hence the stress in each bar is proportional to its Modulus of Elasticity.

17. Adhesion Stress in Reinforced Concrete.[*]—As an example of the principles of the preceding paragraph let us consider a reinforced concrete column, i.e. a column of concrete with vertical steel rods embedded in it. Let E_1 and S_1 refer to the steel, E_2 and S_2 to the concrete, and suppose a load P is uniformly distributed over the ends of the column. Then the load per unit area will be

$$p = \frac{P}{S_1 + S_2}$$

The loads applied to the steel and concrete respectively will be

$$\frac{PS_1}{S_1 + S_2} \text{ and } \frac{PS_2}{S_1 + S_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

But according to § 16 the loads actually carried will be

$$\frac{E_1S_1}{E_1S_1 + E_2S_2}P \text{ and } \frac{E_2S_2}{E_1S_1 + E_2S_2}P. \quad . \quad . \quad . \quad \text{(ii)}$$

The difference between the loads given by (i) and those given by (ii) must be taken up by a drag between the two surfaces in contact. This drag is called the adhesion force, and its intensity per unit area of surface is called the adhesion stress. Its nature will be more clearly understood after studying the following example.

In practice the adhesion stress is usually limited to 100 lbs./in.^{2}

Example.—A reinforced concrete column is 18″ square and has four steel rods 1″ diameter embedded in it. Taking $E = 30 \times 10^6$ lbs./in.^{2} for steel and 2×10^6 lbs./in.^{2} for concrete, find the stress in the steel and concrete when the total load on the column is 100 tons.

Find also the adhesive force between the steel and the concrete.

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[*] See *Engineering*, Feb. 26, 1926.

Let the suffix 1 refer to the steel, and the suffix 2 to the concrete. Then :—

$$S_1 = 4 \times \frac{\pi}{4} \times 1 \text{ in.}^2 = 3\cdot14 \text{ in.}^2 \quad E_1 = 30 \times 10^6 \text{ lbs./in.}^2$$

$$S_2 = (324 - 3\cdot14) \text{ in.}^2 = 320\cdot86 \text{ in.}^2 \quad E_2 = 2 \times 10^6 \text{ lbs./in.}^2$$
$$E_1S_1 = 30 \times 10^6 \text{ lbs./in.}^2 \times 3\cdot14 \text{ in.}^2 = 94\cdot26 \times 10^6 \text{ lbs.}$$
$$E_2S_2 = 2 \times 10^6 \text{ lbs./in.}^2 \times 320\cdot86 \text{ in.}^2 = 641\cdot72 \times 10^6 \text{ lbs.}$$
$$E_1S_1 + E_2S_2 = 736 \times 10^6 \text{ lbs., nearly.}$$

Hence we have by § 16

$$\text{total load on steel} = \frac{94\cdot26 \text{ lbs.}}{736 \text{ lbs.}} \times 100 \text{ tons} = 12\cdot8 \text{ tons.}$$

$$\text{,,} \quad \text{,,} \quad \text{concrete} = \frac{641\cdot7 \text{ lbs.}}{736 \text{ lbs.}} \times 100 \text{ tons} = 87\cdot2 \text{ tons.}$$

$$\text{The stress in the steel} = \frac{12\cdot8 \text{ tons}}{3\cdot14 \text{ in.}^2} = 4\cdot07 \text{ tons/in.}^2$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad \text{concrete} = \frac{87\cdot2 \text{ tons}}{320\cdot86 \text{ in.}^2} = 0\cdot272 \text{ tons/in.}^2$$
$$= 610 \text{ lbs./in.}^2\text{, nearly.}$$

It should be noted that in structural work the compressive stress in the concrete is almost invariably limited to 600 lbs./in.^{2}

Assuming that the load is applied evenly to the ends of the column, the applied load per unit area is $\frac{100}{324}$ tons/in.^{2}

The total load applied to the concrete is therefore

$$\frac{100}{324} \text{ tons/in.}^2 \times 320\cdot86 \text{ in.}^2 = 99\cdot03 \text{ tons.}$$

But the actual load carried has been found to be 87·2 tons.

The difference, 11·8 tons, must be transferred to the steel by adhesion.

The load applied to the steel = 0·97 tons, but the load carried by it is 12·8 tons, which agrees with the amount we have just found to be transferred from the concrete.

18. Temperature Stresses.—When the temperature of a body is raised or lowered, the material will expand or contract. If this expansion or contraction is wholly or partially checked stresses will be set up in the body. In the case of a long bar we can proceed as follows :

Let l_0 = the length of the bar at temperature τ_0.
α = the coefficient of linear expansion of the material.
τ = the rise of temperature.

If the bar is quite free to expand, its length will increase by $\alpha l_0 \tau$, i.e. the length becomes $l_0(1 + \alpha\tau)$. If this expansion is prevented it is as if a bar of length $l_0(1 + \alpha\tau)$ were acted upon by a thrust sufficient to reduce its length to l_0. In this case the compressive strain would be

$$\frac{\alpha l_0 \tau}{l_0(1 + \alpha\tau)} = \frac{\alpha\tau}{1 + \alpha\tau}$$

The corresponding stress will be

$$\frac{\alpha\tau E}{1 + \alpha\tau}$$

or, since $\alpha\tau$ will be small compared with unity, the stress can be taken as

$$\alpha\tau E.$$

Similarly, if there is a fall of temperature equal to τ, the contraction would be $al_0\tau$ if it were not prevented. If the contraction is prevented it will be as if a rod of length $l_0(1 - a\tau)$ were stretched by a tension of such value as to produce an increase of length $al_0\tau$, i.e. to produce a strain

$$\frac{al_0\tau}{l_0(1 - a\tau)} = \frac{a\tau}{1 - a\tau} = a\tau, \text{ nearly.}$$

The corresponding stress will again be $Ea\tau$. Thus in each the stress produced = Young's Modulus multiplied by the unhindered thermal change of length per unit length.

In the case of steel a is about 7×10^{-6} in Fahrenheit units, so that Ea = about 210 lbs./in.^{2}, or roughly one-tenth of a ton per square inch. Hence every 10° F. change of temperature will produce a stress of approximately one ton per square inch.

19. Temperature Stresses in Composite Rods.—Suppose we have a compound rod consisting of a bar of one material alongside a bar of different material, the ends of the two bars being fixed together at a certain temperature. If the temperature of the compound rod is raised (or lowered), stresses will be set up in each bar, since one will try to expand (or contract) more than the other.

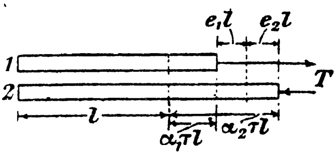

Fig. 9.

Let E_1, a_1, S_1 refer to one bar, and E_2, a_2, S_2 to the other, and let l be the original length of each. Let τ = the increase of temperature, and assume $a_2 > a_1$. Suppose, first, that the bars were quite independent. Then one bar would expand an amount $a_1l\tau$ and the other an amount $a_2l\tau$, each being free from the restraining action of the other. The difference in their lengths would therefore be $(a_2 - a_1)l\tau$.

Now let the first bar be acted on by a tension T and the second by an equal thrust; let e_1 and e_2 be the tensile and compressive strains produced in the two bars. Then

$$e_1 = \frac{T}{E_1S_1}$$

and

$$e_2 = \frac{T}{E_2S_2} \text{ nearly.}$$

$$\therefore \frac{e_1}{e_2} = \frac{E_2S_2}{E_1S_1} \quad \dots\dots\dots \quad (5)$$

Our problem requires that the final lengths of the two rods should be the same. Therefore (see Fig. 9) we must have

$$e_1 + e_2 = (\alpha_2 - \alpha_1)\tau \quad \dots\dots\dots \quad (6)$$

From equations (5) and (6) we can easily find e_1 and e_2; the stresses in the two bars are then E_1e_1 and E_2e_2.

Example.—A gunmetal rod, $\frac{7}{8}''$ diameter, screwed at the ends, passes through a steel tube 1 in. and $1\frac{1}{4}''$ internal and external diameters. The whole is heated to 260° F., and the nuts on the rod are then screwed lightly home on the ends of the tube. Find the intensity of stress in the rod when the common temperature has fallen to 60° F.

Coeff. of expansion per ° F. for steel $= 6 \times 10^{-6}$.
„ „ „ „ „ gunmetal $= 10 \times 10^{-6}$.
Young's Modulus for steel $= 30 \times 10^{6}$ lbs./in.^{2}
„ „ „ gunmetal $= 13{\cdot}4 \times 10^{6}$ lbs./in.^{2}

(R.N.E.C., Keyham, 1923.)

Let the suffix 1 refer to the steel tube.
„ „ 2 „ „ gunmetal rod.

Then

$$E_1S_1 = 30 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}442 \text{ in.}^2 = 13{\cdot}26 \times 10^6 \text{ lbs.}$$
$$E_2S_2 = 13{\cdot}4 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}6 \text{ in.}^2 = 8{\cdot}04 \times 10^6 \text{ lbs.}$$

The strains being e_1 and e_2 we have from (5), p. 17,

$$\frac{e_2}{e_1} = \frac{13{\cdot}26}{8{\cdot}04} = 1{\cdot}65.$$

From (6) we have

$$e_1 + e_2 = 4 \times 10^{-6} \times 200 = 8 \times 10^{-4}.$$

Solving these two equations we get

$$e_1 = 3{\cdot}02 \times 10^{-4}; \; e_2 = 4{\cdot}98 \times 10^{-4}.$$

The stress in the tube $= 30 \times 10^6$ lbs./in.$^2 \times 3{\cdot}02 \times 10^{-4}$.
$= 9060$ lbs./in.^{2}
„ „ „ rod $= 13{\cdot}4 \times 10^6$ lbs./in.$^2 \times 4{\cdot}98 \times 10^{-4}$.
$= 6{,}670$ lbs./in.^{2}

20. Abrupt Changes of Section.—The stresses and strains in a tapered rod have been worked out in § 15 on the assumption that the stress is evenly distributed over all cross sections. This is not strictly accurate for, whenever there is a change of section, the stress increases in intensity towards the boundary; but for all practical purposes when the rate of change of cross section is sufficiently gentle to be called a taper no appreciable error is made in assuming that the stress is distributed uniformly over a cross section. When, however, the section changes abruptly, as at a groove or shoulder, the local increase of stress becomes important, for instance in such cases as are illustrated in Fig. 10.

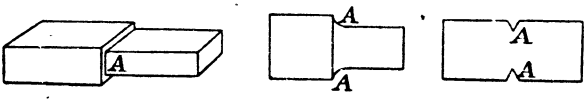

FIG. 10.

In all cases of this sort the stress is greatly increased in the corners A, the ratio of the maximum stress at A to the mean stress in the rod, remote from A, depending on the sharpness of the corners. The effect of this is that the metal on the surface at A will reach the yield point before the rest of the material. Now we know that when the yield point is reached a relatively large increase of strain can take place with only a small increase of stress. Therefore, as the stress in the interior of the metal increases with increasing load, the strain will increase, but the stress on the surface will only increase slightly. In this way the stresses redistribute themselves in the direction of safety. This argument only applies to members which are subject to a constant tension or compression, but when the stress alternates from one to the other the presence of sharp cornered grooves or shoulders is extremely dangerous and should be avoided.

The distribution of stress in a flat tension member whose shape changes from a wide parallel portion to a narrower parallel portion has been studied by Coker* by his photo-elastic method. Fig. 11 † shows graphically the variation of stress intensity across the width of the plate at various sections: the widths of the two parallel portions are 0·855″ and 0·4488″ respectively, the curved parts of the boundaries being circular arcs of 0·3″ radius. Each small graph, drawn across the section of the plate, shows the variation in the intensity of stress across that section; for instance the graph *ab* shows the distribution of stress across the section *AB*. The maximum stress is nearly 20 per cent. greater than the uniform stress in the narrower section. In another case, where the inner corners were radii of $\frac{1}{16}''$ the maximum stress exceeded the mean by nearly 60 per cent. The effects of sharp corners has also been investigated theoretically by C. E. Inglis.‡

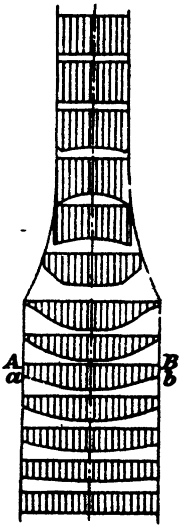

FIG. 11.

21. Work Done During Tension and Compression.—Since the point of application of the load is moved when a rod is stretched or compressed it is evident that work is done. If the stress does not exceed the elastic limit practically the whole of the strain disappears on removal of the load, and the work done is recovered. When the body is in the state of tension or compression the work which has been done on it is therefore stored in the form of *strain energy*, provided the stress is below the elastic limit. Thus within these limits a stretched or compressed rod behaves like a spring, in fact it is a spring.

Beyond the elastic limit most of the strain does not vanish after removing the load, and the work which has been done cannot be regained directly: it has been used up in making the material flow and reappears in the form of heat.

If a variable force P moves its point of application a distance dx, the work done is Pdx. Therefore, within the elastic limit, the work done by tension or compression, i.e. the strain energy, is given by

$$U = \int_0^x Pdx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where x is the total extension. If the cross section is constant and equal to S, the stress p is equal to P/S and the extension dx is equal to lde, where de is the strain. Hence, if l denote the length of the rod,

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* *B.A. Reports*, 1914 and 1921; *Photo-Elasticity*, Coker and Filon, p. 564. See also p. 581 below.

† *B.A. Report*, No. 4, 1921, p. 294.

‡ *Proc. Inst. Naval Architects*, March 14, 1913.

$$U = Sl\int_0^e pde,$$

since we are integrating from zero strain to strain e. Hence $U =$ volume of rod × area of stress-strain curve. Below the elastic limit $e = p/E$, so

$$U = Sl.\int_0^p \frac{p.dp}{E}$$

where p is the final value of the stress. Thus

$$U = Sl\frac{p^2}{2E} = \frac{p^2}{2E} \text{ per unit volume} \quad . \; . \; . \; . \quad (8)$$

Since $p = P/S$ this can be written

$$U = \frac{lP^2}{2ES} = \frac{lPe}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Equations (8) and (9) give the strain energy of a stretched or compressed rod, provided the stress is below the elastic limit.

22. Resilience.—The energy which is stored in a strained body is also called the resilience; if p is equal to the stress at the elastic limit the energy stored is called the proof resilience.

23. Stress Due to Sudden Application of Load.—It has been stated above that the loads are always supposed to be applied gradually in the theory of stress calculations. The reason for this will now be apparent. When the load is applied gradually its initial value is zero and it is increased in proportion to the strain. But now suppose the load is applied suddenly with initial value P, producing an extension δl; let P' be the load which applied gently will produce the same extension and therefore the same stress. The work done by P, $(P.\delta l)$. must equal the strain-energy of the rod in the given state of strain $(P'\delta l/2)$.

$$P\delta l = \tfrac{1}{2}P'\delta l$$
$$\therefore \quad P' = 2P$$

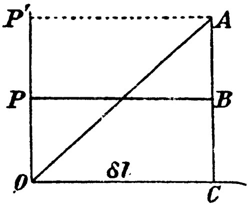

FIG. 12.

That is, it requires double the load applied slowly to produce the same stress and strain as when applied suddenly. In other words, if the load is applied suddenly the stresses and strains are twice what they would be if it were applied slowly. This is shown graphically in Fig. 12. The work done by P is the area $OPBC$, and the work done by P' is the area OAC. These areas must be equal, and therefore $P' = 2P$ as before.

This simple method of treating the problem does not really meet the facts of the case; and, although in this particular instance it leads to a result which errs on the side of safety, this is not by any means always true.

24. Waves of Stress.—When a load is suddenly applied to the end of a rod the problem involves us in dynamical considerations. Consider what happens when one end of a rod is suddenly acted on by a tension, whilst the other end is held fixed. The particles of the rod at the loaded end begin to move under the action of the applied force, and we are at once faced with the question of the effects of the inertia of these particles. As the particles at the loaded end move they will try to drag those next to them, and so on until the tension is felt at the fixed end. If the reader has ever observed a locomotive starting a long stationary goods train, and a similar train running into fixed buffers, it will be helpful to recall the experience. In the case of starting a train all the trucks do not begin to move at once, but each starts up the one behind it, so that a " wave " of tightening up of couplings can be heard and seen pass along the train : the last truck of the train knows nothing of what is going on in front of it until the next truck has been moving long enough to tighten up the coupling chain. So it is when a tension is suddenly applied to the end of a rod : the tension travels along the rod as a " wave," and until the wave reaches any particular section we assume that there is no stress at or beyond this section on account of the blow applied to the free end.

Let us return again to the analogy of the train and examine matters a little farther : as each truck is set in motion by the one in front of it its motion is resisted by the inertia of the truck behind it. But the last truck meets no such resistance, and, as it is acted on by the same pull as the others,* it will tend to move faster than the trucks in front of it ; in other words, the last truck will run after the next in front and push it forwards, and so on until the front of the train is reached. Thus a push " wave " travels up the train, i.e., the wave is reversed in type at the free end of the train. Next, consider the case of the train running into fixed buffers : the engine will be brought to rest first, and gradually each truck in turn will feel the resistance in front and be brought to rest until the last is reached, i.e. a push " wave " travels down the train from front to rear. But the last truck, having nothing behind it, will rebound and start moving backwards, and so exert a pull on the next truck, and thus a pull " wave " travels up the train to the engine. Again we see that the type of wave is reversed at the free end. When the pull " wave " reaches the engine the latter will resist being pulled backwards, and will therefore exert a forward *pull* on the first truck. Thus a pull " wave " meeting the fixed end of the train is sent back as another pull " wave," i.e. the type of wave is not reversed at the fixed end.

So it is with the rod we were considering : a pull suddenly applied to one end travels as a wave of tension to the fixed end, and is reflected from there as another wave of tension ; this reflected wave will be reflected from the free end as a wave of compression, and so on.

We remark here two important rules : (i) a wave is reflected from a fixed end without reversal of type; (ii) at a free end the reflected wave is one of tension if the original wave were one of compression,—and vice versa.

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* Neglecting the effects of ground friction.

We shall now apply approximate analytical processes to investigate the phenomena we have described, and derive formulæ for the stresses. For an exact consideration of the problem the reader is referred to Love's *Theory of Elasticity* or Todhunter and Pearson's *History*.

25. Velocity of Propagation of Stress in a Straight Rod.*—Consider the case of straight rod of uniform section with one end fixed, while a load is suddenly applied to the free end, and neglect the effect of gravity if the rod is not horizontal.

Let w = the weight of the rod per unit volume.
S = the area of the cross section.
p_0 = the intensity of tensile stress which is suddenly applied to the free end.
v = the velocity with which the stress wave travels along the rod.

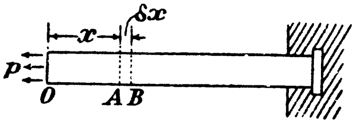

FIG. 13.

Then after a time t the stress p will reach a section A (Fig. 13) distant $x = vt$ from the free end O; after a further time δt it will reach the section B, where $AB = v\delta t = \delta x$.

Let u = the velocity of the particles comprising the section A on account of the stress reaching them.

Then, while the stress wave travels from A to B the particles at A will move a distance $u\delta t$, *towards* O, since the stress is tensile, and after them the particles forming the next section, and so on for all the sections between A and B will move the same distance. That is in time δt the portion AB will lengthen by $u\delta t$, its original length being $v\delta t$.

Hence the strain in AB is

$$e = \frac{u.\delta t}{v.\delta t} = \frac{u}{v}$$

Let p = the stress on the section A at time $t + \delta t$; then

$$p = Ee = E\frac{u}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Now consider the motion of AB: at the beginning of the time interval δt it has no velocity, whilst at the end of the same interval it has acquired a velocity u, if we assume that each layer of particles between A and B acquires the *same* velocity. The weight of the part AB is $wS.\delta x = wS.v\delta t$, so that its gain of momentum in time δt is $u \times \frac{wSv\delta t}{g}$, which is brought about by a force pS. Hence we have

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* This treatment is due to Mr. J. W. Landon.

$$pS\delta t = \frac{uwSv\delta t}{g}$$

$$\therefore \quad p = \frac{\rho uv}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

Hence from (i) and (ii), eliminating u,

$$\mathrm{v} = \sqrt{\frac{\mathrm{Eg}}{\mathrm{w}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is the velocity with which the stress p is transmitted along the rod. Eliminating v from (i) and (ii) gives

$$\mathrm{p} = \mathrm{u}\sqrt{\frac{\mathrm{Ew}}{\mathrm{g}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

So that, if we know u, the velocity of the particles at any time, we can calculate the stress.

The wave of tension which starts from O is reflected from the fixed end as another tension wave; which in turn is reflected from the free end as a compression wave and so on. It will be seen, then, that we cannot form a correct estimate of the strain or stress without considering the reflected waves; we shall, however, postpone a discussion of these to a later chapter.

Example.—A vertical wire is being wound on a drum at a speed of 10 ft./sec. when the free end is suddenly held fixed. Show that the instantaneous stress induced is about 6 tons/in.^{2} $E = 15 \times 10^6$ lbs./in.^{2}; weight of wire = 560 lbs./ft.^{3} (Intercoll. Exam., Cambridge, 1922.)

From equation (11)

$$\begin{aligned} p &= u\sqrt{\frac{Ew}{g}} \\ &= 10 \text{ ft./sec.}\sqrt{\frac{15 \times 10^6 \text{ lbs./in.}^2 \times 560 \text{ lbs./ft.}^3}{32{\cdot}2 \text{ ft./sec.}^2}} \\ &= 10 \text{ ft./sec.}\sqrt{\frac{15 \times 10^6 \times 560}{32{\cdot}2} \times \frac{\text{lbs.}^2\ \text{sec.}^2}{\text{in.}^2 \times \text{ft.}^4}} \\ &= \frac{10 \text{ ft}}{\text{secs.}} \times 16{,}100\ \frac{\text{lbs. secs.}}{\text{ins. ft.}^2} \\ &= \frac{161{,}000}{12} \text{ lbs./in.}^2 = \frac{161{,}000}{12 \times 2{,}240} \text{ tons/in.}^2 = 6 \text{ tons/in.}^2 \end{aligned}$$

26. Maximum Stress.—It must be noted that the formulæ of § 25 tell us nothing about the maximum stress induced as they do not enable us to calculate u, and take no account of the reflected waves. Attempts have been made to obtain formulæ for the stresses by elementary methods, but no such formula has yet been found to give the correct answers, so far as the author is aware. For purposes of calculation the following formulæ, derived from exact analysis, are given here :—

Case 1.—A uniform vertical rod is fixed at the upper end, and a weight W is suddenly attached to the lower end without velocity.

$$\mu = \frac{\text{wt. of body attached}}{\text{wt. of rod}}.$$

$w =$ wt. of rod per unit volume.
$l =$ length of rod.

μ	1	2	4
Intensity of stress .	$3{\cdot}27wl/E$	$5{\cdot}04wl/E$	$9{\cdot}18wl/E$ (at top)

If we treat the problem by the simple method of equating the work done by gravity, as the weight falls a distance equal to the extension of the rod, to the strain energy of the rod, we find that the stress induced is double the statical stress, i.e. it is $2W/S$, where S is the cross section of the rod, neglecting the weight of the rod. This errs on the side of safety.

Case 2.—A uniform horizontal rod is fixed at one end and free at the other. The free end is struck longitudinally by a weight W moving with velocity V. The maximum stress induced is given by Fig. 14, or when $\mu > 20$, by

$$p = V\sqrt{\frac{Ew}{g}}\left(1 + \sqrt{\mu}\right) \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

approximately; it occurs at the fixed end.

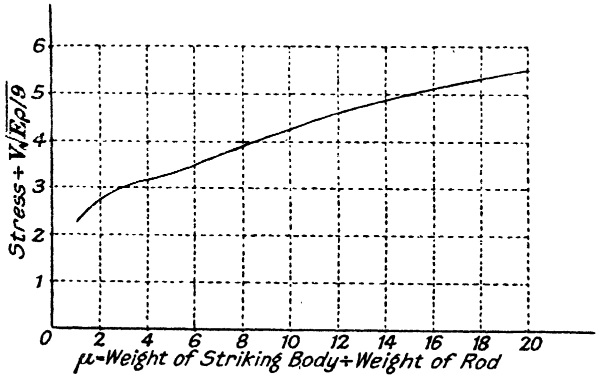

Fig. 14.

Simple methods will not yield a correct answer to this problem.

A source of uncertainty in these calculations is that we do not at present know for certain whether the relations between stress and strain, when these are changing with great rapidity, are the same as when the loads are applied slowly. It must also be borne in mind that the stresses are fluctuating stresses, and this has an important influence on the maximum permissible working stresses.

Example.—A steel rod $1\frac{1}{2}''$ diameter, 10 ft. long, weighing 0·283 lbs./in.^{3} is struck longitudinally at one end by a body weighing 200 lbs., moving at 4 ft./sec. The other end of the rod is fixed. Calculate the maximum stress set up in the rod, taking $E = 30 \times 10^6$ lbs./in.^{2}

The area of the cross section of the rod is $S = 1{\cdot}765$ in.^{2}

The weight is $0{\cdot}283$ lbs./in.$^3 \times 120$ in. $\times 1{\cdot}765$ in.$^2 = 60$ lbs. Hence

$$\mu = \frac{200}{60} = 3{\cdot}333$$

$$\sqrt{\frac{Ew}{g}} = \sqrt{\frac{30 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}283 \text{ lbs./in.}^3}{32{\cdot}2 \times 12 \text{ in./sec.}^2}} = 168 \text{ lbs. secs./in.}^3$$

$$V = 48 \text{ in./secs.}$$

Hence the maximum stress is, from Fig. 14,

$$p = 48\,\frac{\text{in.}}{\text{secs.}} \times 168\,\frac{\text{lbs. secs.}}{\text{in.}^3} \times 3{\cdot}10 = 25{,}000 \text{ lbs./in.}^2$$

It will be instructive to compare this result with that given by a more approximate method. The kinetic energy of the moving body is

$$\frac{200 \times 48^2}{2 \times 32{\cdot}2 \times 12} \text{ in. lbs.} = 598 \text{ in. lbs.}$$

If this were all instantly converted into strain energy in the rod, inducing a stress p, we should have

$$\frac{p^2}{2E} \times \text{vol. of rod} = 598 \text{ in. lbs.}$$

or

$$\frac{p^2 \times 1{\cdot}765 \times 120 \text{ in.}^3}{60 \times 10^6 \text{ lbs./in.}^2} = 598 \text{ in. lbs.}$$

giving

$$p = 13{,}000 \text{ lbs./in.}^2,$$

which is 48 per cent. too low.

Some writers attempt to get a more accurate result by assuming that after impact the velocity of the rod varies uniformly from a value v at the struck end to zero at the fixed end, and, on this assumption, calculating the loss of energy by the principle of the conservation of momentum. If w be the weight of the rod, it is easy to show that its momentum and kinetic energy are $\frac{wv}{2}$ and $\frac{wv^2}{6g}$ respectively, and that $v = V\Big/\left(1 + \frac{1}{2\mu}\right)$. The energy of the system after impact is then $\frac{WV^2}{2g} \cdot \frac{(1 + 1/3\mu)}{(1 + 1/2\mu)^2}$. Proceeding thus we find that the stress in the above case is 11,850 lbs./in.^{2}, so that the error is considerably increased by this method.

27. Stress in a Rotating Ring.—When a circular ring rotates about its axis each element is acted on by centrifugal force and the ring tends to swell, i.e. to increase in diameter. This tendency is resisted by tensile stresses set up between the elements of the ring. In what follows the linear dimensions of the cross section of the ring are supposed to be very small compared with the mean radius of the ring, and the area of the cross section of the ring is assumed to be constant.

Let r = the radius of the ring.

ω = the angular velocity in radians per second.

S = the area of the section of the ring cut by a plane through the axis.

w = the weight of the material per unit volume.

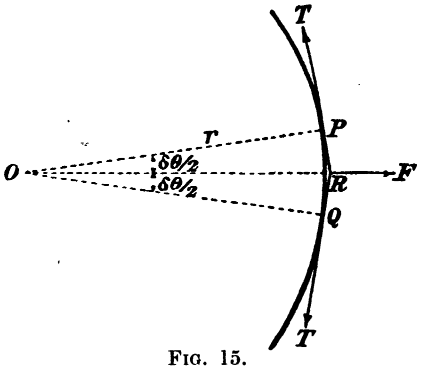

FIG. 15.

Consider an element PQ subtending a small angle $\delta\theta$ at the centre. Let T be the tensions exerted on the ends of PQ by the remainder of the ring; it is clear that T acts along the tangents at P and Q, and, from symmetry, that the tensions are the same at each end of PQ and therefore the same all round the circumference.

The weight of $PQ = wSr\delta\theta$.

The centrifugal force on PQ is

$$F = \frac{wSr\delta\theta . r\omega^2}{g}$$

which acts outwards along the radius OR bisecting PQ.

Resolving along OR we have

$$2T \sin\frac{\delta\theta}{2} = \frac{wSr^2\omega^2}{g}\delta\theta.$$

Since $\delta\theta$ is a very small angle we can write $\frac{\delta\theta}{2}$ for $\sin\frac{\delta\theta}{2}$, so that the equation becomes

$$2T\frac{\delta\theta}{2} = \frac{wSr^2\omega^2}{g}\delta\theta$$

$$\therefore T = \frac{wSr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The tensile stress in the ring is $p = T/S$, that is

$$\mathbf{p} = \frac{\mathbf{wr^2\omega^2}}{\mathbf{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which is independent of the area of the cross section.*

It is important here, as always, to pay careful attention to the units: if w, r, S are measured in inch units g must also be measured in inch units, i.e. $g = 32{\cdot}2 \times 12$ inches/sec.^{2}

It should be noted that this cannot be applied, with any accuracy, to find the stresses in the rims of spoked flywheels, since the spokes exercise a local restraint on the rim and a bending action takes place.

LATERAL CONTRACTION AND EXPANSION DUE TO DIRECT STRESSES

28. Poisson's Ratio.—When the material of a body is under tension (or compression) the stretching (or contracting) in the direction of the applied force is accompanied by a lateral contraction (or expansion) in all directions at right angles to the applied force. If the strain in the direction of the applied stress p be $e = p/E$, the lateral strain is found to be proportional to the direct strain e, and is given by

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* This result is only true if the dimensions of the cross section are small compared with the radius of the ring.

$$\text{lateral strain} = \frac{\mathbf{e}}{\mathbf{m}} = \frac{\mathbf{p}}{\mathbf{mE}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where m is a constant determined by experiment. This constant depends only on the material in question and not at all on the stress. The reciprocal of m, i.e. $\frac{1}{m}$, is called *Poisson's ratio.* It is sometimes convenient to write $1/m = \sigma$. For most metals m has a value between 3 and 4; for steel it is frequently taken as $\frac{10}{3}$.

Example.—A steel bar of rectangular cross section, 2″ × ½″ is subjected to a pull of 10 tons in the direction of its length. Taking $E = 13{,}500$ tons/in.^{2} and $m = \frac{10}{3}$, find the decrease of length of the sides of the cross section and the percentage decrease of area of cross section.

The strain in the direction of the pull is

$$e = \frac{p}{E} = \frac{P}{ES} = \frac{10 \text{ tons}}{13{,}500 \text{ tons/in.}^2 \times 1 \text{ in.}^2} = \frac{1}{1350} = 0{\cdot}00074.$$

The lateral strain is

$$\frac{e}{m} = \frac{3}{10} \times 0{\cdot}00074 = 0{\cdot}000222.$$

Hence the 2″ side decreases by 2 × 0·000222 = 0·000444″.
and " ½″ " " ½ × 0·000222 = 0·000111″.

The new area of the cross section is

$$\begin{aligned} &(2 - 0{\cdot}000444)(0{\cdot}5 - 0{\cdot}000111) \text{ in.}^2 \\ &= 2 \times 0{\cdot}5(1 - 0{\cdot}000222)^2 \\ &= 1 \times (1 - 0{\cdot}000444), \text{ neglecting } (0{\cdot}000222)^2. \end{aligned}$$

Hence the decrease of area is 0·000444 in.^{2}, or 0·0444 per cent.

29. Strain Due to Two Stresses at Right Angles.—If the material of a body be subjected simultaneously to two direct stresses at right angles, the strain produced by either stress is the same as if the other were absent. This is known as the *principle of superposition.*

Thus, let p_x and p_y denote direct stresses in two perpendicular directions, Ox and Oy. Considering extensions as positive,

p_x produces strain $\frac{p_x}{E}$ in direction Ox, $-\frac{p_x}{mE}$ in direction Oy.

p_y " " $-\frac{p_y}{mE}$ " Ox, $\frac{p_y}{E}$ " Oy.

Hence, if e_x and e_y denote the total strains in the directions Ox and Oy, we have, by addition of the above,

$$\left.\begin{aligned} \mathbf{e_x} &= \frac{\mathbf{p_x}}{\mathbf{E}} - \frac{\mathbf{p_y}}{\mathbf{mE}} \\ \mathbf{e_y} &= -\frac{\mathbf{p_x}}{\mathbf{mE}} + \frac{\mathbf{p_y}}{\mathbf{E}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Similarly, if there be three mutually perpendicular stresses p_x, p_y, p_z, the corresponding strains are

$$\left.\begin{aligned} e_x &= \frac{p_x}{E} - \frac{p_y + p_z}{Em} \\ e_y &= \frac{p_y}{E} - \frac{p_z + p_x}{Em} \\ e_z &= \frac{p_z}{E} - \frac{p_x + p_y}{Em} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (17)$$

30. Change of Area and Volume Due to Strain.—If, on any plane, the strains in two directions at right angles be e_x and e_y, the sides δx and δy of an element of area become $\delta x(1 + e_x)$ and $\delta y(1 + e_y)$. Hence the area becomes

$$\delta x \delta y.(1 + e_x)(1 + e_y)$$

or

$$\delta x.\delta y(1 + e_x + e_y),$$

neglecting the product $e_x e_y$, which can be seen from the example on p. 27 to be justifiable. Moreover, it was mentioned on p. 8 that the whole theory of our subject supposes the squares and products of strains to be negligible. We see, then, that the increase of area per unit area is measured by

$$\mathbf{\delta S/S = e_x + e_y} \quad . \; . \; . \; . \; . \; . \; . \quad (18)$$

Similarly, the change of volume per unit volume is measured by

$$\mathbf{\delta V/V = e_x + e_y + e_z} \quad . \; . \; . \; . \; . \; . \; . \quad (19)$$

where e_x, e_y and e_z denote the strains in three mutually perpendicular directions.

The ratio of the change of volume to the original volume is called the *volumetric strain.*

31. Bulk Modulus.—If a body be subjected to hydrostatic pressure such that the stress is the same in all directions, we shall have $p_x = p_y = p_z = p$, say. Then the ratio

$$\frac{p}{\text{volumetric strain}}$$

is known as the Bulk Modulus of the material, and is usually denoted by K.

32. Relation between E and K.—When $p_x = p_y = p_z = p$ we have

$$e_x = e_y = e_z = \frac{p}{E} - \frac{2p}{mE} = \frac{m - 2}{mE} \cdot p.$$

Hence the volumetric strain is

$$e_x + e_y + e_z = \frac{3(m - 2)}{mE} p.$$

But this is equal to p/K by the definition of K, therefore

$$\mathbf{K = \frac{mE}{3(m - 2)}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

From this relation we see that, if m be positive, it must be greater than 2, otherwise K would be negative, a state of affairs which is inconceivable.

33. Modified Values of E when Lateral Strain is Prevented.—If a body be subjected to a two-dimensional distribution of stress defined by p_x and p_y, the corresponding strains are

$$e_x = \frac{p_x}{E} - \frac{p_y}{mE} \text{ and } e_y = -\frac{p_x}{mE} + \frac{p_y}{E}.$$

If one of these strains is prevented, say e_y, we must have

$$p_y - \frac{p_x}{m} = 0, \text{ or } p_y = \frac{p_x}{m}.$$

Then

$$e_x = \frac{p_x}{E} - \frac{p_x}{m^2E} = \frac{m^2-1}{m^2E}\,p_x.$$

This is the same as it would be if p_y were absent and E took on the value $\dfrac{m^2E}{m^2-1}$.

Similarly, if there are three mutually perpendicular stresses p_x, p_y, p_z, and two strains, say e_y and e_z are prevented, we must have

$$p_y - \frac{p_z + p_x}{m} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$p_z - \frac{p_x + p_y}{m} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Hence, by subtraction,

$$(p_y - p_z)\left(1 + \frac{1}{m}\right) = 0$$

Therefore, since m[*] $\neq -1$, $p_y = p_z$. Then, putting $p_z = p_y$ in (i), we get

$$p_y\left(1 - \frac{1}{m}\right) - \frac{p_x}{m} = 0$$

$$\therefore\ p_y = \frac{p_x}{m-1}.$$

Giving this value to p_y and p_z we have

$$e_x = \frac{p_x}{E} - \frac{p_y + p_z}{mE} = \frac{(m+1)(m-2)}{m(m-1)E}\,p_x,$$

which is the same as if p_y and p_z were absent and E were replaced by $\dfrac{m(m-1)E}{(m+1)(m-2)}$.

Example 1.—A piece of steel 9″ long and 1″ × 1″ cross section is subjected to a tensile stress of 10 tons/in.^{2} in the direction of its length. If $m = 10/3$ and $E = 30 \times 10^6$ lbs./in.^{2}, calculate the change of volume. Let e_x denote the strain in the direction of the axis of the rod, then

$$e_x = \frac{10 \times 2{,}240 \text{ lbs./in.}^2}{30 \times 10^6 \text{ lbs./in.}^2} = \frac{224}{3} \times 10^{-5}.$$

---

[*] A result which is proved later.

Let e_y and e_z denote the strains parallel to the sides of the cross section, then

$$e_y = e_z = -\frac{e_x}{m} = -\frac{3}{10} \times \frac{224}{3} \times 10^{-5} = -\frac{67 \cdot 2}{3} \times 10^{-5}.$$

The increase of volume per unit volume is

$$e_x + e_y + e_z = \frac{224 - 134 \cdot 4}{3} \times 10^{-5} = 29 \cdot 87 \times 10^{-5}.$$

The original volume $= 9$ in.^{3}
∴ increase of volume $= 9$ in.$^3 \times 29 \cdot 87 \times 10^{-5}$
$= 0 \cdot 00269$ in.^{3}

Example 2.—The plates of a cylindrical boiler 6′ 0″ diameter and 10′ long are under a tensile stress of 10,000 lbs./in.^{2} in the direction of the circumference and a tensile stress of 5,000 lbs./in.^{2} in an axial direction. With the same values of the constants as above, find the increase in the internal capacity of the boiler. Neglect the compressive stress on the inner surface, which is equal to the steam pressure.

Let x refer to the circumferential direction,
y „ „ axial „

Then $p_x = 10{,}000$ lbs./in.^{2} and $p_y = 5{,}000$ lbs./in.^{2} The strains are

$$e_x = \frac{10{,}000}{E} - \frac{5{,}000}{E \times \frac{10}{3}} = \frac{8{,}500}{E}$$

$$e_y = \frac{5{,}000}{E} - \frac{10{,}000}{E \times \frac{10}{3}} = \frac{2{,}000}{E}.$$

Now the diameter of the boiler increases or decreases in direct proportion to the circumference, so that e_x also denotes the increase of diameter per unit length. Thinking of the space inside the boiler, the strains of the dimensions are

$\frac{2{,}000}{E}$ parallel to the axis

$\frac{8{,}500}{E}$ along any two perpendicular radii.

Hence the increase of volume per unit volume is

$$\frac{2{,}000}{E} + \frac{8{,}500}{E} + \frac{8{,}500}{E} = \frac{19{,}000}{E}.$$

The original volume

$$= \frac{\pi}{4} \times 6^2 \times 10 = 90\,\pi \text{ ft.}^3$$

Hence the increase of volume

$$= \frac{19{,}000}{E} \times 90\,\pi = \frac{19{,}000}{30 \times 10^6} \times 90\,\pi = 0 \cdot 179 \text{ ft.}^3 = 310 \text{ in.}^3$$

Example 3.—A gun tube *A* is 32″ outside diameter, and another tube *B* is to be shrunk on to it. It is desired that the radial pressure between the two tubes should be 5 tons/in.^{2}, and calculation shows that there will then be a tensile circumferential stress of 22·8 tons/in.^{2} at the inside of *B*, and a similar compressive stress of 8·35 tons/in.^{2} at the outside of *A*. Calculate the proper initial difference between the inside radius of *B* and the outside radius of *A*, before heating, if $E = 13{,}000$ tons/in.^{2} There is no axial stress.

The circumferential strain at the outside of *A* is

$$-\frac{8\cdot35}{E}+\frac{5}{E\times\frac{10}{3}}=-\frac{6\cdot85}{E}$$

At the inside of *B* it is

$$\frac{22\cdot8}{E}+\frac{5}{E\times\frac{10}{3}}=\frac{24\cdot3}{E}$$

These quantities will also denote the strains in the direction of any radius. If *B* were removed the outside radius of *A* would increase by $16''\left(\frac{8\cdot35}{E}-\frac{5}{mE}\right)$ and the inside radius of *B* would decrease by $16''\left(\frac{22\cdot8}{E}+\frac{5}{mE}\right)$. Hence the initial difference between the radii should be

$$16''\left(\frac{8\cdot35}{E}-\frac{5}{mE}+\frac{22\cdot8}{E}+\frac{5}{mE}\right)=\frac{16''\times31\cdot15}{13{,}000}=0\cdot038''.$$

EXAMPLES I

1. A tie bar of steel has a cross-section 6″ × $\frac{11}{16}$″ and a load of 18 tons is applied to it. Find the tensile stress produced across a normal section.

2. A specimen of steel 1″ diam. extends 0·0061″ under a load of 8 tons, the original length being 8″. Find the total extension and the strain when the load is 10 tons. When would the relation you apply in this case become inapplicable ? (Special Exam., Cambridge, 1914.)

3. A bar of diameter $\frac{7}{8}$″ is subjected to an axial load of 2 tons, find the stress on the cross section and the percentage extension, taking $E = 30 \times 10^6$ lbs./in.^{2} (Special Exam., Cambridge, 1911.)

4. The observations below were taken, for the load and the extension, during the test of a bar of length 10″ and diameter 0·88″. The breaking load was 15·0 tons and the diameter at fracture 0·74″. Estimate Young's modulus, the elastic limit, the yield point, the breaking stress (nominal) and the percentage contraction of area at the breaking point. (Special Exam., Cambridge, 1911.)

Load (tons) .	0	3	6	9	10	10·5	11·0	11·5	11·8	11·9	12·0
Extension (in.)	0	·0037	·0075	·0112	·0125	·0131	·0140	·0165	·021	·035	·069

5. A steel bar, 1″ diameter and 8″ long, is compressed 0·004″ by a thrust of 12,000 lbs. Determine the extension under a pull of 4,000 lbs. (Special Exam., Cambridge, 1913.)

6. 1,000 ft. of uniform steel rope are hanging down a shaft. Find the elongation of the first 500 ft. at the top if the weight of steel is 480 lbs./ft.^{3} and $E = 30 \times 10^6$ lbs./in.^{2} (Special Exam., Cambridge, 1919.)

7. 1″ diameter stays are to be used for the firebox of a boiler, the pressure being 180 lbs./in.^{2} If the stress in the bolts is not to exceed 4,500 lbs./in.^{2} at what pitch should they be set ?

8. In a tensile test piece the diameter is $\frac{3}{4}$″ and the length under test 8″. The metal yields when under a load of 8 tons, and breaks when the load is 15 tons ; the length is then 10″, and the diameter of the neck 0·4″. Find the yield point, the ultimate strength, the percentage elongation and percentage reduction of area. (R.N.C., Greenwich, 1923.)

9. Two parallel wires, 20 ft. long and 0·25 in.^{2} cross-sectional area, are hung vertically 3″ apart and support a horizontal bar at their lower ends. When a load of 2,000 lbs. is attached to one of the wires, it is observed that the bar is inclined 1·1° to the horizontal. Estimate the value of Young's Modulus for the wire. (R.N.C., Greenwich, 1923.)

10. A length of 400 ft. of steel wire, weighing $1\frac{1}{8}$ lb. per ft., is placed along a horizontal floor and pulled slowly along by a horizontal force applied to one end. If this force measures 110 lbs., estimate the increase in length of the wire due to being towed thus, assuming a uniform coefficient of friction. Take the weight as 486 lbs./ft.^{3}, and $E = 30 \times 10^6$ lbs./in.^{2} (R.N.E. College, Keyham, 1923.)

11. The piston rod of a double acting hydraulic cylinder is 3″ diameter and 10′ 0″ long. The piston has a diameter of 10″ and is subjected to 1,000 lbs./in.^{2} water pressure on one side, and 40 lbs./in.^{2} on the other. On the return stroke these pressures are interchanged. Find the maximum stress that occurs in the rod, and the change in length of the rod between two strokes, allowing for the area of the piston rod on one side of the piston. $E = 30 \times 10^6$ lbs./in.^{2} (R.N. College, Greenwich, 1922.)

12. Part of a spinner in front of the airscrew of an aeroplane is a circular wooden ring of semi-circular cross section. The outside diameter of the ring is 36″ and it rotates at 2,000 r.p.m. In addition to the centrifugal forces acting on the ring there is an external radial load of 30 lbs. per inch outwards. What must be the diameter of the cross section of the ring if the stress is not to exceed 2,500 lbs./in.^{2}? Take the weight of the wood as 0·0173 lbs./in.^{3}

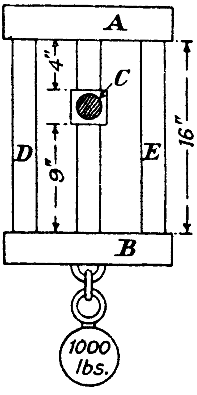

FIG. 16.

13. The framework shown in Fig. 16, which consists of three rods connecting two yokes A and B, is supported on a horizontal shaft passing through the bearing C. The rods have the same section and the same Modulus of Elasticity. The yokes are supposed to be absolutely rigid and the rods initially free from stress. A load of 1,000 lbs. is then attached to B. Find the pull set up in the two outer rods, and the thrust and pull in the upper and lower portions of the middle rod. (Mech. Sc. Trip., 1911.)

14. A hexagonal slab of steel 3 tons in weight is supported horizontally on six vertical legs, all of equal section. Two of the legs, on one diagonal, are of brass ($E_1 = 12 \times 10^6$ lbs./in.^{2}), and the remainder of steel ($E = 30 \times 10^6$ lbs./in.^{2}), and all are carefully adjusted so that each bears one-sixth of the total weight of the slab. If a load of 2 tons is now placed in the centre of the slab, which may be assumed perfectly rigid, find the distribution of load on the legs. (R.N.E. College, Keyham, 1921.)

15. To enable two walls, 20 ft. apart, to give mutual support they are stayed together by a 1″ diameter steel tension rod with screwed ends, plates and nuts. The rod is heated to 300° F. when the nuts are screwed up. If the walls yield, relatively, $\frac{3}{16}$″ when the rod cools to 60° F. find the pull of the rod at that temperature. The coefficient of expansion of steel $= 6 \times 10^{-6}$ per °F., and $E = 13{,}000$ tons/in.^{2} (R.N.E. College, Keyham, 1922.)

16. A steel tube 1·25″ diameter, 0·104″ thick, and 12′ 0″ long, is covered and lined throughout with copper tubes 0·08″ thick. The three tubes are firmly united at their ends. The compound tube is then raised in temperature 200° F. Find the stresses in the steel and copper, and the increase in length of the tube; also what must be the magnitude of the

forces which, applied to the ends of the tube, will prevent its expansion? Assume $E = 30 \times 10^6$ lbs./in.^{2} for steel and 16×10^6 lbs./in.^{2} for copper, and the coefficients of linear expansion of steel and copper 0·000,006 and 0·000,0095 per ° F. respectively. (R.N.E. College, Keyham, 1920.)

17. A steel rod, $\frac{3}{4}''$ diameter, screwed at each end, is placed inside a brass tube $10''$ long, whose internal diameter is $\frac{3}{4}''$ and external diameter $1\frac{1}{2}''$. Nuts and washers at each end are adjusted so that there is no end play at 60° F. Calculate the stress set up in the steel and in the brass when the temperature of both is raised to 160° F. Coefficient of expansion of steel $6{\cdot}5 \times 10^{-6}$ per ° F., and of brass $10{\cdot}5 \times 10^{-6}$ per ° F.; E for steel 30×10^6 lbs./in.^{2}, and for brass $14{\cdot}5 \times 10^6$ lbs./in.^{2} (Private Studentship of Naval Architecture, 1922.)

18. The hoisting rope for a mine shaft is to lift a cage of weight W, and itself weighs w per unit length. If the rope is to be tapered so that the stress on every section is p, prove that the law of the taper is

$$S = \frac{W}{p}\varepsilon^{\frac{wx}{p}}$$

where S is the area of the cross section of the rope at a height x above the cage, and ε is the base of Naperian logarithms.

19. A bar of steel $3'' \times 1''$ cross section is subject to an axial pull of 18 tons. Calculate the decrease in the lengths of the sides of the cross section, if $E = 13{,}500$ tons/in.^{2} and $m = 3{\cdot}5$.

20. A cube of iron, the length of whose side is 100 in., is subjected to a uniform pressure of 10 tons/in.^{2} on two opposite faces. The other faces are prevented, by lateral pressure, from extending more than $0{\cdot}02''$. Find the lateral pressure, taking E and m as in Ex. 19.

21. A tube is to be shrunk on to a solid shaft $8''$ diameter, the radial pressure between the two being 2 tons/in.^{2} It is calculated that the tensile hoop stress at the bore of the tube is 5·2 tons/in.^{2} Taking $E = 13{,}700$ tons/in.^{2} and $m = 10/3$, find the correct internal diameter of the tube before heating.

22. In a $13{\cdot}75''$ gun, the calculated stresses at the bore are: Radial pressure, 18 tons/in.^{2}; hoop stress, 26 tons/in.^{2}; axial tensile stress, 3 tons/in.^{2} Taking $E = 12{,}000$ tons/in.^{2} and $m = 10/3$, find the increase of diameter of bore.

23. At the inside of a certain gun tube, the stresses at the moment of firing are, circumferential compressive stress 2 tons/in.,^{2} radial compressive stress 8 tons/in.^{2}, longitudinal tensile stress 1 ton/in.^{2} Taking $m = 10/3$, find the strain-energy per unit volume, and compare this with the strain-energy per unit volume in simple tension with a stress of 30 tons/in.^{2} (H.M. Dockyard Schools, 1931.)

24. A beam weighing 100 lbs. is held in a horizontal position by three vertical wires, one attached to each end of the beam, and one to the middle of its length. The outer wires are brass $\frac{1}{20}''$ diameter, and the centre one is steel $\frac{1}{40}''$ diameter. Find the intensity of stress in each wire. All the wires are the same length. E for brass $= 12{\cdot}5 \times 10^6$ lbs./in.,^{2} E for steel is 30×10^6 lbs/in.^{2} (H.M. Dockyard Schools, 1931.)

CHAPTER II

DISPLACEMENT DIAGRAMS AND REDUNDANT FRAMES

34. Displacement Diagrams.—The principles set out in the previous chapter at once enable us to solve two problems which the methods of pure statics leave unsolved, namely to find the deformation of a simple framework under a given system of forces, and to find the loads in the members of a framework which has a greater number of bars than is required to enable it to keep its shape.

We shall first show how to find the deformation of a framework, with pin-joints, when acted upon by a given system of external forces. It is most convenient to employ graphical methods, and the process is usually called drawing a displacement diagram. The student who is familiar with the methods employed for drawing velocity and acceleration diagrams for link mechanisms will find no difficulty at all in drawing displacement diagrams for frameworks.

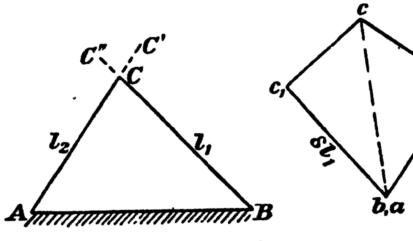

Fig. 17.

Suppose we have a triangular frame ABC (Fig. 17) in which the side AB is so fixed that its length cannot change, whilst the corner C is acted on by any forces in the plane ABC. The joints A, B, C are pin-joints.

First, by ordinary statics, find the loads in AC and BC; then, by formula (4, p. 4) find the elongations (positive or negative) of AC and BC.

Let δl_1 and δl_2 be the elongations of BC and CA.

Now, but for the constraint offered by BC, the elongation of AC would carry C to C' in the line AC produced, such that $CC' = \delta l_2$; similarly, but for AC, the elongation of BC would carry C to C'' on BC produced, such that $CC'' = \delta l_1$. On account of the restraint offered by the one rod to the other they must undergo rotations until C' and C''

coincide. These rotations being very small we can regard C' and C'' as moving short distances at right angles to AC' and BC''. We therefore proceed as follows:

Take a point a to represent A and, since the displacement of B relative to A is zero, the point a can also represent B, and so is labelled b as well.

Draw $bc_1 = \delta l_1$ parallel to BC, and $ac_2 = \delta l_2$ parallel to BC. If either rod had contracted instead of extending, BC say, then the corresponding stretch δl_1 would be drawn the other way.

From c_1 and c_2 draw lines at right angles to bc_1 and ac_2 to meet at c.

Then ac is the total displacement of c relative to A or B.

In more complicated frameworks we have simply to repeat the above process for each triangle of the frame, taking any convenient angular point and one side through that point, as the a point and line from which to measure the displacements.

Example.—In Fig. 18 *BCDFG* represents the front truss of the wings of a certain aeroplane. The loads in each member and the dimensions are given in the table below. It is required to find the vertical displacements of D, F and G relative to C for five times normal load, B and C being regarded as fixed points. The external loads are *upward* vertical forces at C, D, F, B, E, G, due to the air pressure on the wings. The loads are given in the following table.

Member.	T Normal Load lbs.	l in.	E lbs./in.^{2}	S in.^{2}	Elongation (in.)	
					$\frac{lT}{ES}$	$5\frac{lT}{ES}$
CD	− 1,650	90	$1·6 \times 10^6$	4·80	− ·0193	− ·0965
DF	− 512	108	„	3·40	− ·0102	− ·0510
BE	+ 512	90	„	3·50	+ ·00825	+ ·0412
EG	0	108	„	3·50	0	0
DE	− 550	66	„	2·10	− ·0108	− ·0540
FG	− 160	66	„	1·65	− ·004	− ·020
BD	+ 1,435	115·5	30×10^6	0·093	+ ·0594	+ ·297
EF	+ 600	127	„	0·069	+ ·0368	+ ·184

In the table l denotes the length, and S the cross-sectional area of a member, and T denotes the normal load in the member. The rest of the table is occupied with the calculation of the elongation (lT/ES) of each member; tensions and stretches are taken as positive; thrusts and contractions as negative. The displacement diagram in the lower part of the figure is drawn as follows.

The displacement of C relative to B is zero, so take a point b, c to represent both, and commence with the triangle *BCD*. Draw bd_1 = 0·297 in. parallel to BD and cd_2 = 0·0965″ in the direction DC (right to left since DC contracts). Then draw d_2d and d_1d, at right angles to bd_2 and bd_1 respectively, to meet at d. Then bd is the total displacement of D, the vertical component being d_2d which is found to be 0·638″.

We next find the displacement of E: BE stretches 0·0412″ so draw be_1 = 0·0412″ in the direction BE. DE shortens by 0·054″ so that E moves upwards relative to D by this amount; therefore draw de_2 upwards

= 0·054 in. Then, by drawing e_1e perpendicular to be_1 and e_2e perpendicular to de_2 the point e is found.

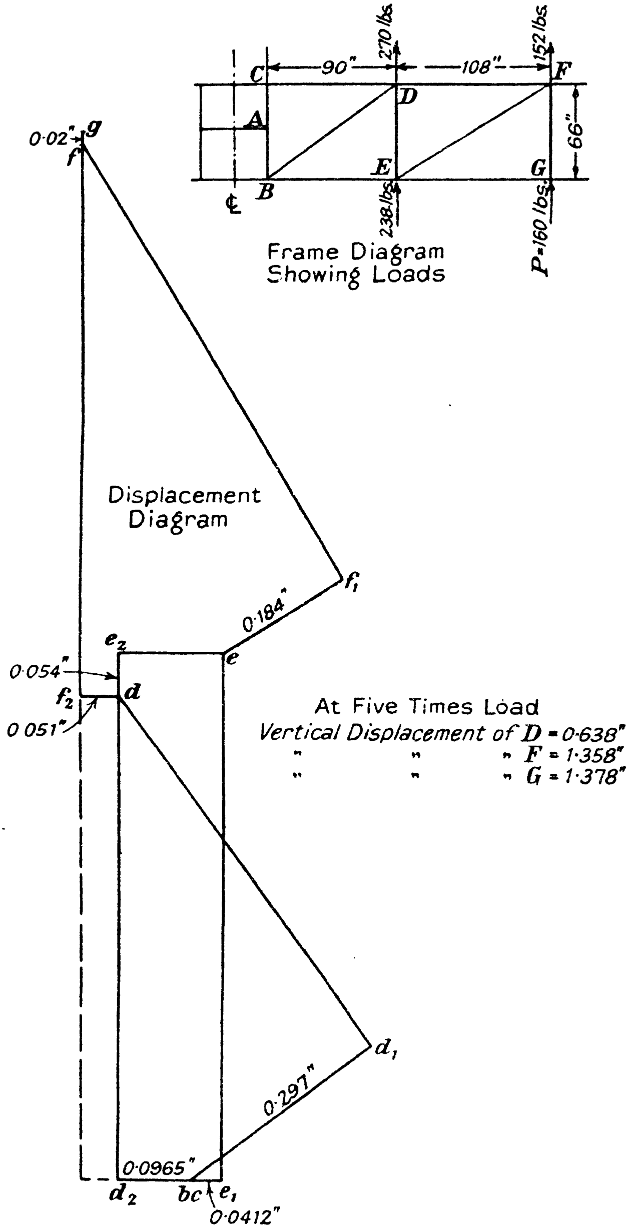
C
90"
270 lbs
108"
152 lbs
F
D
A
66"
E
G
B
℄
238 lbs
P = 160 lbs
Frame Diagram
Showing Loads
0·02"
g
f
Displacement
Diagram
f_1
0·184"
e_2
e
0·054"
f_2
d
0 051"
At Five Times Load
Vertical Displacement of D = 0·638"
" " " F = 1·358"
" " " G = 1·378"
d_1
0·297"
0·0965"
d_2
bc
e_1
0·0412"

FIG. 18.

Finally, starting at e and d and proceeding in a similar manner f is found and then g. Measurement shows that the vertical displacements of F and G, relative to C, are 1·358 in. and 1·378 in. respectively.

35. Application of the Principle of Virtual Work.—The following method of calculating displacements will sometimes be useful, particularly when the component displacement of one particular point only, in a given direction, is required. For example, consider the pin-jointed framework shown in Fig. 19, where the loads P_1, P_2 and P_3 are supported by the reactions R_1 and R_2. Suppose we require the vertical component of the displacement F, then we proceed as follows:

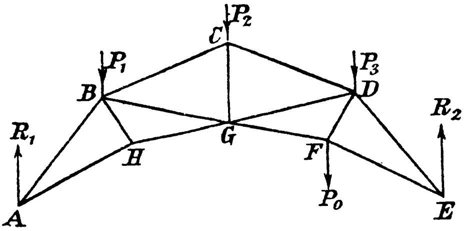

FIG. 19.

First find the force in every member due to the external forces P_1, P_2, etc.

Let T_1 denote the tension in any member. Next, introduce at F any convenient[*] force P_0, *in the direction in which the displacement of F is required*, and find the forces in all the members due to P_0.

Let T_0 denote the tension in any member, due to P_0. Then the total force in any member will be the sum of the corresponding T_1 and T_0; denote this by T. The increase in length of the member will be $\frac{lT}{ES}$, hence the work done on it on account of P_0 will be $\frac{1}{2}T_0\frac{lT}{ES}$, if P_0 be increased uniformly from zero to its final value.

The total work done by P_0 will therefore be given by $\frac{1}{2}\Sigma T_0\frac{lT}{ES}$.

But this can also be expressed in the form $\frac{1}{2}P_0x$, where x is the displacement of F in the direction of P_0, due to all the forces P_0, P_1, P_2, etc. Therefore

$$\frac{1}{2}P_0x = \frac{1}{2}\Sigma T_0\frac{lT}{ES}$$

$$\therefore\ x = \Sigma\frac{T_0}{P_0}\cdot\frac{lT}{ES} \quad . \ . \ . \ . \ . \ . \ . \quad (1)$$

The ratio $\frac{T_0}{P_0}$ represents the numerical value of the force, in the particular member to which it refers, for unit load at F.

If now P_0 be made zero, T becomes T_1, and the corresponding value of x is the displacement of F, in the direction P_0, due to the external forces P_1, P_2, etc.

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[*] In numerical cases P_0 will usually be taken as unit force, such as one pound or one ton.

Example.—Apply this method to the example on p. 35 to find the vertical displacement of G. The calculations are summarized in the following table :—

Member.	T_1 (lbs.)	T_0/P_0 =	$l\frac{T_1}{ES}$ (in.)	$\frac{T_0}{P_0}\frac{lT_1}{ES}$ (in.)
CD	− 1,650	− 3	− 0·0193	0·05790
DF	− 512	− 1·64	− 0·0102	0·01670
BE	+ 512	+ 1·64	+ 0·00825	0·01355
EG	0	0	0	0
DE	− 550	− 1	− 0·0108	0·01080
FG	− 160	− 1	− 0·004	0·00400
BD	+ 1,435	+ 1·75	+ 0·0594	0·10400
EF	+ 600	+ 1·92	+ 0·0368	0·07060

$$\Sigma \frac{T_0}{P_0}\frac{lT_1}{ES} = 0{\cdot}27755.$$

The second and fourth columns are taken from the table on p. 35. The third column is obtained by finding the algebraic value of the force in each member when a vertical upward force of one pound acts at G. The last column is found by multiplying together the figures of the two previous columns. The sum of the figures of the last column gives the upward vertical displacement of G. The displacement at five-times load is thus

$$5 \times 0{\cdot}27755'' = 1{\cdot}388'' \text{ approximately.}$$

FRAMED STRUCTURES WITH REDUNDANT BARS

36. Simply-stiff and Redundant Frames.

—By a framed structure we mean a structure which is composed of straight bars joined together at their extremities, and unless it be expressly stated to the contrary we assume that the joints are such that no bending can be transmitted from one bar to another ; this is commonly expressed by saying that the structure is pin-jointed.

If the structure has just sufficient bars to prevent collapse without the application of external forces it is called a simply-stiff frame ; when there are more bars than this the frame is said to be redundant. Definite relations exist which must be satisfied by the number of bars and joints if a framework is to be simply-stiff.

37. Conditions for Simple Stiffness.

—Let b = the number of bars, and j the number of joints in a simply-stiff frame.

(i) PLANE FRAMEWORKS.—Consider first the case where the axes of all the bars are in one plane : let us build up the frame, starting with one bar. To fix any other point in reference to the ends of this bar we must use two more bars, thus making a triangular frame, i.e. $b = 3$ and $j = 3$. If we wish to add another point to the frame we must again add two more bars, thereby fixing the new point with reference to one side of the triangle ; we now have $b = 5$, $j = 4$. Proceeding in this we shall find that b and j are always connected by the relation*

$$b = 2j - 3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(ii) THREE-DIMENSIONAL FRAMEWORKS.—We shall now take the case when the axes of the bars are not all in one plane. The simplest plane frame that we can have is a triangle, *ABC* say; if we wish to anchor a fourth point *D* in space with reference to this triangle we can first join it to two corners of the triangle by two bars, *DA* and *DB* say, but this will still leave *D* free to swing round the side *AB* of the triangle: to fix *D* completely we must also join it to *C*, so that altogether we must add three bars (making $b = 6$) to fix a fourth point ($j = 4$) relative to a triangle. Similarly a fifth point will require three more bars to fix its position relative to one of the triangles formed by the first six rods, making $b = 9$ and $j = 5$. Continuing in this manner we shall see that the relation between b and j is

$$b = 3j - 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If b is less than the value given by these formulæ the frame will collapse, if greater the excess gives the number of redundant bars. Thus a plane framework consisting of four joints and six bars would have one redundant member, since it is only necessary to have $2 \times 4 - 3 = 5$ bars.

38. Self-strained Frameworks.—If a framework is simply stiff none of the members will have any stresses until external forces are applied, but when there are redundant bars this is not necessarily true. Suppose we wish to add extra bars to a simply stiff frame and that these are slightly too long or too short to fit exactly into their places: the original frame will have to be strained to accommodate the new members. When the latter are in position and the applied constraint is removed the tendency of the original framework to resume its original configuration will strain the new members. In such a case the new framework will be in a state of stress before any external forces are applied to it, and we say that the frame is self-strained. If the new members fit exactly so that no straining is necessary to get them into position, the new framework of course will not be self-strained.

39. Stresses in Redundant Frames.—When the frame is simply stiff the forces in all the bars can be found by the ordinary methods of statics, but when there are redundant bars this is not possible without the aid of the principles of elasticity. Here we shall be concerned only with redundant frames, and we make the following assumptions: (i) the structure is pin-jointed, so that the forces acting on any member must act along the axis of that member. (ii) The members are such that they obey Hooke's Law, not only as regards the material of which they are made, but also as regards their own load-extension relations. With these assumptions we shall now proceed to the theory which leads to a method of determining the stresses in redundant frameworks.

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* Or thus: The equilibrium of each joint provides two equations of equilibrium, so that there will be $2j$ equations. But these will contain implicitly the three equations of equilibrium of the whole structure, so that there are only $2j-3$ independent equations, which is therefore the number of bars necessary for equilibrium.

40. Strain Energy of a Framework.

—In what follows we shall use the following notation:

U = strain energy.
P = an external force acting at a joint of a frame.
x = the displacement of the point of application of P, in the direction in which P acts, as P is increased gradually from zero to its final value.
T = the tension in a member.
l = length of a member.
S = area of the (uniform) cross section of a member.
E = Young's Modulus.

Then, by equation (9) § 21, the strain energy of the frame is given in terms of the tensions in the members by the equation

$$U = \Sigma \frac{lT^2}{2ES} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the summation extending to all the members of the frame.

Again by the principle of the conservation of energy, the strain energy given to the frame by the application of the external forces P must equal the work done by these forces, i.e. $\frac{1}{2}\Sigma Px$, the summation extending to all the external forces.

If the framework is not self-strained we can then write

$$U = \tfrac{1}{2}\Sigma Px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus we can express the strain energy in terms of the tensions in the bars or in terms of the external forces as we please.

41. Theorem Relating to the Strain Energy of a Framework.

—*If the strain energy of a framework be expressed as a function of the external forces, the differential coefficients of this function, with respect to these forces, give the displacements of the points of application of these forces in their own directions.*

If the framework is initially self-strained let U_0 be its strain energy, and let U' be the strain energy added by the gradual application of the external forces of which P is a type. Then we can write

$$U = U_0 + U' = U_0 + \tfrac{1}{2}\Sigma Px \quad . \quad . \quad . \quad . \quad (i)$$

where U denotes the total strain energy of the frame.

Now suppose the external forces, such as P, undergo indefinitely small increments δP. The displacements of their points of application suffer small increments such as δx. Then the increase in the strain energy of the frame must equal the work done by the forces P, that is

$$\delta U = \Sigma(P.\delta x) \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

But, from (i), we must have, since U_0 is a constant,

$$\delta U = \tfrac{1}{2}\Sigma d(Px) = \tfrac{1}{2}\Sigma(P.\delta x) + \tfrac{1}{2}\Sigma(x.\delta P) \quad . \quad . \quad (iii)$$

From (ii) and (iii) it follows that

$$\Sigma(P.\delta x) = \tfrac{1}{2}\Sigma(P.\delta x) + \tfrac{1}{2}\Sigma(x.\delta P).$$

$$\therefore \Sigma(x.\delta P) = \Sigma(P.\delta x) \quad . \quad . \quad . \quad . \quad (iv)$$

Now the right side represents the work done, or the increase of strain energy, on account of the increases δP of the external forces P, and therefore the left-hand side must equally well represent the same quantity. But the increase of U, the strain energy, arising from increments δP, must be given by

$$\delta U = \Sigma\frac{\partial U}{\partial P}\delta P.$$

Therefore, from (iv), we have

$$\Sigma\frac{\partial U}{\partial P}\delta P = \Sigma x.\delta P.$$

This must be true for *any* force of the system which suffers a small increment. Hence

$$\frac{\partial U}{\partial P} = x.$$

Since U is given in terms of the loads in the several members by (4), this can be written

$$\frac{d}{dP}\Sigma\frac{lT^2}{2ES} = x$$

or

$$\Sigma\frac{l}{ES}T\frac{\partial T}{\partial P} = x \qquad \text{. . . . . . .} \qquad (6)$$

Example.—Let us apply this method of finding displacements to the example on p. 35; in particular let us find the vertical displacement of G, a five-times load as before.

We must denote the external force at G by P and at the end of the work give P its proper value.

The calculations are most conveniently done in a table, thus:

Member.	Normal Load T (lbs.)	$\frac{\partial T}{\partial P}$	$T\frac{\partial T}{\partial P}$	$\frac{l}{ES}$ in./lbs.	$\frac{l}{ES}\cdot T\frac{\partial T}{\partial P}$ (in.)
CD	$-(3P+1{,}170)$	-3	$9P+3{,}510$	$11{\cdot}7 \times 10^{-6}$	$(105P + 41{,}100)10^{-6}$
DF	$-(1{\cdot}64P+250)$	$-1{\cdot}64$	$2{\cdot}69P+409$	$19{\cdot}9$,,	$(53{\cdot}5P + 8{,}140)$,,
BE	$1{\cdot}64P+250$	$1{\cdot}64$	$2{\cdot}69P+409$	$16{\cdot}1$,,	$(43{\cdot}3P + 6{,}590)$,,
EG	0	0	0	$19{\cdot}3$,,	0
DE	$-(P + 390)$	-1	$P+390$	$19{\cdot}6$,,	$(19{\cdot}6P + 7{,}650)$,,
FG	$-P$	-1	P	$25{\cdot}0$,,	$(25{\cdot}0P + 0)$,,
BD	$1{\cdot}75P + 1{,}155$	$1{\cdot}75$	$3{\cdot}06P+2{,}020$	$41{\cdot}4$,,	$(126{\cdot}5P + 83{,}700)$,,
EF	$1{\cdot}92P + 293$	$1{\cdot}92$	$3{\cdot}69P + 563$	$61{\cdot}4$,,	$(226{\cdot}0P + 34{,}600)$,,

The values of T, for the external forces shown in Fig. 18, with P written instead of 160 lbs. at G, are found by the ordinary methods of statics. The second, third, and fourth columns are then filled in. The fifth column is filled in from the data in the table on p. 35, and then the last column is obtained by multiplying together the figures in the two previous columns.

The last column gives, by addition,

$$\Sigma\frac{l}{ES}T\frac{\partial T}{\partial P} = (598{\cdot}9\,P + 181{,}780)10^{-6}, \text{ inches.}$$

Giving P its value 160 lbs., this gives 0·277″, which is therefore the vertical displacement of G. At five-times load the displacement will therefore be 1·385″ (on p. 36 we found it to be 1·378″).

It will be seen that this method is essentially the same as that employed on p. 37.

42. Second Theorem Relating to the Strain Energy of a Frame.—*The differential coefficient of the strain energy of a framework with respect to a couple gives the angular rotation of the arm of the couple about its axis.*

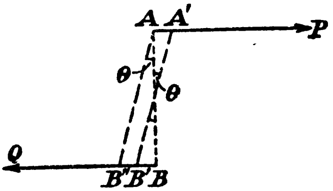

FIG. 20.

In Fig. 20 let P and Q be equal and opposite forces acting at A and B at right angles to AB. Then when we write down the expression for the strain energy of the framework we must regard Q as a function of P for purposes of differentiation, afterwards putting $Q = P$ and $dQ/dP = 1$.

Let $U =$ the strain energy of the frame, then after strain let AA' and BB' be the displacements of A and B in the directions of P and Q, and let the angle turned through by $AB = \theta$. Draw AB'' parallel to $A'B'$, then

$$\theta = \frac{BB''}{AB} = \frac{AA' + BB'}{AB}$$

Now by the previous theorem we have

$$AA' = \frac{\partial U}{\partial P} \text{ and } BB' = \frac{\partial U}{\partial Q}$$

$$\therefore \theta = \frac{1}{AB}\left(\frac{\partial U}{\partial P} + \frac{\partial U}{\partial Q}\right)$$

Now let M denote the moment of the couple formed by P and Q, then

$$M = P.AB = Q.AB.$$

or

$$P = Q = \frac{M}{AB}.$$

We can regard U as a function of M, and P and Q can also be regarded as functions of M. Hence we can write

$$\begin{aligned}\frac{dU}{dM} &= \frac{\partial U}{\partial P}\cdot\frac{dP}{dM} + \frac{\partial U}{\partial Q}\cdot\frac{dQ}{dM}\\ &= \frac{\partial U}{\partial P}\cdot\frac{1}{AB} + \frac{\partial U}{\partial Q}\cdot\frac{1}{AB}\\ &= \frac{1}{AB}\left(\frac{\partial U}{\partial P} + \frac{\partial U}{\partial Q}\right) = \theta\end{aligned}$$

From the expression found above for θ, this gives

$$\frac{dU}{dM} = \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which proves the proposition.

43. The Theorem of Least Work for a Framework which is not Self-strained.*—*The forces in the members of a framework which is not initially self-strained can be found from the conditions that the strain energy is a minimum.*

Suppose the framework has n redundant bars; it is clear that so far as concerns the rest of the framework we may replace each of these n bars by forces at their extremities equal to the tensions or thrusts which they are exerting. For instance if T_{AB} is the thrust in a bar connecting the joints A and B (Fig. 21) we can remove the bar AB and apply at A and B forces equal to T_{AB} acting away from each other, as shown in Fig. 21A.

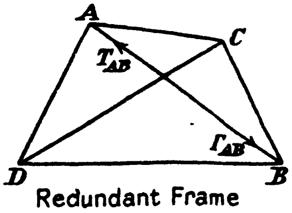

FIG. 21.

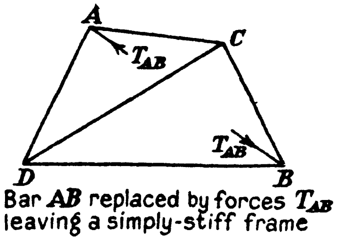
A
C
T_{AB}
T_{AB}
D
B
Bar AB replaced by forces T_{AB} leaving a simply-stiff frame

FIG. 21A.

The effect on the rest of the structure will be the same in each case.

In this way we shall be left with a simply-stiff frame under the action of the original external forces and the additional external forces representing the tensions or thrusts in the redundant bars.

By the ordinary methods of statics we can then express the loads in all the remaining bars as functions of all these external forces, and hence we can express the strain energy of the simplified frame as a function of the same quantities.

Thus let $P, Q, R \ldots.$ stand for the external forces, and $T_1, T_2, T_3 \ldots.$ for the loads in the redundant bars. Then we shall have

$$U_1 = f(P, Q, R \ldots . T_1, T_2 \ldots .)$$

where U_1 represents the strain energy of the simplified framework.

It follows from § 41 that $\frac{\partial U_1}{\partial T_1}$ gives the amount by which the points of application of the forces T_1 separate. But in the complete frame this is given by the extension of the bar carrying the load T_1, i.e. by $\frac{l_1 T_1}{ES_1}$. Here it is necessary to pay attention to sign: suppose T_1 is a thrust, then in the simple frame the external forces T_1 will push the corresponding joints away from each other, but the fact of T_1 being a thrust means that this member of the complete frame is in compression, i.e. has contracted. A similar argument will hold when the member is in tension. Therefore we write:

---

* The theorems given here are due to Castigliano, whose work has been translated into English by E. S. Andrews (*Elastic Stresses in Structures*, Scott, Greenwood & Son, 1919). The proofs given here follow the lines adopted by Castigliano. See also a very useful little book, *The Mechanics of Internal Work*, by Church (Wiley & Sons); *Theory of Elasticity*, by R. V. Southwell (O.U.P.); *The Analysis of Engineering Structures*, by A. J. Sutton-Pippard and J. F. Baker (Edward Arnold).

$$\frac{\partial U_1}{\partial T_1} = -\frac{l_1 T_1}{ES_1}$$

or

$$\frac{\partial U_1}{\partial T_1} + \frac{l_1 T_1}{ES_1} = 0$$

which can be written

$$\frac{\partial}{\partial T_1}\left(U_1 + \frac{l_1 T_1^2}{2ES_1}\right) = 0$$

or

$$\frac{\partial}{\partial T_1}\left(U_1 + \sum_1^n \frac{l_1 T_1^2}{2ES_1}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

since the partial differential coefficients of the other terms included in the summation will be zero.

Now the terms included under the Σ sign in (i) give the strain energy of the redundant bars, so that the quantity within the brackets represents the strain energy of the whole framework, U say. A similar equation will be obtained by considering the other redundant bars.

We shall thus have n equations of the type

$$\frac{\partial U}{\partial T_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which will enable us to calculate the loads $T_1, T_2 \ldots T_n$ in the n redundant bars. We can then go back and find the loads in all the other bars.

These equations express the conditions that the strain energy of the whole frame should be a minimum or a maximum, and the latter is obviously precluded by the nature of the problem, so that the strain energy must be a minimum.

44. Method of Calculating Stresses in Redundant Frames which are not Self-strained.—From the theorem just proved we deduce the following method for calculating the loads and stresses in the members of a redundant framework.

Let there be n redundant bars; then the process is as follows:

(i) Choose any n members of the frame which could be omitted without causing collapse.

(ii) By resolving the forces at the joints express the loads in the remaining bars in terms of the loads applied to the frame and the loads $T_1 \ldots T_n$ in the redundant bars.

(iii) We have to form the equations

$$\frac{\partial U}{\partial T_1} = 0, \quad \frac{\partial U}{\partial T_2} = 0 \ldots \frac{\partial U}{\partial T_n} = 0$$

where U is the strain energy of the *whole* frame including the redundant members. Now

$$U = \tfrac{1}{2}\Sigma\frac{lT^2}{ES}$$

$$\left.\begin{aligned} \therefore \frac{\partial U}{\partial T_1} &= \Sigma\frac{lT}{ES}\cdot\frac{\partial T}{\partial T_1} = 0 \\ \frac{\partial U}{\partial T_2} &= \Sigma\frac{lT}{ES}\cdot\frac{\partial T}{\partial T_2} = 0 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

Therefore, instead of actually forming the function U and differentiating it we find the values of $\frac{\partial T}{\partial T_1}, \frac{\partial T}{\partial T_2} \ldots \frac{\partial T}{\partial T_n}$ for each member and so form the equations (9). We then solve these equations for $T_1, T_2 \ldots T_n$. In all but the most simple cases it will be best to do the calculation in the form of tables.

Having found $T_1 \ldots T_n$ we can go back and find the loads in the remaining members.

If E and S are the same for all the members we can reduce the equations to the form

$$\Sigma lT\frac{\partial T}{\partial T_1} = 0. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9a)$$

A concrete example will make the method clear.

Example.—The framework $ABCFED$ (Fig. 22) is loaded by the three forces W, P, Q. All the bars have the same cross section and are of the same material, and are pin-jointed at their ends. Find the loads in all the members, in terms of W.

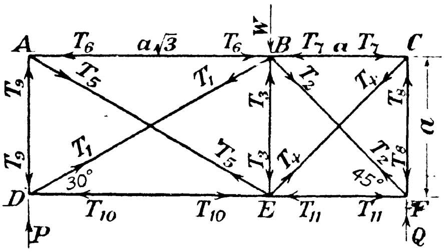

FIG. 22.

This is a plane frame with 6 joints, therefore 9 bars are necessary to prevent collapse, whereas actually there are eleven. Let us regard BD and BF as the redundant bars, and let the tensions in them be T_1 and T_2.

Let $T_3 \ldots T_{11}$ denote the loads in the other bars as shown. Then the first step is to express these in terms of T_1, T_2, P, Q, W.

Resolving vertically and horizontally at each joint we obtain the results given on page 46 in the second column. The third column is filled in from the diagram. Columns 4 and 5 give the partial differential coefficients of the loads T, with respect to T_1 and T_2, multiplied by the lengths of the corresponding members. Columns 6 and 7 are obtained by multiplying column 2 by columns 4 and 5 respectively.

1	2	3	4	5	6	7
T	In terms of P, Q, W, T_1, T_2	length l	$l\frac{\partial T}{\partial T_1}$	$l\frac{\partial T}{\partial T_2}$	$lT\frac{\partial T}{\partial T_1}$	$lT\frac{\partial T}{\partial T_2}$
T_3	$W + \frac{1}{2}T_1 + \frac{1}{\sqrt{2}}T_2$	a	$\frac{1}{2}a$	$\frac{a}{\sqrt{2}}$	$\left(\frac{W}{2} + \frac{1}{4}T_1 + \frac{1}{2\sqrt{2}}T_2\right)a$	$\left(\frac{W}{\sqrt{2}} + \frac{1}{2\sqrt{2}}T_1 + \frac{1}{2}T_2\right)a$
T_8	$Q + \frac{1}{\sqrt{2}}T_2$	a	0	$\frac{a}{\sqrt{2}}$	0	$\left(\frac{Q}{\sqrt{2}} + \frac{1}{2}T_2\right)a$
T_{11}	$\frac{1}{\sqrt{2}}T_2$	a	0	$\frac{a}{\sqrt{2}}$	0	$\frac{1}{2}T_2a$
T_9	$P + \frac{1}{2}T_1$	a	$\frac{a}{2}$	0	$\left(\frac{P}{2} + \frac{1}{4}T_1\right)a$	0
T_{10}	$\frac{\sqrt{3}}{2}T_1$	$a\sqrt{3}$	$\frac{3a}{2}$	0	$\frac{3\sqrt{3}}{4}T_1a$	0
T_4	$\sqrt{2}Q + T_2$	$a\sqrt{2}$	0	$a\sqrt{2}$	0	$(2Q + \sqrt{2}T_2)a$
T_5	$2P + T_1$	$2a$	$2a$	0	$(4P + 2T_1)a$	0
T_6	$\sqrt{3}P + \frac{\sqrt{3}}{2}T_1$	$a\sqrt{3}$	$\frac{3a}{2}$	0	$\left(\frac{3\sqrt{3}}{2}P + \frac{3\sqrt{3}}{4}T_1\right)a$	0
T_7	$Q + \frac{1}{\sqrt{2}}T_2$	a	0	$\frac{a}{\sqrt{2}}$	0	$\left(\frac{1}{\sqrt{2}}Q + \frac{1}{2}T_2\right)a$
T_1	T_1	$2a$	$2a$	0	$2T_1a$	0
T_2	T_2	$a\sqrt{2}$	0	$a\sqrt{2}$	0	$\sqrt{2}T_2a$

We next add up the two last columns and equate the results to zero. We obtain :

$$\frac{3\sqrt{3}(\sqrt{3}+1)}{2}T_1 + \frac{1}{2\sqrt{2}}T_2 + \frac{W}{2} + \frac{3\sqrt{3}(\sqrt{3}+1)}{2}P = 0 \quad \text{(i)}$$

$$\frac{1}{2\sqrt{2}}T_1 + (2 + 2\sqrt{2})T_2 + \frac{W}{\sqrt{2}} + (2 + \sqrt{2})Q = 0 \quad . \quad \text{(ii)}$$

We also have the equations of equilibrium of the structure as a whole :

$$P + Q = W$$

and, taking moments about D,

$$a(1 + \sqrt{3})Q = W.a\sqrt{3}$$

which give

$$P = \frac{W}{1 + \sqrt{3}} = 0{\cdot}366W \; ; \quad Q = \frac{W\sqrt{3}}{1 + \sqrt{3}} = 0{\cdot}634W$$

Substituting for P and Q in (i) and (ii) and simplifying we get :

$$7{\cdot}098T_1 + 0{\cdot}3535T_2 = -\,3{\cdot}10W$$
$$0{\cdot}50T_1 + 6{\cdot}828T_2 = -\,4{\cdot}06W$$

which give :

$$T_1 = -\,0{\cdot}465W$$
$$T_2 = -\,0{\cdot}564W.$$

We now go back and substitute for T_1 and T_2, and incidentally for P and Q, in column 2 above and thus obtain the values of the loads in the other members :

$T_3 = 0{\cdot}368W$ $\qquad$ $T_6 = 0{\cdot}232W$ $\qquad$ $T_9 = 0{\cdot}134W$
$T_4 = 0{\cdot}334W$ $\qquad$ $T_7 = 0{\cdot}235W$ $\qquad$ $T_{10} = -\,0{\cdot}403W$
$T_5 = 0{\cdot}267W$ $\qquad$ $T_8 = 0{\cdot}235W$ $\qquad$ $T_{11} = -\,0{\cdot}399W$

By following this system the loads in complicated pin-jointed structures, which are free from initial stresses, can be found with no more difficulty than arises from the laborious arithmetic.*

45. Frameworks which are Self-strained.—Let us imagine that the framework is first built up as far as the inclusion of all the necessary bars, and that we then attempt to put in the first redundant bar and find it is too short by an amount λ_1. In order to get it into position we shall have to push together the joints between which it is to be placed. Then, when it is in position, the bar will stretch slightly and the joints will go part of the way back to their original position, leaving the new bar with a tension T_1, say.

The redundant bar will have stretched an amount $\dfrac{l_1 T_1}{E_1 S_1}$.

At the same time it will be exerting a pull T_1 on each of the joints, and these will have been pulled together by an amount $\dfrac{\partial U_1}{\partial T_1}$, where U_1 is the strain energy (due to the external loads, including T_1) of the frame without the redundant bar.

These two lengths together must make up the amount by which the redundant bar was originally too short, i.e.

$$\frac{\partial U_1}{\partial T_1} + \frac{l_1 T'_1}{ES_1} = \lambda_1$$

$$\text{i.e. } \frac{\partial}{\partial T_1}\left(U_1 + \frac{l_1 T_1{}^2}{2ES_1}\right) = \lambda_1 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and there will be a similar equation for each redundant bar. These, added to the equations of equilibrium of the joints, will enable us to find the forces in all the members of the complete framework under the action of a given external load system, just as in the case when there were no initial stresses.

Equation (i) can be written

$$\frac{\partial U}{\partial T_1} = \lambda_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where U is the strain energy of the whole frame including the redundant bars, expressed in terms of the forces in these bars and the external forces.

46. Alternative Method : Use of Displacement Diagrams.—The stresses in redundant frames may also be estimated by the use of displacement diagrams. Supposing the frame to possess one redundant member AB the method of procedure is as follows :—

---

* It will be seen that the determination of the loads involves a knowledge of the cross sections of all the members. In design work it usually happens that we cannot fix the size of the members until we know the loads, so that a process of trial and error is involved. A method of direct design of redundant frameworks has been given by A. J. Sutton Pippard : Aeronautical Research Committee. *R. & M.*, 793, H.M. Stationery Office, 4*d*. Reference may also be made to a paper by the same author on the torsional stresses in aeroplane fuselages : *R. & M.*, 736, 9*d*. Papers on three-dimensional frameworks have been published by R. V. Southwell in the same series : *R. & M.*, 737, 790, 791, 1*s*. 3*d*. the set.

(i) Suppose the bar AB removed from the frame and determine the loads in the remaining members due to the external forces.

(ii) Work out the elongations of all the members and draw a displacement diagram * to determine by how much A and B separate; suppose the separation $= a$.

(iii) Apply at A and B, in the line AB, equal and opposite forces P,† and determine the forces in all the members on account of the forces P only.

(iv) Draw another displacement diagram and determine the separation of A and B under the action of the forces P; suppose it is equal to b.

Let Q be the load in the redundant bar AB. Then, if external forces were absent, equal and opposite forces Q at A and B would cause these points to separate by a distance Qb/P.

The external forces without Q cause A and B to separate by an amount a.

Both together will therefore cause a separation equal to $a + \frac{Q}{P}b$.

But this must equal the elongation of the bar AB under a pull Q, i.e. $\frac{AB.Q}{ES}$, where S is the area of the cross section of AB. Hence we have

$$a + \frac{Q}{P}b = \frac{AB.Q}{ES} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is an equation from which to find Q. When Q has been found the loads in the other members are obtained by adding to the loads found in (i) Q/P times the loads found in (iii) above.

The method will be made clear by an example.

Example.—Find the loads in all the members of the framework shown in Fig. 23, the members AC and BD being independent. $AC = 5'$ and

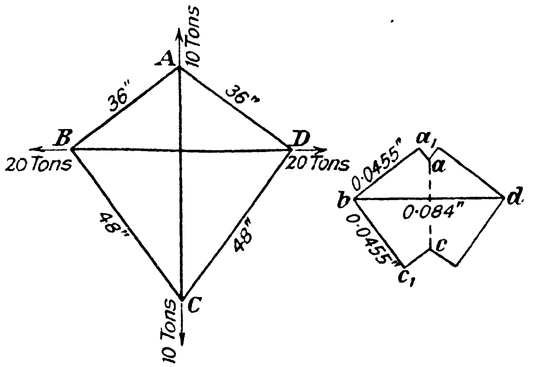

FIG. 23.

Displacement diagram for external forces; AC removed.

$BD = 4{\cdot}8'$. The sectional area of each member is 0·5 in.², and $E = 13{,}200$ tons/in.²

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* Or a may be found by the method of § 41.

† E.g. P may be one ton: see example.

We can regard either *AC* or *BD* as the redundant member. Let us take *AC* and suppose it removed. Then by the methods of statics we find that the loads in the other members, due to the pulls of 10 and 20 tons, are

AB and *AD* : 8·33 tons, tension.
CB and *CD* : 6·25 „ „
BD : 9·60 „ „

Now take *B* as a fixed point and *BD* as a fixed direction and draw the displacement diagram. The elongations are: *AB* and *AD*, 0·0455″; *CB* and *CD*, 0·0455″; *BD*, 0·0840″. It is found that

$$a = \text{increase of distance } AC = 0\cdot0455''.$$

Next apply forces of 1 ton at *A* and *C* as shown in Fig. 24; find the loads in all the members and draw the corresponding displacement diagram. The loads and elongations are found to be:

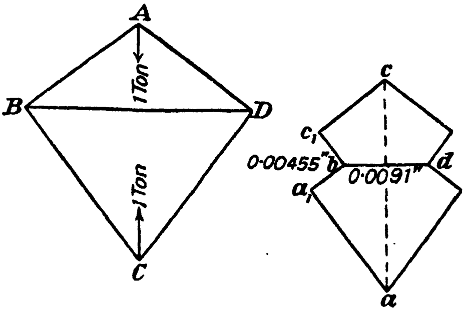

FIG. 24.

AB and *AD* : load = 0·833 tons comp.; elongation = – 0·00455″.
CB and *CD* : „ = 0·625 „ „ „ = – 0·00455″.
BD : „ = 1·04 „ tension; „ = + 0·0091″.

The increase in *AC* is found by measurement to be $b = -0\cdot018''$.

Let Q be the actual load in the member *AC* of the original frame, then we must have

$$0\cdot0455 - 0\cdot018Q = \frac{Q \times 60}{0\cdot5 \times 13{,}200},$$

whence

$$Q = 1\cdot68 \text{ tons.}$$

The loads in the other members are then:

$AB : 8\cdot33 - 1\cdot68 \times 0\cdot833 = 7\cdot13$ tons.
$BC : 6\cdot25 - 1\cdot68 \times 0\cdot625 = 5\cdot20$ „
$BD : 9\cdot60 + 1\cdot68 \times 1\cdot04 = 11\cdot35$ „

EXAMPLES II

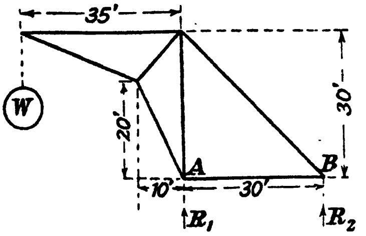

FIG. 25.

1. Fig. 25 represents the framework of a crane which carries a load at its outer extremity and is supported by forces R_1, R_2 acting at A and B in the direction shown. The sections of the various members are so proportioned that all tie bars are stressed up to 6 tons per square inch, and all struts to 3 tons per square inch. Construct a diagram showing the displacement of all the joints of the framework relative to A. The bar AB is supposed to remain horizontal. (Intercoll. Exam., Cambridge, 1908.)

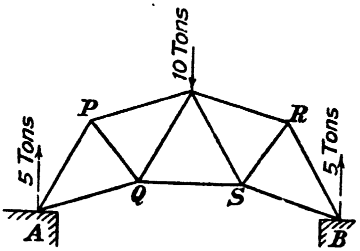

FIG. 26.

2. In the framework shown in Fig. 26 the bars PQ, RS are each 8 ft. long. All other members of the framework are 11 ft. long.

If the section of each member is ·1 sq. in. and E for the material is 30×10^6 lbs. per square inch, determine by means of a displacement diagram the horizontal displacement of A relatively to B. The reactions at these ends are to be taken as vertical in direction. (Intercoll. Exam., Cambridge, 1910.)

3. In the framework shown in Fig. 27, the sections of the various members are as follows:

$BC = 8$ sq. in.,

$AB = CD = 4$ sq. in.,

$EA = EC = EB = ED =$ 3 sq. in.

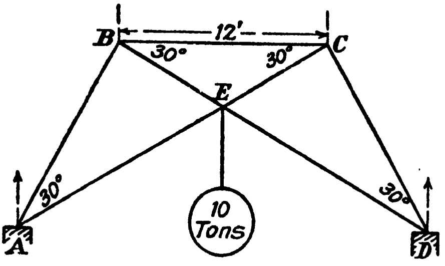

FIG. 27.

The framework is supported by vertical reactions at A and D. Take E = 14,000 tons per square inch and determine the elongation or contraction for each bar.

Draw the displacement diagram for the framework and determine the "spread" of the frame produced by the load. (Intercoll. Exam., Cambridge, 1913.)

4. The framework shown in Fig. 28 is supported on wheels. Find graphically the stresses in the various bars of the framework. Given that all tie bars are stressed up to 8 tons per square inch and all struts support a compression of 2 tons per square inch, draw up a table giving the sections of the various bars.

By means of a displacement diagram determine the increase of the span *AB* produced by the load. Take $E = 14{,}000$ tons per square inch. (Intercoll. Exam., Cambridge, 1914.)

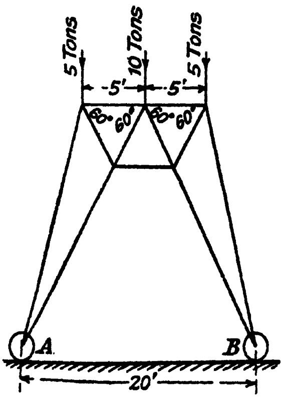

FIG. 28.

5. The double cantilever shown in Fig. 29 is supported by vertical reactions at *A* and *B*. The figure is drawn to a scale of 20 ft. to 1 in. and the numbers on the bars denote their sectional areas in square inches. Taking *E* for the material as 30×10^6 lbs. per square inch, draw the displacement diagram for the framework and determine the vertical displacement of the points *E* and *F*. (Mech. Sc. Trip., 1911.)

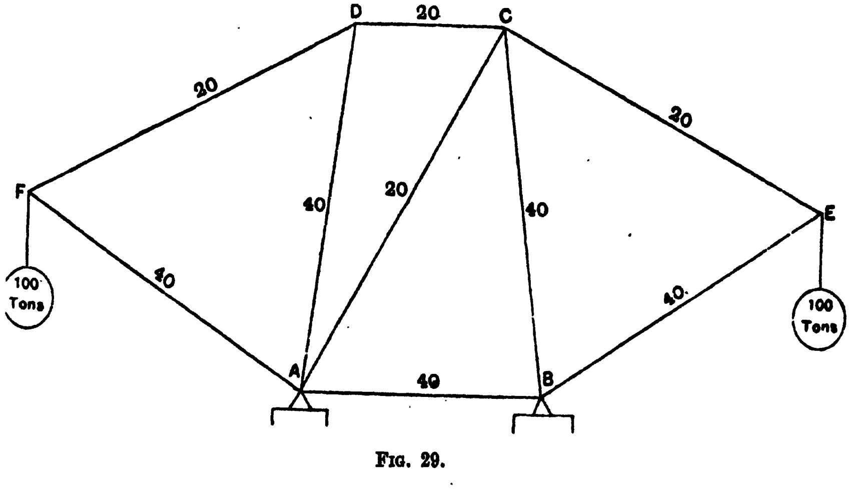

FIG. 29.

6. The framework shown in Fig. 30 carries the loads W_1, W_2 and W_3 as shown and rests on two supports at A and B. The bars composing the frame all have the same cross section and are made of the same material.

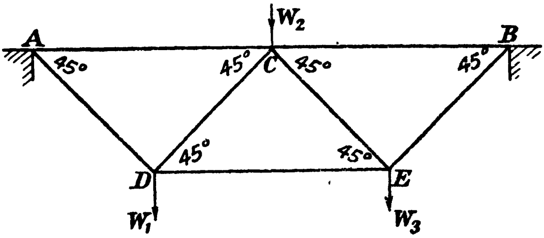

FIG. 30.

$AC = CB = DE = l_1$, and $AD = DC = CE = EB = l_2$. Prove that on account of the loads the point D sinks a distance

$$\frac{l_1}{ES}\left(\frac{7}{8}W_1 + W_2 + \frac{5}{8}W_3\right) + \frac{l_2}{ES}\left(\frac{9}{8}W_1 + \frac{11}{8}W_2 + \frac{1}{2}W_3\right),$$

where S is the area of the cross section of each bar.

If $l_1 = 20$ ft., $l_2 = 14{\cdot}14$ ft., $W_1 = 8$ tons, $W_2 = 2$ tons, $W_3 = 4$ tons, $S = 2$ in.^{2}, $E = 30 \times 10^6$ lbs./in.^{2}, find how much D sinks.

7. One-half of the wing structure of an aeroplane is shown diagrammatically in Figs. 31 and 32. AB and CD are the rear spars; $A'B'$ and $C'D'$ are the front spars; BC and $B'C'$ are rear and front struts between the upper and lower wings. The other members operating are steel wires BD, $A'C'$, BC'. The external loads are as shown. In plan view each wing is as shown in

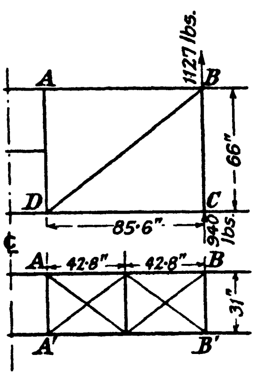

FIG. 31.

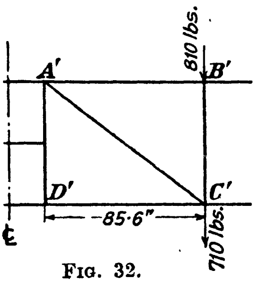

FIG. 32.

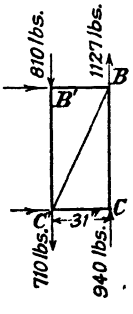

Fig. 31, the cross members being wooden struts and the diagonals steel wires. The cross section of each front spar is 3·59 in.^{2} and of each rear spar 2·47 in.^{2}; $BD = 0{\cdot}0538$ in.^{2}; $A'C' = 0{\cdot}0269$ in.^{2}; the other wires are all 0·0158 in.^{2} For steel $E = 30 \times 10^6$ lbs./in.^{2}, for the wood $E = 1{\cdot}4 \times 10^6$. Considering only the structure $ABB'C'D'A'$ and neglecting the strain energy of the struts, find the load in the wire BC'.

8. In the double cantilever framework shown in Fig. 33, determine the stress in the member *AB*. The numbers on each of the bars give their section area in square inches; the reactions at *C* and *D* are vertical, and *E* is the same for all the bars. (Mech. Sc. Trip., B., 1911.)

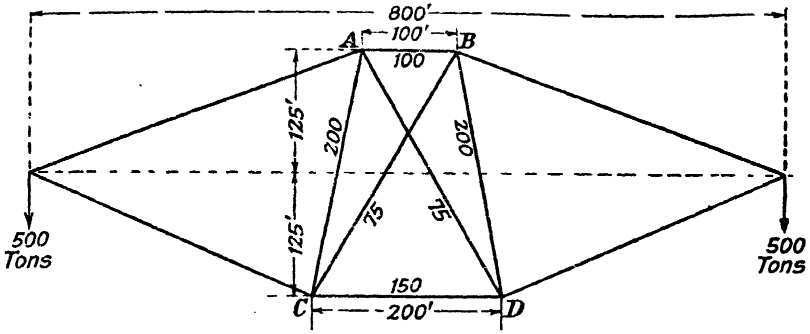
800'
A
100'
B
100
125'
125'
200
200
75
75
150
C
200'
D
500 Tons
500 Tons

FIG. 33.

9. The framework shown in Fig. 34 contains a redundant bar. The members are all of the same section and material and are freely jointed together at their extremities. Find the stress in the bottom horizontal member. (Mech. Sc. Trip., B., 1910.)

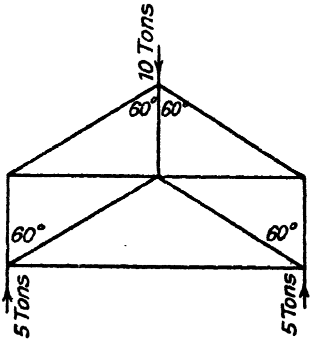
10 Tons
60°
60°
60°
60°
5 Tons
5 Tons

FIG. 34.

CHAPTER III

SHEARING STRESSES

47. Shearing Stress.—In our first chapter we made a detailed study of two kinds of stress only: tensile and compressive stress, which are collectively referred to as "direct" stresses. There is, however, a third kind of stress which we must study. In dealing with tension and compression we confined our attention to the action between the particles of the body on the two sides of an imaginary plane at right angles to the line of action of the resultant pull or thrust. Let us now consider the actions between particles separated by a plane which is not at right angles to the resultant applied force. For instance, in Fig. 35 (i) suppose the rod *AB* is subjected to a push *P* applied to its ends, and let *CD* be an imaginary oblique section of the rod. What is the action between the two parts of the rod across the plane *CD*? Clearly the equilibrium of the part *A* demands that the *resultant* force exerted by the part *B* should be a thrust *P*, as shown at (ii). But this resultant action can be resolved into two components, one at right angles to, and one tangential to, the plane *CD*, as shown at (iii). Similarly the reaction of the part *A* on the part *B* across the plane *CD* can be represented by two components, as shown at (iv); clearly the component at right angles to *CD*, of the action of *B* on *A*, is equal to the corresponding component of the reaction of *A* on *B*, and likewise the two components tangential to *CD* must be equal.

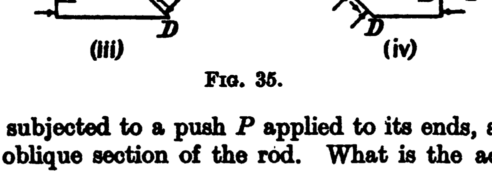

FIG. 35.

We see, then, that the action and reaction between the two portions of the rod can be regarded as made up of a pressure between them at right angles to *CD*, and a tendency to slide over each other along the surface *CD*. The former of these mutual actions gives rise to a direct stress, such as we have studied in the first chapter; the latter constitutes a *shearing* action.*

In one extreme case, when *CD* is at right angles to the resultant thrust, the shearing action disappears, and in the other extreme case, when *CD* is parallel to the resultant thrust, the whole action between the two parts of the rod is zero, both the compressive and shearing actions being zero, if the load be distributed uniformly over the ends.

These considerations show us that, in order to specify the stress at any point in a body, even under such a simple load as a pure thrust, it is necessary to state the direction in which the stress is measured. Thus at any point in the rod, if the stress is measured in the direction of the axis of the rod it will be a compressive stress, if at right angles to any oblique cross section it will again be a compressive stress but of a different value; if it be measured along any oblique plane it will be a shearing stress. Referring to Fig. 35 it will be equally true to say that the stress at any point in *CD* is a certain compressive stress in the direction *AB*, or to say that it consists of a different compressive stress at right angles to *CD* accompanied by a shearing stress along *CD*. Under certain circumstances the stress at a point in a body can be completely specified by stating it as a shearing stress in a certain direction. In such cases there is said to be a state of "pure shear."

As a further illustration of shearing stresses suppose we have a block of material (Fig. 36) glued to a table, and a thin plate glued to the top of the block, the upper and lower faces of the block being parallel: if we pull on the end of the plate with a force *F* the plate will tend to slide along the top of the block, and the block to slide along the table. The table resists the sliding of the block, and the block resists the sliding of the plate. The technical way of stating this is to say that there is a shearing force acting along the surfaces of separation.

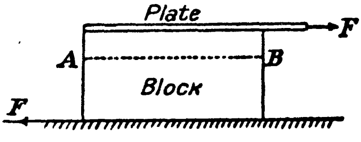

FIG. 36.

Again, if we consider the block divided by any imaginary plane *AB*, parallel to the upper and lower faces, the top part of the block will be trying to slide over the bottom part. In other words, this plane is subject to a *shearing stress*, and so is every plane such as *AB*. It is in this manner, i.e. the tendency of one part of a body to slide over the neighbouring part, that obvious shearing actions are met with in practice.

To make this clear we shall consider one or two instances. First, consider the case of two flat plates (Fig. 37 (i)) held together by a single rivet and pulled with forces *F*. We can imagine the rivet divided into two portions by the plane *AB* shown dotted; then the top half of the rivet is tending to slide over the bottom half, i.e. a shearing stress exists at every point in this plane.

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* The reader may find it helpful to think of a stick of wood sawn in half along *CD*, the two parts being placed in contact on a table, and then to imagine what will happen when the ends are pushed towards each other.

Next consider (Fig. 37 (ii)) a shaft *A* with a collar *C* held in a bearing *B*, one end of the shaft being pushed with a force *F*. In this case there is a tendency for the shaft to be pushed bodily through the collar, i.e. there is a shearing action over the surface of the cylinder indicated by the dotted lines *Y*. There is also the possibility of the collar pulling through the bearing, shear taking place on the surface of the imaginary cylinder *X*. Which happens first will depend on the relative capacities of the two surfaces to resist shear.

As a third example we will take the case of a steel bolt through the end of a bar of wood, as shown in Fig. 37 (iii), the bolt being pulled by forces *F*. Suppose that the grain of the wood runs parallel to the length of the bar. Then, if the forces *F* are large enough the whole block *ABCD* will be pushed out, shear taking place along the planes *AB* and *CD*.

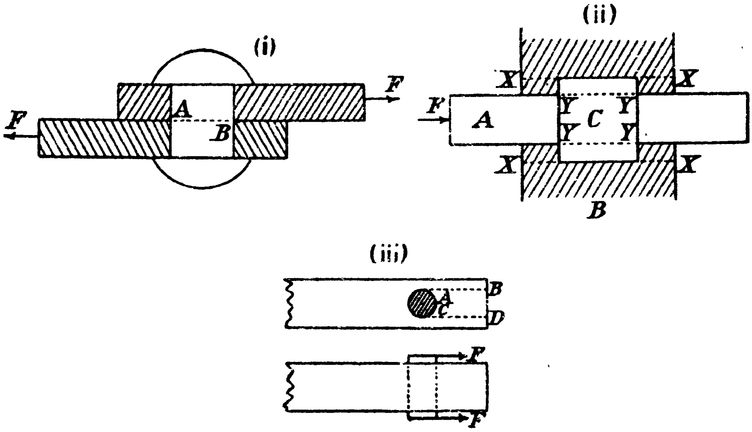

FIG. 37.

48. Complementary Shear Stresses.—Let us return now to the consideration of the block discussed above (p. 55). We have seen that all planes, such as *AB* (Fig. 36), parallel to the top and bottom of the block, are subjected to shearing stress. Consider the material between two such planes, which are close together, and imagine that it consists of a number of rectangular elements, as shown by the full lines in Fig. 38. Under the action of the shearing forces *F*, which, together, constitute a couple, the elementary blocks will tend to take up the positions shown

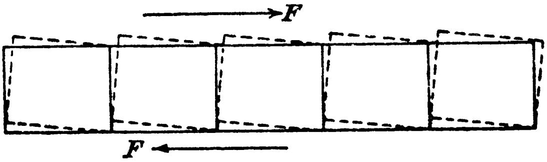

FIG. 38.

by the dotted lines in Fig. 38. It will be seen that there is a tendency for the vertical faces of the blocks to slide over each other. Actually the ends of the blocks do not slide over each other in this way, but the tendency to do so shows that the shearing stress in horizontal planes is accompanied by shearing stresses in vertical planes perpendicular to the applied shearing forces. This is true for all cases of shear: a given shearing stress acting on one plane is always accompanied by a *complementary* shearing stress on planes at right angles to the given stress and to the plane on which it acts.

Consider, now, the equilibrium of one of the elementary blocks * referred to above. Let q be the intensity of shearing stress over the faces AB and CD (Fig. 39), and let q' be the intensity of the complementary shearing stress on the faces BC and AD.

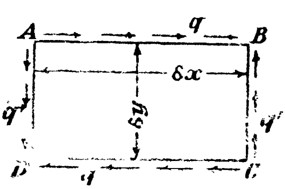

FIG. 39.

Let the thickness of the block perpendicular to the plane of the paper be unity.

The total shearing force on $AB = q.\delta x$, and that on the face CD is the same. These two forces constitute a couple of moment $q.\delta x.\delta y$ in a clockwise direction. Similarly, the stresses q' give rise to a couple of moment $q'.\delta y.\delta x$ in an anti-clockwise direction. For the equilibrium of the element these two couples must be equal. Hence we must have $q' = q$.

We see then, that, whenever there is a shearing stress over a plane passing through a given line, there must be an *equal* complementary shearing stress on a plane perpendicular to the given plane and passing through the given line. The directions of the two shearing stresses must be either both towards, or both away from, the line of intersection of the two planes in which they act.

It is extremely important to realize the existence of the complementary shear stress, for its necessary presence has a direct effect on the maximum stress in the material, as we shall see later. As a simple practical illustration of their importance we give the example shown in Fig. 41: the pieces P and Q are under the action of the forces F, and sliding is prevented by the wooden dowel CD. This dowel is obviously in shear across the

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* It may occur to the reader to ask why we take a rectangular element. Surely, he may say, the couple due to the stresses q could be balanced equally well by shearing stresses on oblique planes, as in Fig. 40. True, it could, and it is easy to show that in the same way as above, that $q'' = q$, whatever the angle θ is. It might appear from this that we could take our balancing shear stresses on any plane we please, but in general the simple relationship $q'' = q$ will not hold. We frequently have shear stresses and direct stresses acting simultaneously. Suppose there were two direct stresses of intensity p acting at right angles to the faces AB and $C'D'$; they would give rise to a couple $p\delta x.\delta y$ cot θ, so that for equilibrium we should have $q''\delta y.\delta x = q\delta x.\delta y - p\delta x.\delta y$ cot θ, and the value of q'' would depend on p except when $\theta = \pi/2$. Thus it will be most convenient always to take $\theta = \pi/2$, so that the complementary shear stress is equal to q whatever direct stresses may be acting as well.

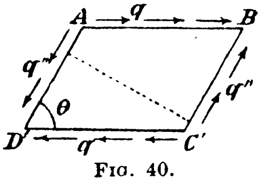

FIG. 40.

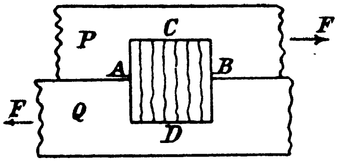

FIG. 41.

plane AB; therefore there must be equal shearing stresses in the vertical planes, such as CD. If this be the direction of the grain, failure of the piece may occur from the shearing stress parallel to the grain being too large, since wood is weak in shear along the grain.

49. The Shearing Stresses on a Cross Section must always Act in Directions Tangential to the Boundary.

—Let XYP be the boundary of the cross section of a body which is subjected to shear in the plane XYP. If possible let the shear stress at P be q, as shown in Fig. 42. Then, if the direction of q is not along the tangent at P, the stress q can be resolved into components q_1 along the normal at P, and q_2 along the tangent. The shear stress q_1 demands an equal complementary stress acting in the tangent plane at P, at right angles to q_1 and q_2. But, if no external forces are applied to the surface, this is impossible. Therefore, if the surface of the body is not acted on by tangential forces, we conclude that the component stress q_1 must be zero, that is the shearing stress at P must act along the tangent to the bounding curve at P. We see, then, that in such cases as rivets, etc., the distribution of shearing stress over the section must be somewhat as shown in Fig. 43, rather than as in Fig. 44, and we infer that the intensity of shearing stress must vary over the section.

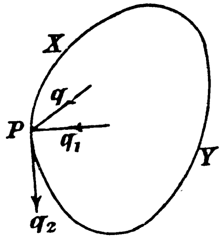

FIG. 42.

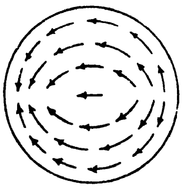

FIG. 43.

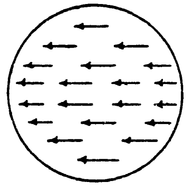

FIG. 44.

50. Measurement of Shear Stress.

—As in the case of direct stresses, the intensity of shear stress is reckoned per unit area. If the distribution is uniform the shear stress is equal to the shearing force divided by the area resisting shear; if the distribution is not uniform the stress is taken as the limiting value of $\delta F/\delta S$, where δF is the shearing force acting on a small area δS which contains the point under consideration.

The occurrence of shear stress, and its estimation, will be better understood after studying the following examples, in all of which the load is assumed to be uniformly distributed. This assumption is usually not true, but leads to sufficiently reliable results.

Example 1.—Three steel plates are held together by a single rivet as shown in Fig. 45, the load transmitted is 14 tons, and the diameter of the rivet is $\frac{5}{8}$ in. Find the average shear stress in the rivet.

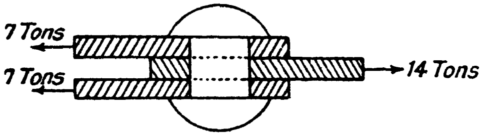

Fig. 45.

The tendency to shear is across the planes indicated by the dotted lines. The area resisting shear is *twice* the cross sectional area of the rivet.

Let q = the shear stress. Then

$$2q \times \frac{\pi}{4} \times \left(\frac{5}{8}\right)^2 \text{in.}^2 = 14 \text{ tons}$$

$$q = \frac{14 \text{ tons}}{0{\cdot}612 \text{ in.}^2} = 22{\cdot}9 \text{ tons/in.}^2$$

Example 2.—A cotter joint between two steel bars is shown in Figs. 46 and 47. The shearing strength of steel can be taken as 23 tons/in.^{2} Find which part of the joint is likely to shear first, the load being a tension.

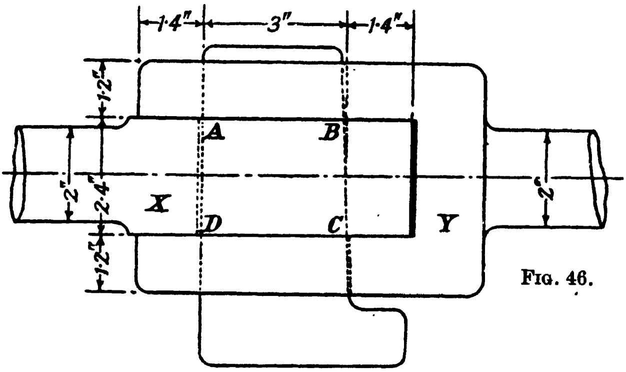

Fig. 46.

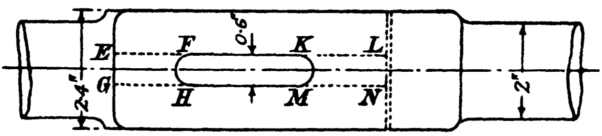

Fig. 47.

Shearing failure may occur in the following ways:

(i) Shear of the cotter in the planes AB and CD (Fig. 46).

The area resisting shear $= 2.\ FKMH = 2 \times 3{\cdot}0 \times 0{\cdot}6 = 3{\cdot}6$ in.^{2}

∴ the load required to cause failure in this way

$$= 3{\cdot}6 \text{ in.}^2 \times 23 \text{ tons/in.}^2 = 82{\cdot}8 \text{ tons.}$$

(ii) By the cotter pulling through the ends of the socket Y, i.e. by shear in the planes EF and GH (Fig. 47). There are four areas resisting shear. The total area in $4 \times 1{\cdot}2 \times 1{\cdot}4 = 6{\cdot}7$ in.^{2}

∴ the load required to cause failure $= 6{\cdot}7$ in.^{2} $\times$ 23 tons/in.^{2} $= 154$ tons.

(iii) By the cotter pulling through the ends of the rod X, i.e. by shear in the planes KL and MN (Fig. 47).

The area resisting shear $= 2 \times 1{\cdot}4 \times 2{\cdot}4 = 6{\cdot}72$ in.^{2}

$\therefore$ the load required to cause failure $= 6{\cdot}72$ in.$^2 \times 23$ tons/in.$^2 = 154$ tons. Hence shear failure will occur first in the cotter itself (i).

Example 3.—A lever is keyed to a shaft 1·5″ diameter, the width of the key being 0·5″ and the length 2″. What load can be applied at a radius of 36″ without causing the shear stress in the key to exceed 4 tons/in.^{2}?

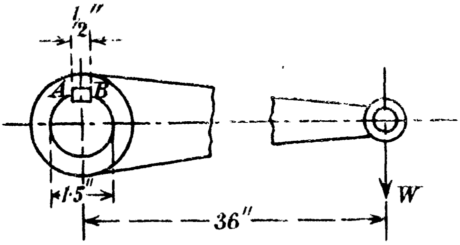

FIG. 48.

If the key shears it will do so in the plane AB (Fig. 48).

Let W tons be the load.

The torque produced $= 36W$ tons. ins.

This will produce a shearing force in the plane AB

$$= \frac{36 \text{ in.} \times W \text{ tons}}{0{\cdot}75 \text{ in.}} = 48W \text{ tons.}$$

The area resisting shear $= 0{\cdot}5 \times 2 = 1$ in.^{2}

$\therefore$ the shearing force must not exceed 4 tons.

$\therefore$ W must not exceed $\frac{1}{12}$ ton = 187 lbs

51. Shear Strain.—If a piece of material be subjected to shearing stress, an element such as $ABCD$ (Fig. 49), which was originally rectangular, will become oblique. The relative motion of the faces AB and CD will be one of sliding in their own planes, and lines such as AD and BC, originally perpendicular to AB and CD, will become inclined to their original directions at an angle φ. This angle φ is defined as the shear strain and is measured in radians.

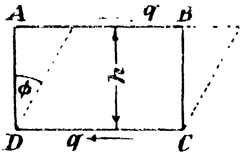

FIG. 49.

52. Modulus of Rigidity.—As in the case of direct stresses, the shear strain is found to be proportional to the shearing stress, within certain limits, for most metals. In fact a stress-strain diagram for shear exhibits all the principal features of a stress-strain diagram for tension.

Thus within the limits of proportionality we can write

$$q = C\varphi \qquad (1)$$

where q = the shear stress

φ = „ „ strain

and C is an experimental constant for the material; this constant is called the *modulus of rigidity* of the material, and is roughly of the same order of magnitude as Young's *Modulus*, e.g. for steel $E = 30 \times 10^6$ lbs./sq. in. and $C =$ about $12{\cdot}5 \times 10^6$ lbs./sq. in.

53. Strain Energy due to Shear.—When a body is in a state of shear stress, the points of application of the forces causing the stress undergo displacement; therefore work is done exactly as in the case of tension or compression. If the strains are kept within the elastic limit the work done is recoverable and is stored in the form of strain energy. Referring to Fig. 49, let δS denote the area of either of the faces AB or CD, and let h be the height of the little block. The force acting on either face is $q.\delta S$, and the couple due to them is $q.\delta S.h$. As the stress is gradually increased from zero to the value q, bringing about the strain φ, the work done by the stress couple is $\frac{1}{2}qh.\delta S.\varphi$, i.e. $\frac{1}{2}q\varphi$ per unit volume, φ being always small.

Thus the strain energy or resilience, due to shear, per unit volume, is, from (1),

$$U = \tfrac{1}{2}q\varphi = \frac{q^2}{2C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C is the modulus of rigidity.

As in the case of direct tension and compression, loads suddenly applied will produce approximately double the stress that would be produced if the same load were applied gradually.

EXAMPLES III

1. Rivet holes $\frac{7}{8}''$ diameter are punched in a steel plate $\frac{3}{8}$ in. thick. The shearing strength of the plate is 21·5 tons/in.^{2} Find the compressive stress in the punch at the time of punching.

2. The diameter of the bolt circle of a flanged coupling for a shaft 5″ diameter is 15″. There are six bolts 1″ diameter. What horse-power can be transmitted at 150 r.p.m. if the shear stress in the bolts is not to exceed 4 tons/in.^{2} ?

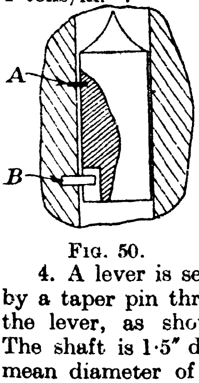

Fig. 50.

3. The pellet (Fig. 50) carrying the striking needle of a fuse has a mass of 0·2 lb.; it is prevented from moving longitudinally relative to the body of the fuse by a copper pin A of diameter 0·02″. It is prevented from turning relative to the body of the fuse by a steel stud B. A fits loosely in the pellet so that no stress comes on A due to rotation. If the copper shears at 10 tons/in.^{2}, find the retardation of the shell necessary to shear A. (R.N.C., Greenwich, 1923.)

4. A lever is secured to a shaft by a taper pin through the boss of the lever, as shown in Fig. 51. The shaft is 1·5″ diameter and the mean diameter of the pin is $\frac{3}{8}$ in. What torque can be applied to the lever without causing the shear stress in the pin to exceed 8,500 lbs./in.^{2} ?

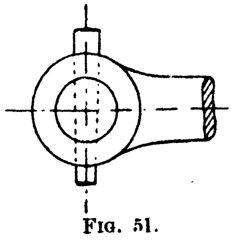

Fig. 51.

5. A cotter joint between two round rods in tension is shown in Fig. 52. Taking the tensile strength of the rods as 23 tons/in.^{2}, the shearing strength of the cotter 18 tons/in.^{2}, the permissible bearing pressure between surfaces in contact 45 tons/in.^{2}, the shearing strength of rod ends 12 tons/in.^{2}, calculate suitable dimensions for the joint so that it may be equally strong against the possible types of failure. Take the thickness of the cotter $= d/4$, and the taper of the cotter 1 in 48.

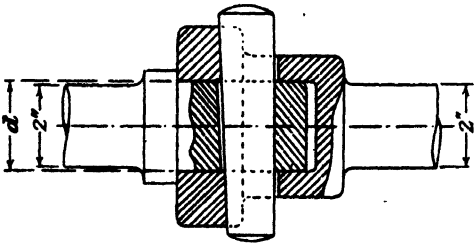

FIG. 52.

6. A horizontal arm, capable of rotation about a vertical shaft, carries a mass of 5 lbs., bolted to it by a $\frac{1}{2}''$ bolt at a distance 18″ from the axis of the shaft. The axis of the bolt is vertical. If the ultimate shear strength of the bolt is 3 tons/in.^{2}, at what speed will the bolt snap ? (R.N.E.C., Keyham, 1919.)

7. A shaft is subjected to a twisting moment which produces a shearing stress at the surface of 10 tons/in.^{2}, in planes perpendicular to the axis of the shaft. A small square is scratched on the surface of the shaft with two of its sides parallel to the axis of the shaft. Taking $C = 12\cdot5 \times 10^6$ lbs./in.^{2}, find the change in the angle at the corners of the square.

8. A copper disc 4″ diameter and 0·005″ thick, is fitted in the casing of an air compressor, so as to blow and safeguard the cast iron case in the event of a serious compressed air leak. If pressure inside the case is suddenly built up by a burst cooling coil, calculate at what pressure the disc will blow out, assuming that failure occurs by shear round the edges of the disc, and that copper will normally fail under a shear stress of 8 tons/in.^{2} (R.N.E.C., Keyham, 1929.)

CHAPTER IV

RIVETED JOINTS

54. Introductory.—In the majority of built-up structures the several members are united by riveted joints, and the strength of the joints is just as important as the strength of the members themselves. Unfortunately, the strength of riveted joints cannot be calculated with any degree of certainty, and practical design usually depends on empirical formulæ based on experience. Certain theoretical considerations, however, are useful as guides, and provide an interesting application of the principles which we have already studied.

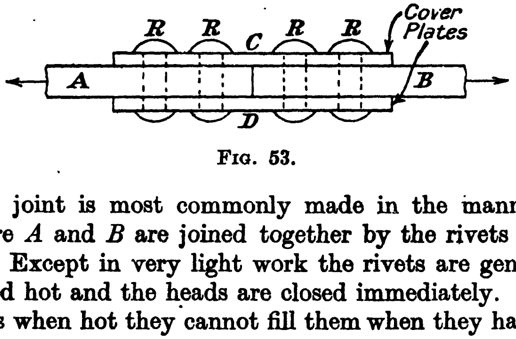

FIG. 53.

A riveted joint is most commonly made in the manner shown in Fig. 53, where *A* and *B* are joined together by the rivets *R* and cover plates *C*, *D*. Except in very light work the rivets are generally placed in position red hot and the heads are closed immediately. If the rivets fill their holes when hot they cannot fill them when they have cooled, on account of lateral contraction. Consequently, the members *A* and *B* will tend to slide over the cover plates, when a load is applied, until the rivets are all pressed up against their holes. On the other hand, the contraction of the rivets presses the cover plates against the members *A* and *B* with considerable force, so that any tendency to slide must be resisted by large friction forces between *A*, *B* and *C*, *D*.

Evidently it is desirable that the load on the joint should be less than that required to overcome friction and produce slip, otherwise there will be backlash if the load is reversed, and the joint will become "sloppy."

It is also evident that, if the rivets are carrying any load, they must be acting as stops to check the relative movements of the plates. But, as long as the friction forces are great enough, there will be no relative movement, and the rivets will not bear any load; as soon as slip occurs, the rivets are acted on by shearing forces in the planes separating the cover plates from the main plates.

At the time of writing it is practically impossible to make any reliable calculation of the friction forces which may be reckoned on in any given joint, and in England theoretical calculations are made for the strength of the plates and rivets, friction being ignored. On the Continent, however, the friction theory finds more favour, design resting largely on experimental figures given by Bach. According to Bach, when slipping begins, the friction forces vary from about 14,000 to 30,000 lbs. per square inch of rivet section, for each pair of surfaces in contact, increasing slightly with the length of the rivet, but not quite in proportion to the number of rows of rivets in multiple riveting. In a paper to the Institute of Naval Architects, 1923,* J. Montgomerie gives the following figures for the pull which produces slip, per square inch of rivet section:

Thickness of plates	0·6″	0·75″	1·0″
Slip pull ÷ area of rivet section	7·5 to 8	6·2	5 tons/in.^{2}

These figures were obtained for joints made under ordinary shipyard conditions, and show a decrease in the frictional resistance as the length of the rivet increases, which is contrary to Bach's results. The reason for this is probably the increased difficulty of getting good contact between the plates as the thickness increases. When special precautions were taken to ensure good contact the figure for 1″ plates was raised to 7 tons/in.^{2}

In these tests two plates of equal thickness were riveted together,

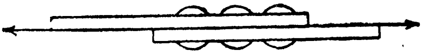

FIG. 54.

as shown in Fig. 54, and subjected to a load acting in the plane separating the two plates.

We shall consider the theoretical calculation of the strength of riveted joints, neglecting friction, which at present is the commonly adopted theory in this country.† Its chief recommendation is that it will produce a joint that will not fail even if slip does occur.

55. Possible Types of Failure of Simple Riveted Joints, Neglecting Friction.

—To explain the general principles it will be sufficient to consider the joint shown in Fig. 55.

Let d = the diameter of the rivets.

x = the pitch of the rivets, i.e. the distance between their centres measured parallel to the line separating the plates.

q_1 = the ultimate shearing stress for the rivets.

q_2 = the ultimate shearing stress for the plates and cover plates.

f = the ultimate tensile stress for the plates and cover plates.

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* See *Engineering*, June 8, 15, 1923.

† For other papers on the subject see *Annales des Ponts and Chaussées*, articles by Considère (1886), Dupuy (1895); *Zeit. d. Ver.* (1897); Iron and Steel Institute, Vol. XCVI (E. B. Wolff); C. Batho in *Engineering*, Sept. 3, 1920.

p_b = the maximum bearing pressure allowed between the plates and the rivets.

P = the load per unit length of joint, as shown in Fig. 55.

The other dimensions of the joint are as shown in the figure.

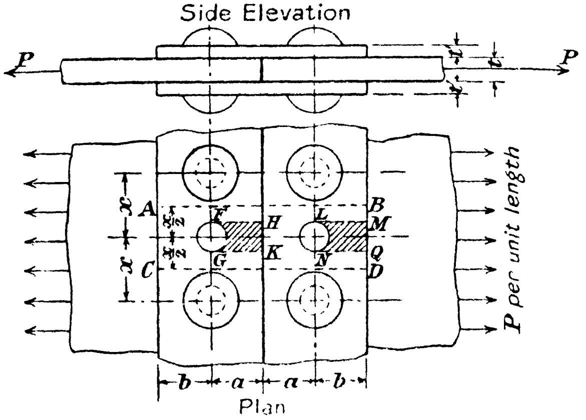

FIG. 55.

Let us consider that portion of the joint which lies between AB and CD: the total force acting across AC and BD is Px.

Then the joint may fail in any of the following ways:

(i) The rivets may shear (Fig. 56). The resistance of each rivet to shear is $2 \times \frac{\pi d^2 q_1}{4}$, so that the relation

$$Px \leqslant \frac{\pi d^2 q_1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

must be satisfied.

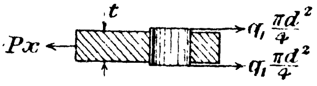

FIG. 56.

(ii) The permissible bearing pressure may be exceeded (Fig. 57). To prevent this we must have

$$\left.\begin{array}{l} Px \nless p_b.d.t \\ \text{or} \nless 2p_b d.t' \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whichever is the smaller.

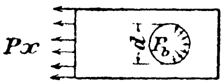

FIG. 57.

The bearing pressure between the plates and the rivets arises from the former being pressed up against the latter, and, as we are neglecting friction, the whole load is transmitted in this way, as shown in Fig. 58, which refers to the left-hand rivets in Fig. 55.

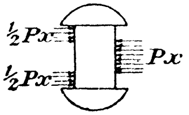

FIG. 58.

(iii) The shaded area $FHKG$ may be pushed out of the plates which are being joined together. This is resisted by the shearing stresses on two rectangular areas perpendicular to the plates, the length of each area being a and the depth t. Hence for safety we must have

$$Px \nless 2atq_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(iv) Similarly, the shaded area *LMNQ* may be pushed out of the cover plates, and the condition that this should not happen is

$$Px < (2bt'q_2) \times 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The factor 2 outside the bracket is due to there being two cover plates.

(v) The plates may fail in tension across the line *XY* (Fig. 59). The resistance to tension is $f \times (x - d)t$, so that we require

$$Px < ft(x - d) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

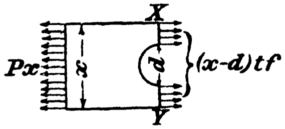

FIG. 59.

(vi) The plates may tear in the manner shown in Fig. 60 at *Z*, but the probability of this is not open to calculation.

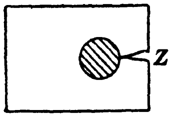

FIG. 60.

Example.—Two steel plates $\frac{3}{8}''$ thick are to be joined by a single riveted butt joint with cover plates. The rivets are to be $\frac{5}{8}''$ diameter and the tensile and shearing stresses are to be 6 and 4·8 tons/in.^{2} respectively. Find the proportions of the joint so that it shall be equally strong in shear and tension, and calculate the bearing pressure between the rivets and plates.

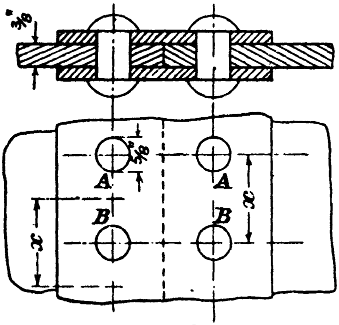

FIG. 61.

The arrangement of the joint will be as shown in Fig. 61. Let x = the "pitch" of the rivets, and let the load on the joint be P tons per inch so that the load on a length x is xP tons.

It will be logical to make each cover plate half * the thickness of the plates which are being joined, i.e. $\frac{3}{16}''$.

If the joint fail in tension it will be along the lines *AB*. The area resisting tension is

$$\tfrac{3}{8}(x - \tfrac{5}{8}) \text{ sq. in.}$$

With a tensile stress of 6 tons/in.^{2} the total load on *AB* will be

$$6 \text{ tons/in.}^2 \times \tfrac{3}{8}\,(x - \tfrac{5}{8}) \text{ in.}^2 = \tfrac{9}{4}\,(x - \tfrac{5}{8}) \text{ tons.}$$

Hence we must have

$$Px = \tfrac{9}{4}x - \tfrac{45}{32} \text{ tons} \quad . \quad . \quad . \quad . \quad (\text{i})$$

If the joint fail by the rivets shearing, the area resisting shear, in a length x, is

$$2 \times \left(\frac{\pi}{4} \times \frac{25}{64}\right) = \frac{25\pi}{128} \text{ sq. in.}$$

Therefore, with a shear stress of 4·8 tons per sq. in. we can have

$$Px = 4{\cdot}8 \text{ tons/in.}^2 \times \frac{25\pi}{128} \text{ in.}^2 = 4{\cdot}8 \times \frac{25\pi}{128} \text{ tons} \quad . \quad (\text{ii})$$

---

* In practice the cover plates are usually made thicker than this as it is found that the cover plates seem to be weaker than the plates joined.

If the joint is to be equally strong in shear and tension we must have, from (i) and (ii),

$$\frac{9}{4}x - \frac{45}{32} = 4{\cdot}8 \times \frac{25\pi}{128}$$

$$\frac{9}{4}x = 2{\cdot}94 + 1{\cdot}4 = 4{\cdot}34$$

$$x = 1{\cdot}93''.$$

Then from (ii), we have $Px = 2{\cdot}95$ tons.

The area of each rivet resisting crushing is $\frac{5}{8}'' \times \frac{3}{8}'' = \frac{15}{64}$ in.^{2} Hence the bearing pressure is

$$2{\cdot}95 \text{ tons}/\tfrac{15}{64} \text{ in.}^2 = 12{\cdot}6 \text{ tons/in.}^2$$

Again, since the cover plates are each half the thickness of the plates being joined, equations (3) and (4), p. 65, will give $a = b$ and

$$2a \times \tfrac{3}{8}'' \times 4{\cdot}8 \text{ tons/in.}^2 = 2{\cdot}95 \text{ tons},$$

whence $a = 0{\cdot}82'' = b$.

56. Group-Riveted Joints.*—When two tension members are joined together by cover plates riveted in the manner shown in Fig. 62

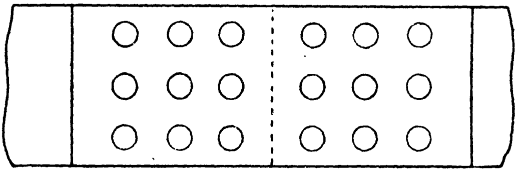

FIG. 62.

the joint is said to be group-riveted. The greatest efficiency of joint is obtained when the rivets are arranged as shown in Fig. 63, where it is supposed six rivets are required each side of the join. The loss of cross section in the main members, on the line *A*, is that due to one rivet hole. If the load is assumed to be equally distributed among the rivets, the

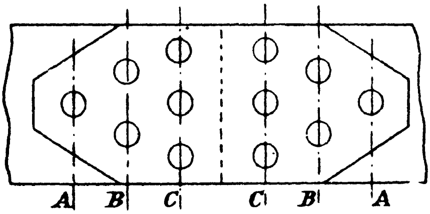

FIG. 63.

rivet on the line *A* will take one-sixth of the total load, so that the tension in the main plates, across *B*, will be $\frac{5}{6}$ of the total. But this section is reduced by two rivet holes, so that, relatively, it is as strong as the section *A*, and so on: the reduction of the nett cross section of the main plates increases as the load carried by these plates decreases. Thus a more efficient joint is obtained than when the rivets are arranged as in Fig. 62.

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* See *Engineering*, Dec. 6, 1918, for an article on the rigidity of riveted joints; the effect of rivet holes is dealt with Sept. 1, 1911, Sept. 19, 1913, Sept. 8, 1922.

57. Eccentric Loads.—It is obvious that a uniform distribution of load among the rivets cannot be attained unless the line of action of the resultant force acting on the joint passes through the centroid of the rivet-holes, and, at the same time, the rivets are symmetrically disposed with regard to the resultant force. In very many cases these conditions are not satisfied and it is desirable to form an idea of the load distribution.

In Fig. 64 let $ABCD$ be a plate riveted to a member EF, and let P be the resultant of the forces acting on AB. Let G be the centroid of the rivet-holes.

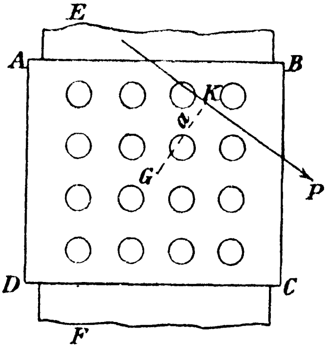

FIG. 64.

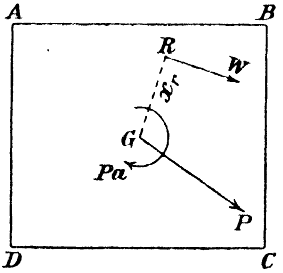

FIG. 65.

Let n = the total number of rivets.

a = the perpendicular distance of G from the line of action of P.

Now we can replace P in Fig. 64 by an equal parallel force P acting through G, combined with a couple Pa, as shown in Fig. 65.

We assume that the deformation of the plate is negligible in comparison with that of the rivets, and that the load on each rivet is proportional to the displacement of the corresponding hole in $ABCD$, relative to the member EF, and acts in the same direction.

Consider the loads on the rivets due to P and Pa separately :—

On account of the force P each rivet will carry a load P/n acting in a direction parallel to that of P.

On account of the couple Pa the displacement of any rivet-hole R will be at right angles to GR (Fig. 65) and proportional to the distance GR. Therefore, according to our assumption, the load on the rivet at R will be proportional to GR and perpendicular to it.

Let $GR = x_r$, W_r = the load on the rivet at R, and let

$$W_r = kx_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where k is a constant. Then we must have

$$Pa = \sum_{r=1}^{r=n} W_r x_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii) we get

$$Pa = \sum_{r=1}^{r=n} kx_r^2 = k \sum_{r=1}^{r=n} x_r^2$$

$$\therefore k = \frac{Pa}{\sum_{r=1}^{r=n} x_r^2}$$

Hence, from (i),

$$W_r' = Pa . \frac{x_r}{\sum_{r=1}^{r=n} x_r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation gives the load on each rivet due to the couple Pa. The total load on the rivet is the vector resultant of this and the load P/n.

Example.—Fig. 66 shows a bracket riveted to a vertical stanchion and loaded with a vertical load of 5 tons. Assuming that the total shearing stress in a rivet is proportional to the relative displacement of the bracket and the stanchion in the neighbourhood of the rivet, find the load carried by each of the rivets. (Intercoll. Exam., Cambridge, 1922.)

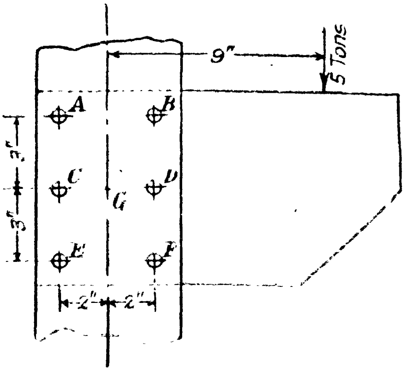

FIG. 66.

The centroid of the rivets is evidently at the point marked G in the figure.

For A, $x = AG = \sqrt{9 + 4} = \sqrt{13}$ in.
B, $x = BG = \sqrt{13}$ in.
C and D, $x = 2$ in.
E and F, $x = \sqrt{13}$ in.

$\Sigma x_r^2 = 13 + 13 + 13 + 13 + 4 + 4 = 60$ in.²
$n = 6$
$a = 9$ in.
$P = 5$ tons.
$\therefore Pa = 45$ tons. ins.

Then the loads on rivets A, B, F, E, due to the couple Pa are each

$$45 \text{ tons/in.} \times \frac{\sqrt{13} \text{ in.}}{60 \text{ in.}^2} = 2{\cdot}71 \text{ tons},$$

at right angles to GA, GB, GF, GE respectively.

The corresponding loads on the rivets C and D are each

$$45 \text{ tons/in.} \times \frac{2 \text{ in.}}{60 \text{ in.}^2} = 1{\cdot}50 \text{ tons},$$

perpendicular to GC and GD respectively.

The load on every rivet, due to the force of 5 tons is ⅚ ton = 0·833 ton, vertically downwards. Thus the resultant loads on all the rivets are found, as shown in Fig. 67, by drawing parallelograms of forces.

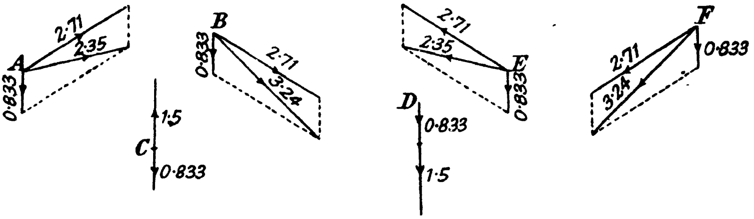

FIG. 67.

The resultant load on rivets A and E = 2·35 tons.
" " " " B " F = 3·24 "
" " " " C = 0·667 "
" " " " D = 2·333 "

EXAMPLES IV

1. A double-riveted butt-joint connects two ⅜″ plates with one cover strap. The diameter of the rivets is ¾″, and the distance between rivet centres along the pitch line is 5″. Assuming that the other unstated dimensions are adequate, calculate the strength of the joint per foot, in tension, allowing 5 tons/in.² shear stress in the rivets, and a tensile stress of 6 tons/in.² in the plates. (Special Exam., Cambridge, 1907.)

2. Two boiler plates ½″ thick are connected by a double-riveted lap joint, formed by ¾″ rivets with a pitch of 2½″. Determine the least tensile stress in the rivets which will enable the joint to remain tight under a tension of 12 tons per foot, along the joint, if the coefficient of friction is 0·2. (Special Exam., Cambridge, 1913.)

3. A riveted joint is formed by two cover plates shaped in the manner represented in Fig. 68. When the joint is subjected to a large pull, it may

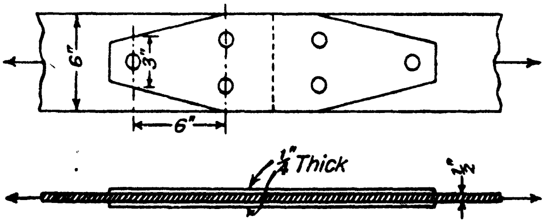

FIG. 68.

be assumed that the resistance is all provided by the shear in the rivets, and that the resistance exerted by a rivet is proportional to the relative slip of the corresponding rivet holes.

When the four centre rivets only are in position, it is found that with a pull of 20 tons the slip at the rivet holes is 0·01″.

With all six rivets in position show that the loads taken by the outer rivets exceed the loads taken by each of the four centre rivets in the ratio of 27 : 25. E for the plates is 14,000 tons/in.^{2} (Mech. Sc. Trip., 1922.)

4. A flat steel bar is attached to a gusset plate by eight rivets in the manner represented in Fig. 69. At the section AB the gusset plate exerts on the flat bar a vertical shearing force S and a counter-clockwise couple M.

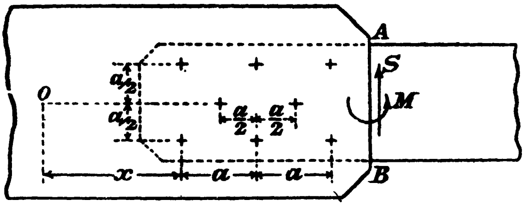

FIG. 69.

Assuming that the gusset plate, relative to the flat bar, undergoes a minute rotation about a point O on the line of the two middle rivets, also that the loads on the rivets are due to and proportional to the relative movement of the plates at the rivet holes, prove that

$$x = -a \cdot \frac{4M + 3aS}{4M + 6aS}.$$

Prove also that the horizontal and vertical components of the load on the top right hand rivet are

$$\frac{2M + 3aS}{24a} \text{ and } \frac{4M + 9aS}{24a}$$

respectively. (Mech. Sc. Trip., 1923.)

5. A steel strip of cross section 2″ × ½″ is bolted to two copper strips, each of cross section 2″ × ⅜″, as shown in Fig. 45, there being two bolts on the line of pull. Show that, neglecting friction and the deformation of the bolts, a pull applied to the joint will be shared by the bolts in the ratio 3 to 4.

Assume that E for steel is twice E for copper.

(Intercoll. Exam., Cambridge, 1923.)

6. *Two flat bars are riveted together in the manner shown in Fig. 70, x being the pitch of the rivets in a direction at right angles to the plane of the figure. Assuming that the rivets themselves do not deform, show that the load taken by the rivets (1) is $\frac{tPx}{t + 2t'}$, and that the rivets (2) are free from load.

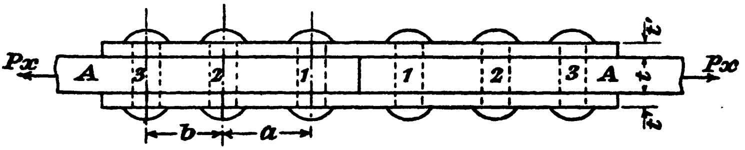

FIG. 70.

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* In connection with the distribution of load in riveted joints, see papers by Montgomerie and Batho (see footnote, p. 64); also a paper by the author in *Aeronautics*, 1918.

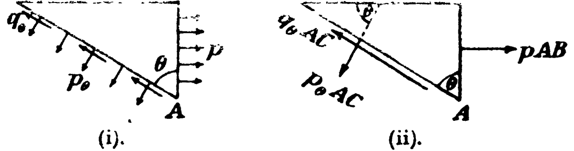
CHAPTER V

ANALYSIS OF STRESS AND STRAIN. COMPOUND STRESSES

ANALYSIS OF STRESS

58. Introductory.—Up to the present we have confined our attention to considerations of simple tensile, compressive, or shearing stresses. But in a very large number of practical cases we have to deal with a combination of these simple stresses, and some standard of comparison becomes necessary. The strengths and elastic properties of materials are usually determined by simple tensile or compressive tests. How are we to make use of the results of such experiments when we know that stress in a given case is made up of a tensile stress in one direction, a compressive stress in some other direction, and a shearing stress in a third direction? Clearly we cannot make tests of a material under all the possible combinations of stress and tabulate the results. The analytical processes which we explain below will show us that, when all the stresses act in directions parallel to one plane, the most complicated stress system is equivalent to a combination of two mutually perpendicular direct stresses. In the next chapter we consider the relationship between the strength of a material acted on by two such stresses and its strength under simple tension. The analysis which follows must therefore be regarded as having a very direct and important bearing on practical engineering, and not merely a display of mathematical acrobatics.

59. Stress-components on any Plane due to a Direct Stress on a Given Plane.—Suppose that the plane *AB* (perpendicular to the plane of the paper) in a body is acted on only by a direct stress p (Fig. 71 (i)), and that it is required to find the stress components on a plane AC making an angle θ with AB.



(i). (ii).

FIG. 71.

Through B draw a plane BC at right angles to the plane AB and to the plane of the paper, and consider the triangular wedge *ABC*, the thickness perpendicular to the plane of the paper being unity.

Let p_θ = the direct stress on the plane *AC*,
q_θ = „ shear „ „ „ „ „

their directions being positive when they act according to the arrows in Fig. 71.

Consider the equilibrium of *ABC*: the total actions on *AB* and *AC* are shown in Fig. 71 (ii).

Resolve at right angles to *AC*:

$$\begin{aligned} p_\theta.AC &= p.AB.\cos\theta \\ &= p.AC\cos^2\theta \\ \therefore\ \mathbf{p_\theta} &= \mathbf{p.\cos^2\theta} \quad \dots\dots\dots \quad (1) \end{aligned}$$

Resolve in the direction *AC*:

$$\begin{aligned} q_\theta.AC &= p.AB\sin\theta \\ &= p.AC\sin\theta\cos\theta \\ \therefore\ \mathbf{q_\theta} &= \mathbf{p\sin\theta.\cos\theta} = \tfrac{1}{2}p\sin 2\theta \quad \dots\dots \quad (2) \end{aligned}$$

Then p_θ and q_θ are the component stresses on the plane *AC* due to the stress p on the plane *AB*. The stress at *A* can be specified (i) by p on *AB*, or (ii) by p_θ and q_θ on *AC*.

The shear component q_θ is numerically a maximum when $\sin 2\theta = \pm 1$, which gives $\theta = 45°$ or $135°$. We then have $p_\theta = q_\theta = \tfrac{1}{2}p$.

60. Stress-components on any Plane due to a Shearing Stress on a Given Plane.—Next suppose that the plane *AB* is acted on only

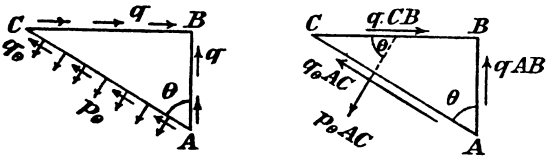


FIG. 72.

by a shearing stress q (Fig. 72), and proceed exactly as before. From § 48, p. 57, we know that there must be an equal shearing stress on the plane *CB*. Let p_θ and q_θ be the stress components on the plane *AC*.

Then consider the equilibrium of *ABC*; resolving at right angles to *AC* we have

$$\begin{aligned} p_\theta.AC &= q.CB.\cos\theta + q.AB.\sin\theta \\ &= q.AC\sin\theta\cos\theta + q.AC\cos\theta.\sin\theta. \\ \therefore\ \mathbf{p_\theta} &= \mathbf{2q\sin\theta\cos\theta = q\sin 2\theta} \quad \dots\dots \quad (3) \end{aligned}$$

Resolving parallel to *AC* gives

$$\begin{aligned} q_\theta.AC &= q.CB\sin\theta - q.AB\cos\theta \\ &= q.AC\sin^2\theta - q.AC\cos^2\theta \\ \therefore\ \mathbf{q_\theta} &= \mathbf{q(\sin^2\theta - \cos^2\theta) = -q\cos 2\theta} \quad \dots\dots \quad (4) \end{aligned}$$

(3) and (4) give the stress components on the plane *AC*.

We see from (3) that p_θ is numerically a maximum when $\theta = 45°$

or 135°, and that then q_θ is zero. When θ is 45° we get $p_\theta = q$, and, when θ is 135°, $p_\theta = -q$. Thus two complementary shearing stresses are equivalent to two equal and opposite direct stresses acting on planes at 45° to the planes of the shearing stresses, and numerically equal to the shearing stresses. Hence we see that the stresses at O specified by Figs. 73 and 74 are equivalent to one another.

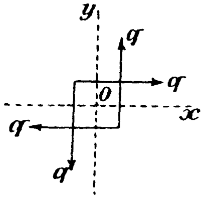


FIG. 73.

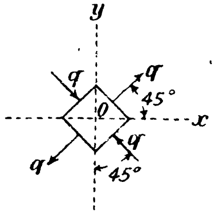


FIG. 74.

Example.—A bar of cross section $\frac{7}{8}'' \times \frac{7}{8}''$ is subjected to an axial pull of 2 tons. Calculate the normal stress and shear stress on a plane the normal to which makes an angle of 60° with the axis of the bar, the plane being perpendicular to one face of the bar.

We have $\theta = 60°$

$P = 2$ tons

$S = 0{\cdot}765$ in.²

$$\therefore p = \frac{2 \text{ tons}}{0{\cdot}765 \text{ in.}^2} = 2{\cdot}62 \text{ tons/in.}^2$$

The normal stress on the oblique plane is

$$p \cos^2 60° = 2{\cdot}62 \text{ tons/in.}^2 \times \tfrac{1}{4} = 0{\cdot}655 \text{ tons/in.}^2$$

The shear stress on the oblique plane is

$$\tfrac{1}{2}p \sin 120° = \tfrac{1}{2} \times 2\ 62 \text{ tons/in.}^2 \times \frac{\sqrt{3}}{2} = 1{\cdot}135 \text{ tons/in.}^2$$

61. General Two-dimensional Stress System.—When the stresses at any point in a body all act parallel to one plane, the most general stress system will consist of two mutually perpendicular direct stresses, and two equal complementary shearing stresses. For, consider the small rectangular block of material $ABCD$ (Fig. 75) and suppose that the forces acting on the four sides are P, Q, R and S. Each of these forces can be resolved into components perpendicular to and tangential to the sides of the block, as shown in Fig. 76, and similarly with any other forces applied to the sides of the block.

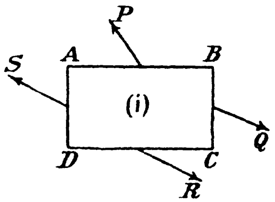


FIG. 75.

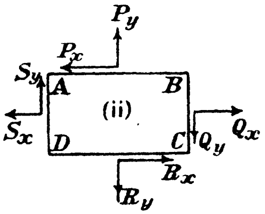


FIG. 76.

The components perpendicular to the sides give rise to direct stresses, whilst those tangential to the sides produce shearing stresses. According to § 48, the shearing stresses on perpendicular faces must be equal, and evidently the direct stresses on opposite faces must be equal if the block is not acted on by a " body force " such as gravity or inertia. Hence the stress at any point, *B* for instance, can always be reduced to two direct stresses acting on perpendicular planes, such as *AB* and *BC*, together with equal shearing stresses on these planes.

62. Stress-components on any Plane in a General Two-dimensional Stress-system.—Let the stress system consist of direct stresses p_x and p_y on the planes *AB* and *BC* at right angles to each other (Fig. 77), together with shear stresses q on these two planes.

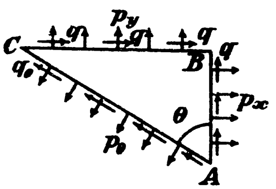


FIG. 77.

Let p_θ and q_θ be the stress components on a plane *AC* making an angle θ with *AB*.

As in §§ 60 and 61, consider a triangular element of the material with unit thickness perpendicular to the plane of the paper.

$$\begin{aligned} &\text{The stress } p_x \text{ produces direct stress on } AC &&= p_x \cos^2\theta \\ &\quad\text{,,} \quad p_y \quad \text{,,} \quad \text{,,} \quad \text{,,} &&= p_y \cos^2(90^\circ-\theta) \\ & &&= p_y \sin^2\theta \\ &\quad\text{,,} \quad q \quad \text{,,} \quad \text{,,} \quad \text{,,} &&= q\sin 2\theta \end{aligned}$$

Hence we have, by addition,

$$\begin{aligned} p_\theta &= p_x\cos^2\theta + p_y\sin^2\theta + q\sin 2\theta \\ &= \tfrac{1}{2}(p_x+p_y) + \tfrac{1}{2}(p_x-p_y)\cos 2\theta + q\sin 2\theta \quad . \quad . \quad (5) \end{aligned}$$

Again,

$$\begin{aligned} &\text{The stress } p_x \text{ produces shear stress on } AC &&= \tfrac{1}{2}\,p_x\sin 2\,\theta \\ &\quad\text{,,} \quad p_y \quad \text{,,} \quad \text{,,} \quad \text{,,} &&= -\tfrac{1}{2}p_y\sin 2(90^\circ-\theta) \\ & &&= -\tfrac{1}{2}p_y\sin 2\theta \\ &\quad\text{,,} \quad q \quad \text{,,} \quad \text{,,} \quad \text{,,} &&= -q\cos 2\theta \end{aligned}$$

Hence

$$q_\theta = \tfrac{1}{2}(p_x-p_y)\sin 2\theta - q\cos 2\theta \quad . \quad . \quad . \quad . \quad (6)$$

63. Principal Planes.—We see from (6) that the stress component q_θ is zero when

$$\tan 2\theta = \frac{2q}{p_x-p_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

i.e. when

$$2\theta = \tan^{-1}\left(\frac{2q}{p_x-p_y}\right) \quad \text{or} \quad \tan^{-1}\left(\frac{2q}{p_x-p_y}\right) + 180^\circ$$

$$\theta = \tfrac{1}{2}\tan^{-1}\left(\frac{2q}{p_x-p_y}\right) \text{ or } \tfrac{1}{2}\tan^{-1}\left(\frac{2q}{p_x-p_y}\right) + 90^\circ \quad . \quad (8)$$

Thus there are two planes, separated by 90°, on which the shearing stress is zero, in any two-dimensional stress system. These planes are called the *principal planes*, and the corresponding values of p_θ are called the *principal stresses*. Comparing (5) with (6) we see that the condition that q_θ vanishes is also the condition that p_θ is a maximum or a minimum.* Thus the principal stresses are the maximum and minimum direct stresses in the material.

From the above we see that we can always replace the stress-system (p_x, p_y, q) by two mutually perpendicular direct stresses only, these stresses acting on planes making angles θ with the plane on which p_x acts, when θ is given by (7). This is a result of very great practical importance, and we shall give below formulæ for the values of these stresses.

64. To Find the Principal Stresses.—Let p_1 and p_2 be the principal stresses. When $\tan 2\theta = \dfrac{2q}{p_x - p_y}$, we have (see Fig. 78)

$$\sin 2\theta = \frac{2q}{\sqrt{(p_x - p_y)^2 + 4q^2}}$$

and

$$\cos 2\theta = \frac{p_x - p_y}{\sqrt{(p_x - p_y)^2 + 4q^2}}$$

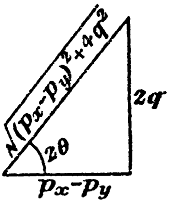


FIG. 78.

Substituting these values in (5) gives

$$\mathbf{p_1 = \tfrac{1}{2}(p_x + p_y) + \tfrac{1}{2}\sqrt{(p_x - p_y)^2 + 4q^2}} \quad . \quad . \quad . \quad (9)$$

When $2\theta = \tan^{-1}\left(\dfrac{2q}{p_x - p_y}\right) + 180°$, the signs of $\sin 2\theta$ and $\cos 2\theta$ are both changed. Therefore, if p_2 denote the corresponding value of p_θ, we have

$$\mathbf{p_2 = \tfrac{1}{2}(p_x + p_y) - \tfrac{1}{2}\sqrt{(p_x - p_y)^2 + 4q^2}} \quad . \quad . \quad . \quad (10)$$

Equations (9) and (10) give the principal stresses.

65. The Principal Stresses Found from First Principles.—On account of the importance of the subject we add the following method of finding the principal stresses from the definition that they are the direct stresses across planes on which the shearing stress is zero.

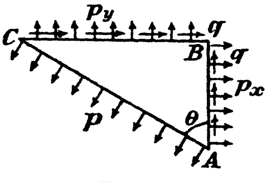


FIG. 79.

Referring to Fig. 79, let AC be a principal plane, and let p be the direct stress on this plane. Then, resolving in the direction BC, we have

$$\begin{aligned} p.AC\cos\theta &= p_x.AB + q.BC \\ &= p_x.AC\cos\theta + q.AC\sin\theta \\ \therefore p - p_x &= q\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)} \end{aligned}$$

Resolving in the direction AB we have

$$\begin{aligned} p.AC\sin\theta &= p_y.BC + q.AB \\ &= p_y.AC\sin\theta + q.AC\cos\theta \\ \therefore p - p_y &= q\cot\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)} \end{aligned}$$

---

* Since $\dfrac{dp_\theta}{d\theta} = -(p_x - p_y)\sin 2\theta + 2q\cos 2\theta$, which must vanish for a maximum or minimum value of p.

Multiplying (i) by (ii) we get

$$(\mathrm{p} - \mathrm{p}_x)(\mathrm{p} - \mathrm{p}_y) = \mathrm{q}^2 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is a quadratic for p, the roots of which are the expression for p_1 and p_2 given above. This is perhaps the most useful form to remember for purposes of calculation.

66. Maximum Shear Stresses.—The planes perpendicular to the plane *ABC*, Fig. 77, on which the shear stresses are a maximum, are given by

$$\frac{dq_\theta}{d\theta} = 0$$

i.e.

$$(p_x - p_y) \cos 2\theta - 2q \sin 2\theta = 0$$

$$\therefore \tan 2\theta = \frac{p_y - p_x}{2q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If a denote the values of θ given by (7) when the normal stresses have stationary values, and β the values of θ given by (12) when the shear stress is a maximum, we see that

$$\tan 2\beta = -\cot 2a \qquad \therefore\ 2\beta = 2a + \frac{\pi}{2} \qquad \therefore\ \beta = a + \frac{\pi}{4}$$

Hence the planes of maximum shear stress are inclined at 45° to the principal planes. Since the maximum value of $A \cos x + B \sin x$ is $\sqrt{A^2 + B^2}$ we see from (6) that the maximum shear stress is

$$\mathrm{q}_{\max} = \tfrac{1}{2}\sqrt{(p_x - p_y)^2 + 4q^2} = \tfrac{1}{2}(\mathrm{p}_1 - \mathrm{p}_2) \quad . \quad . \quad . \quad (13)$$

The normal stress on the planes of maximum shear stress $= \frac{1}{2}(p_x + p_y) = \frac{1}{2}(p_1 + p_2)$.

It can be shown that in a general three-dimensional stress system there are three mutually perpendicular principal planes, three corresponding principal stresses, and that the greatest shearing stress is equal to half the difference between the greatest and least of these. In a two-dimensional system the three principal stresses are p_1, p_2, 0. If p_1 and p_2 have the same sign and $p_1 > p_2$, p_1 will be the greatest, and zero the smallest principal stress, so that the true maximum shear stress is $\frac{1}{2}p_1$; if p_1 and p_2 have opposite signs p_1 will be the greatest, and p_2 the smallest principal stress, and the greatest shearing stress will be $\frac{1}{2}(p_1 - p_2)$ as found above. In all cases the greatest shearing stress occurs on a plane bisecting the angle between the planes of greatest and least principal stresses.

Example 1.—The propeller shaft of a ship is subjected to a longitudinal thrust of 6·5 tons/in.^{2}, and there is a shear stress (due to torsion) of 3·75 tons/in.^{2} Find the principal stresses and the maximum shear stress.

In this case if we take $p_x = -6{\cdot}5$ tons/in.^{2}, we have $p_y = 0$ and $q = 3{\cdot}75$ tons/in.^{2}

By (9) and (10) the principal stresses are

$$-\frac{6\cdot5}{2} \pm \tfrac{1}{2}\sqrt{6\cdot5^2 + 4 \times 3\cdot75^2} \text{ tons/in.}^2$$
$$= -3\cdot25 \pm \tfrac{1}{2}\sqrt{42\cdot25 + 56\cdot25}$$
$$= -3\cdot25 \pm 4\cdot96$$
$$= -8\cdot21 \text{ and } +1\cdot71 \text{ tons/in.}^2$$

The former is compressive and the latter tensile.

$$\text{The maximum shear stress} = \frac{8\cdot21 + 1\cdot71}{2} = 4\cdot96 \text{ tons/in.}^2$$

Example 2.—In an I girder of a bridge, at a point near the top of the web, there is a tensile stress of 4 tons/in.² and a shearing stress of 3 tons/in.² Calculate (i) the maximum direct stress, (ii) the maximum shear stress, also (iii) the tensile stress which would produce the same maximum shear stress.

(i) From equations (9) and (10) the principal stresses are

$$2 \pm \tfrac{1}{2}\sqrt{2^2 + 4 \times 3^2} = 2 \pm 3\cdot16 \text{ tons/in.}^2$$

The maximum stress is 5·16 tons/in.², tensile.

(ii) The maximum shear stress = 3·16 tons/in.²

(iii) In direct tension the tensile stress is double the maximum shear stress produced (§ 60). Therefore in this case the simple tensile stress which would produce the same shear stress is 2 × 3·16 tons/in.² = 6·32 tons/in.²

ANALYSIS OF STRAIN

67. Strain in any Direction due to Strain in a Given Direction.

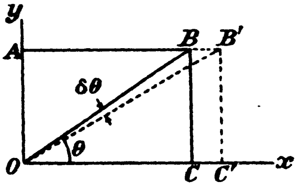


FIG. 80.

—Let the material of a body receive a given strain in one direction only, then it is required to find the corresponding strain in any other direction. Thus, in Fig. 80, let the strain in the direction Ox be e_x, and in the direction Oy zero, so that the sides AB and OC of the small rectangle $OABC$ are changed into AB' and OC' while the sides OA and BC remain unaltered. It is required to find the strain in the direction OB.

Let e_θ = the strain in OB.

We have

$$OB^2 = OC^2 + CB^2$$

Differentiating, we have, since CB does not change,

$$OB \times \delta.OB = OC \times \delta.OC.$$

Divide by OB^2, then

$$e_\theta = \frac{\delta.OB}{OB} = \frac{OC}{OB^2} \times \delta.OC = \frac{OC^2}{OB^2} \times \frac{\delta.OC}{OC}$$

Hence

$$\mathbf{e_\theta = e_x \cos^2 \theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

Similarly if the strain in direction Oy be e_y whilst that in direction Ox is zero, the strain in direction OB will be $e_y \sin^2 \theta$.

If both e_x and e_y exist simultaneously, we shall have, by the principle of superposition,

$$e_\theta = e_x \cos^2 \theta + e_y \sin^2 \theta.$$

68. To Find the Direct Strain in Any Direction due to a Given Shear Strain.—In Fig. 81, let the shear strain be e_{xy}, so that the rectangle $OABC$ is strained into the parallelogram $OA'B'C$, the angles AOA' and BCB' being e_{xy}. It is required to find the direct strain in the direction OB. We have

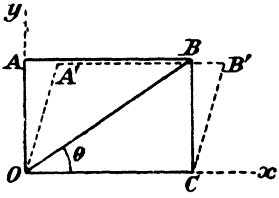


FIG. 81.

$$OB^2 = OC^2 + CB^2 - 2.OC.CB \cos OCB.\text{*}$$

Differentiate this, remembering that OC and CB do not change in length:

$$OB \times \delta.OB = OC.CB \sin \widehat{OCB} \times \delta.\widehat{OCB}$$
$$= OC.CB.e_{xy}$$

since originally $\widehat{OCB} = 90°$.

Dividing by OB^2 we have

$$\frac{\delta.OB}{OB} = \frac{OC}{OB} \cdot \frac{CB}{OB} . e_{xy}$$

or

$$\mathbf{e}_\theta = \mathbf{e}_{xy} \sin \boldsymbol{\theta} . \cos \boldsymbol{\theta} = \tfrac{1}{2} e_{xy} \sin 2\theta \quad . \quad . \quad . \quad . \quad (15)$$

where e_θ is the strain in OB.

69. General Two-Dimensional Strain.—Combining the results of §§ 67 and 68, if the three strains e_x, e_y, and e_{xy} exist simultaneously, the strain in direction OB is

$$e_\theta = e_x \cos^2 \theta + e_y \sin^2 \theta + e_{xy} \sin \theta \cos \theta$$

or

$$e_\theta = \tfrac{1}{2}[(e_x + e_y) + (e_x - e_y) \cos 2\theta + e_{xy} \sin 2\theta] \quad . \quad . \quad . \quad (16)$$

From this we see that e_θ is a maximum or minimum when

$$\tan 2\theta = \frac{e_{xy}}{e_x - e_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This gives two values of θ separated by $\frac{\pi}{2}$.

If e_x and e_y be both positive, the smaller value leads to

$$e_{max.} = \tfrac{1}{2}(e_x + e_y) + \tfrac{1}{2}\sqrt{(e_x - e_y)^2 + e_{xy}^2} \quad . \quad . \quad . \quad (18)$$

and the larger to

$$e_{min.} = \tfrac{1}{2}(e_x + e_y) - \tfrac{1}{2}\sqrt{(e_x - e_y)^2 + e_{xy}^2} \quad . \quad . \quad . \quad (19)$$

These equations give the principal strains; they should be compared with the equations for the principal stresses. The principal strains will be denoted by e_1 and e_2, so that e_1 is given by (18) and e_2 by (19) above.

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* The equation must be written in this general form since the angle OCB is a variable.

70. Maximum Shear Strain.—Since the state of stress is the same as that due to the principal stresses, and similarly the strains in the directions of the principal stresses are e_1 and e_2, we can deal with the case as represented by these strains.

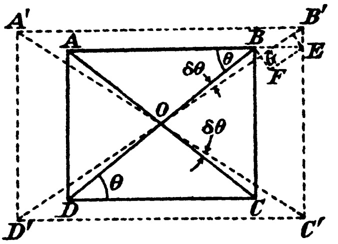

FIG. 82.

Let $ABCD$ (Fig. 82) be a rectangular element of the material, the sides AB and BC being in the direction of the principal strains e_1 and e_2, which have been taken positive when tensile. After strain, $ABCD$ will have become $A'B'C'D'$, and each diagonal will have been rotated an amount $\delta\theta$, so that the shear strain is $2\delta\theta$.

The stretch of $AB = 2EB = e_1.AB$.

$\therefore\ EB = \frac{1}{2}e_1.AB = e_1.OB \cos\theta$

Similarly $EB' = e_2 OB \sin\theta$

Draw BG perpendicular to OB', and EF perpendicular to BG. Then to the first order of small quantities

$$\delta\theta = \frac{BG}{OB} = \frac{EB \sin\theta - EB' \cos\theta}{OB}, \text{ from the figure,}$$

$$= e_1 \cos\theta \sin\theta - e_2 \sin\theta \cos\theta$$

$$= \tfrac{1}{2}(e_1 - e_2) \sin 2\theta$$

Therefore the shear strain

$$= 2\delta\theta = (e_1 - e_2) \sin 2\theta$$

which is a maximum when $\theta = \frac{\pi}{4}$. Hence the *maximum shear strain is equal to the difference between the two principal strains* and occurs on planes making 45° with them. Notice that the maximum shear *stress* is equal to *half* the difference between the principal *stresses* (§ 67).

Substituting the values of e_1 and e_2 from (18) and (19) we have

$$e_1 - e_2 = \sqrt{(e_x - e_y)^2 + e_{xy}^2}$$

This gives the maximum shear strain (or " slide " as it is sometimes called) in terms of e_x, e_y, and e_{xy}.

Example 1.—A flat bar 3″ × ½″ is subjected to an axial pull of 12 tons. One side of the bar is polished and fine lines are ruled on it to form a square of 2″ side, one diagonal of the square being along the middle line of the polished side. If $E = 30 \times 10^6$ lbs./in.^{2} and $m = \frac{10}{3}$, calculate the alteration in the sides and angles of the square. (Intercoll. Exam., Cambridge, 1909.)

The area of the cross section of the bar = 1·5 in.^{2}

$$\therefore \text{ the longitudinal stress} = \frac{12 \times 2{,}240 \text{ lbs.}}{1{\cdot}5 \text{ in.}^2} = 17{,}920 \text{ lbs./in.}^2$$

From § 70, since $e_1 = p/E$ and $e_2 = -p/mE$, we see that the change in the angles of the square will be

$$e_1 - e_2 = \frac{p}{E}\cdot\frac{m+1}{m} = \frac{17{,}920 \text{ lbs./in.}^2}{30 \times 10^6 \text{ lbs./in.}^2} \times \frac{13}{10} = 0{\cdot}0007765 \text{ radian}$$

$$= 0^\circ\, 2'\, 40{\cdot}2''.$$

The angles which are in the line of pull will be diminished by this, and the others increased by the same amount.

$$\text{The longitudinal strain} = \frac{17{,}920 \text{ lbs./in.}^2}{30 \times 10^6 \text{ lbs./in.}^2} = 0{\cdot}0005973$$

$$\text{The lateral strain} = -\,0{\cdot}0005973 \div \frac{10}{3} = -\,0{\cdot}000179.$$

Therefore the strain of each side

$$\begin{aligned} &= 0{\cdot}0005973 \cos^2 45^\circ - 0{\cdot}000179 \sin^2 45^\circ \\ &= 0{\cdot}000{,}2987 - 0{\cdot}000{,}0895 \\ &= 0{\cdot}000{,}2092 \end{aligned}$$

Then the alteration in the length of the sides will be

$$0{\cdot}000{,}2092 \times 2'' = 0{\cdot}000418''$$

Stress-strain Relations

71. Principal Strains.—We have seen above that the state of stress at any point in the material is the same as that which would be produced by two mutually perpendicular stresses p_1 and p_2, whose directions have been found in § 63. We have taken p_1 and p_2 positive when they are tensile. The total tensile strain in the direction of p_1 is, by the principle of superposition (p. 27),

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

whilst that in the direction of p_2 is

$$e_2 = \frac{p_2}{E} - \frac{p_1}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Substituting the values of p_1 and p_2 given by (9) and (10) we have

$$e_1 = \frac{1}{2E}\left(1 - \frac{1}{m}\right)(p_x + p_y) + \frac{1}{2E}\left(1 + \frac{1}{m}\right)\sqrt{(p_x - p_y)^2 + 4q^2} \quad (20)$$

$$e_2 = \frac{1}{2E}\left(1 - \frac{1}{m}\right)(p_x + p_y) - \frac{1}{2E}\left(1 + \frac{1}{m}\right)\sqrt{(p_x - p_y)^2 + 4q^2} \quad (21)$$

If p_x and p_y are both positive the former of these gives the maximum, and the latter the minimum, direct strain in the material.

72. Single Direct Stress Required to Produce same Maximum Strain as a Given Stress System.—If the material is acted on by a single stress $\bar{p}$, the strain is $\bar{p}/E$, therefore, if this is to be the same as e_1, we must have

$$\bar{p} = \frac{m-1}{2m}(p_x + p_y) + \frac{m+1}{2m}\sqrt{(p_x - p_y)^2 + 4q^2} \quad (22)$$

Thus the stress which would produce the maximum strain if acting alone is not the same as the maximum stress, which is (p. 76)

$$\tfrac{1}{2}(p_x + p_y) + \tfrac{1}{2}\sqrt{(p_x - p_y)^2 + 4q^2}.$$

73. Relations Between E, C, K and m.—Suppose that a cube of the material of a body be under the action of a shearing stress q (Fig. 83). Then (§ 60) we know that in the direction of the diagonal BD there is a tensile stress of intensity q.

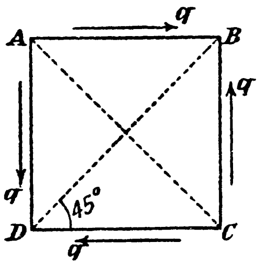

FIG. 83.

This will produce a tensile strain $\frac{q}{E}$ in BD, and a compressive strain $\frac{q}{mE}$ in AC.

At the same time there is a compressive stress q in the direction AC, producing a compressive strain $\frac{q}{E}$ in AC and a tensile strain $\frac{q}{mE}$ in BD. Hence the strain in either diagonal is

$$\frac{q}{E} + \frac{q}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

tensile in BD and compressive in AC.

Again, the shear strain due to q, is

$$e_{xy} = \frac{q}{C}$$

and (§ 68) the strain in the direction BD is

$$e_{xy} \sin 45° \cos 45° = \tfrac{1}{2}e_{xy} = \frac{q}{2C} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii) we have

$$\frac{q}{2C} = \frac{q}{E} + \frac{q}{mE}$$

or

$$\mathbf{C} = \frac{\mathbf{mE}}{\mathbf{2(m+1)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This relation shows that if m be negative it must be less than -1, and we have already seen (p. 28) that, if positive, it must be greater than 2.

Combining (23) with the relation between E and K found on p. 28 we get, on eliminating E,

$$\frac{3K - 2C}{6K + 2C} = \frac{1}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If we eliminate m we get

$$E = \frac{9KC}{C + 3K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence, if we know two out of the four elastic constants we can calculate the other two.

74. Strain Energy of Combined Stresses.—If p_1 and p_2 are the principal stresses, the corresponding strains are given by (i) and (ii) of § 71. Hence, the total strain energy per unit volume is

$$U = \tfrac{1}{2}p_1e_1 + \tfrac{1}{2}p_2e_2$$

$$= \tfrac{1}{2}p_1\left(\frac{p_1}{E} - \frac{p_2}{mE}\right) + \tfrac{1}{2}p_2\left(\frac{p_2}{E} - \frac{p_1}{mE}\right)$$

$$\therefore U = \frac{1}{2E}\left(p_1^2 + p_2^2 - \frac{2p_1p_2}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If we substitute the values of p_1 and p_2 given by (9) and (10) we obtain as the expression for the strain energy :

$$U = \frac{p_x^2 + p_y^2}{2E} - \frac{p_xp_y}{mE} + \frac{q^2}{E}\left(1 + \frac{1}{m}\right)$$

and since $E/C = 2(m + 1)/m$, this can be written

$$U = \frac{p_x^2 + p_y^2}{2E} - \frac{p_xp_y}{mE} + \frac{q^2}{2C} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

STRESSES IN THREE DIMENSIONS

75. Principal Stresses in Three-Dimensional System.—It can be shown that in a general three-dimensional stress system there are always three mutually perpendicular planes on which the shearing stress is zero, i.e. that there are three principal planes. The corresponding direct stresses on these planes are the principal stresses. For a proof of this statement, and the equations for finding the principal stresses, the reader is referred to the Appendix, page 543.

76. Strains in Three-Dimensional Stress Systems.—If the principal stresses be p_1, p_2 and p_3, the strains in these directions, i.e. the principal strains, will be given by (17) of § 29 :

$$e_1 = \frac{1}{E}\left\{p_1 - \frac{1}{m}(p_2 + p_3)\right\} \quad . \quad . \quad . \quad . \quad . \quad (i)$$

$$e_2 = \frac{1}{E}\left\{p_2 - \frac{1}{m}(p_3 + p_1)\right\} \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

$$e_3 = \frac{1}{E}\left\{p_3 - \frac{1}{m}(p_1 + p_2)\right\} \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

Exactly in the same manner as § 70 we can show that the greatest slide, or shear strain, is $e_1 - e_3$ if * $e_1 > e_2 > e_3$; i.e. the greatest slide is

$$e_1 - e_3 = \frac{1}{E}\left(1 + \frac{1}{m}\right)(p_1 - p_3) = \frac{p_1 - p_3}{2C}$$

Hence the maximum shear strain is proportional to the difference between the greatest and least principal stresses, and it follows that the maximum shearing stress is $\frac{1}{2}(p_1 - p_3)$.

---

* If $p_1 > p_2 > p_3$ it follows that $e_1 > e_2 > e_3$: thus $e_1 - e_2 = (m + 1)(p_1 - p_2)/mE$ and $e_2 - e_3 = (m + 1)(p_2 - p_3)/mE$, both of which are positive.

77. Strain Energy in Three Dimensions.—It is easy to show, in the manner of § 74, that the strain energy per unit volume, expressed in terms of the principal stresses, is

$$\frac{1}{2E}\left\{p_1^2 + p_2^2 + p_3^2 - \frac{2}{m}(p_2p_3 + p_3p_1 + p_1p_2)\right\} \quad . \quad . \quad (28)$$

EXAMPLES V

1. A tie bar of steel has a cross section 6″ × ¾″ and a load of 18 tons is applied to it. Find the stress normal to a plane making an angle of 30° with the cross section, and the shearing stress along this plane. (Special Exam., Cambridge, 1914.)

2. A rectangular plate ¼″ thick is 8″ long and 6″ broad. A total pull of 5 tons is applied, uniformly distributed along the longest sides. Find the intensity of the tangential and normal stresses on the section joining opposite angles. What total force on the other pair of sides will produce a pure shear in the plate ? (Special Exam., Cambridge, 1919.)

3. A rivet is under the action of a shearing stress of 4 tons/in.^{2} and a tensile stress, due to contraction, of 3 tons/in.^{2} Determine the magnitude and direction of the greatest tensile and shear stresses in the rivet. (R.N.E.C., Keyham, 1920.)

4. A propeller shaft is subjected to an end thrust producing a stress of 6 tons/in.^{2}, and the maximum shearing stress arising from torsion is 4 tons/in.^{2} Calculate the magnitudes of the principal stresses. (Intercoll. Exam., Cambridge, 1906.)

5. At a point in a vertical cross section of a beam there is a resultant stress of 5 tons/in.^{2}, which is inclined upwards at 35° to the horizontal. On the horizontal plane through the point there is only shear stress. Find, in magnitude and direction, the resultant stress on the plane which is inclined at 40° to the vertical and 95° to the given resultant stress. (Mech. Sc. Trip., 1919.)

6. A plate is subjected to two mutually perpendicular stresses, one compressive of 3 tons/in.^{2}, the other tensile of 5 tons/in.^{2}, and a shear stress, parallel to these directions, of 3 tons/in.^{2} Find the principal stresses and stretches, taking Poisson's ratio as 0·3 and $E = 13{,}500$ tons/in.^{2} (Mech. Sc. Trip., 1916.)

7. At a point in a material under stress there is a shear stress, on a vertical plane, of 4 tons/in.^{2}, and also a compressive stress on the same plane of 1 ton/in.^{2} Find the principal stresses and the maximum shear stress at the point. (R.N.E.C., Keyham, 1922.)

8. Being given the magnitudes and directions of the two principal stresses, find the direction of the plane on which the resultant stress is most inclined to the normal of that plane. Find the magnitude and inclination of this resultant when the principal stresses are 8 tons/in.^{2} tension, and 4 tons/in.^{2} compression. (R.N.E.C., Greenwich, 1923.)

9. At a point in a material the three principal stresses acting in directions Ox, Oy, Oz, have the values 5, 0, − 3 tons/in.^{2} respectively. Determine the normal and shear stresses for a plane perpendicular to the xz plane inclined at 30° to the xy plane. (Intercoll. Exam., Cambridge, 1913.)

10. A cube of iron, the length of whose edge is 100″, is subjected to a uniform pressure of 10 tons/in.^{2} on two opposite faces ; the other faces are prevented by lateral pressure from extending more than 0·02″. Deter-

mine the pressure on these faces, and the maximum shearing stress in the block. $E = 30 \times 10^6$ lbs./in.^{2}, $m = 10/3$. (Mech. Sc. Trip., 1912.)

11. In designing a structure, if it be decided that the greatest longitudinal strain must not exceed that produced by a simple tensile stress of 6 tons/in.^{2}, show that the maximum permissible shearing stress is $6m/(m + 1)$ tons/in.^{2}

With this limitation in regard to strain, calculate the number of ft. lbs. of energy which can be stored per ton of material, (i) when it is subjected to a simple tensile stress, (ii) when it is subjected to a pure shearing stress. Take $E = 30 \times 10^6$ lbs./in.^{2}, $m = 10/3$, and the weight of the material 480 lbs./ft.^{3} (Mech. Sc. Trip., 1906.)

12. Normal forces, applied to the sides of a prism whose section is a square of 2″ side, alter its section to a rectangle of sides $2 \pm 0{\cdot}01''$. Write down the value of the direct strains and deduce the value of the greatest shear strain in the material. If $m = \frac{10}{3}$ and E is 13,500 tons/in.^{2}, what normal stresses would be required to produce the specified strains ? Intercoll. Exam., Cambridge, 1905.)

13. A uniform rod of circular section 1″ diameter is subjected to an axial pull of 4 tons. If the stress is uniformly distributed across a section, calculate the normal and tangential stress intensities on a section inclined at 60° to the axis of the rod. (R.N.E.C., Keyham, 1927.)

CHAPTER VI

FAILURE OF MATERIALS UNDER STEADY COMPOUND STRESSES

(For failure under alternating compound stresses see p. 541.)

78. Introductory.—We now enter upon the discussion of a subject about which there are several diverse opinions, and the importance of which does not seem to be recognized sufficiently by engineers in England and America. The matter in question is: what is the physical cause of the failure of materials under stress? Let us first be clear about the meaning we attach to the terms we shall employ in this discussion. First, with regard to the word "failure": it must be clearly understood that when we write "failure" we do not mean "rupture"; we mean a passing beyond the elastic limit, and we consider a material to have failed when it has taken up permanent set. The equations of mathematical elasticity, and therefore the deductions from them, remain true only within the limits of linear elasticity; that is to say, they cease to be true long before rupture occurs, at least in the case of the ductile materials used in engineering. Furthermore, it is known that, after the elastic limit has been passed, the material suffers a change of state in the physical sense, and that permanent set is the result of sliding movements within the individual grains of metal.* Consequently, no theories of the cause of rupture, which occurs after these changes have taken place, can be deduced from mathematical elasticity.

Next as regards the expression "cause of failure": when a bar of material is tested by pure tension we know that many phenomena are occurring at the same time; sections of the bar at right angles to the line of pull are subject to normal stress; sections oblique to the line of pull also experience shear stress which is a maximum when the planes are inclined 45° to the line of pull; the bar suffers strains in lateral and longitudinal directions, the latter being the larger; and a certain amount of energy has been put into the bar. When the elastic limit is reached all these quantities, the maximum normal stress, the maximum shear stress, the maximum strain, and the strain energy, have certain values. Which occurrence is the cause of the linear relation between stress and strain ceasing to hold? This is the question which we have to answer before we can apply the results of a simple tension test to determine when the elastic limit will be reached in a state of complex stress.

79. The Various Theories of Failure.—The following is a list of the various theories which have been advanced in reply to the question we are considering.

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* This theory was first put forward by Beilby and has been developed by Ewing and Rosenhain.

(i) **Maximum Principal Stress Theory.**—According to this theory the elastic limit is reached, in a state of complex stress distribution, when the maximum principal stress equals the elastic limit under simple tension. This idea was held by all the earlier writers on the subject, but is now usually associated with the name of Rankine.

(ii) Maximum Strain Theory.*—The criterion of failure is stated to be the maximum strain in the material. It is now usually known as St. Venant's theory.

(iii) **Maximum Shear Stress, or Stress-Difference Theory.**—According to J. J. Guest † and Tresca ‡ failure occurs on account of the maximum shear stress, and therefore depends on the difference between the greatest and least principal stresses reaching a certain critical value.

(iv) Mohr's Theory.§—Yield depends on both the difference of the principal stresses and on their sum. Priority is claimed by Guest,‖ who states his rules thus: (*a*) "the shear stress is the main factor in elastic failure, (*b*) the volumetric stress, $\frac{1}{2}(p_1 + p_3)$—the same as the normal stress on the plane of maximum shear—is to be taken into account, (*c*) for practical purposes the intermediate stress is without effect. . . ." (Here we assume $p_1 > p_2 > p_3$.)

(v) Becker's Theory.—In 1916 Dr. Albert Becker¶ published his theory that the elastic limit is fixed by both maximum shear stress and maximum strain.

(vi) Haigh's Theory.—The above theories are all the results of experiment and observation and are entirely empirical in nature, with little or no philosophical foundation. The first attempt to set the matter on a logical basis is due to Prof. Haigh.**

(vii) Misés-Hencky Theory††.—Yield occurs for a definite value of the strain-energy due to the three shear stresses on planes bisecting the angles between the principal planes.

It is outside our present scope to enter into details of the theory, and the following brief statements must suffice. Permanent strain in a material is associated with a change of physical structure, and in general a change of state requires that energy be supplied to or abstracted from the body. In producing the change of state associated with permanent strain, energy is supplied to the body in the form of work done by the forces causing the stresses. If the same change of state can be brought about by various combinations of stresses, the work done will be the same in each case provided the actions are reversible in a thermodynamic sense. That is to say, whether a material reaches a state of permanent strain by the application of a simple tensile stress or of combined stresses, the work done will be the same if the actions are reversible. This leads at once to the idea of a relation between the elastic limits under simple and complex stress systems. From these considerations Prof. Haigh arrives at his strain energy theorem, which may be stated as follows:

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* Todhunter and Pearson's *History of Elasticity*, Vol. II, pt. i, p. 107.

† *Phil. Mag.*, July, 1900.

‡ Inst. Mech. E., 1878, and *Comptes Rendus*.

§ *Zeits. v. Deutschen Eng.*, Nov., 1900.

‖ *Automobile Engineer*, Feb., 1938.

¶ Bulletin 85, Illinois University.

** B.A. Reports, 1919 and 1921.

†† *Göttingen Nachrichten*, 1913.

If a body is brought to the elastic limit of the material by stresses which are increased gradually from zero to their final values, the strain-energy per unit volume attains a nearly constant limiting value independent of the simple or complex nature of the applied stresses.

80. The Significance of these Theories.—This will be made more clear by considering a numerical case. Suppose we have a specimen of a certain steel which in simple tension shows an elastic limit of 25 tons/in.^{2}, and for which $m = 10/3$, $C = 11{\cdot}5 \times 10^6$ lbs./in.^{2}, and $E = 30 \times 10^6$ lbs./in.^{2}

In the tension test the maximum shear stress at the elastic limit is 12·5 tons/in.^{2} (see p. 73); and the strain energy per unit volume is (see p. 20)

$$\frac{p^2}{2E} = \frac{625 \times 2240^2}{2 \times 30 \times 10^6} = 52 \text{ lb. inch units.}$$

Consider first the case of a specimen of the same metal in a state of simple shear, and let q be the shear stress which will bring about the elastic limit. We shall apply four of the above theories:

(i) By § 60 the principal stresses are equal to q and $-q$, so that if this be the deciding factor we shall have $q = p = 25$ tons/in.^{2}

(ii) The maximum strain is (§ 29)

$$e_1 = \frac{1}{E}\left(1 + \frac{1}{m}\right)q = \frac{13}{10}\frac{q}{E}$$

In the simple tensile test the strain was

$$\frac{25 \times 2240 \text{ lbs./in.}^2}{30 \times 10^6 \text{ lbs./in.}^2} = 0{\cdot}00187.$$

Hence, if the maximum strain is the criterion, we must have

$$\frac{13}{10}\frac{q}{E} = 0{\cdot}00187$$

which gives

$$q = \frac{10 \times 30 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}00187}{13 \times 2240 \text{ lbs./tons}} = 19{\cdot}2 \text{ tons/in.}^2$$

(iii) If the shear stress has to be the same as in simple tension we shall have $q = 12{\cdot}5$ tons/in.^{2}

(iv) The strain energy per unit volume is $q^2/2C$, and, according to Haigh's theory, this must $= 52$ lbs. in. units. This gives

$$q = 15{\cdot}4 \text{ tons/in.}^2$$

As another instance, take the case of a plate of the same material under equal tensions in perpendicular directions. Let $p_x = p_y = p$ be the stress which will cause the material to reach the elastic limit.

(i) The principal stresses are p, p, 0, so that the greatest principal stress is p; hence the maximum-stress theory will give $p = 25$ tons/in.^{2}

(ii) The stress which will bring about the same maximum strain as in simple tension (see § 72) is

$$\bar{p} = \frac{m-1}{2m} 2p = \frac{7}{10}p.$$

If $\bar{p} = 25$ tons/in.^{2}, we shall have $p = 35{\cdot}7$ tons/in.^{2}

(iii) The difference between the greatest and least principal stresses is p, so that the greatest shear stress is equal to $\frac{1}{2}p$. Hence, according to Guest's theory we must have $p/2 = 12{\cdot}5$ tons/in.^{2}, or $p = 25$ tons/in.^{2}

(iv) The strain energy per unit volume is by putting $p_1 = p_2 = p$ in equation (26) of § 74.

$$\frac{p^2}{E}\left(1 - \frac{1}{m}\right)$$

With Haigh's theory this must equal 52 lbs. in. units, which gives

$$p = 21{\cdot}1 \text{ tons/in.}^2$$

Summing up, then, the results of applying the various theories, we have the following figures for the stresses which will cause the material to reach the elastic limit:

	Simple Shear.	Equal perpendicular Tensions.
Max. Stress Theory . .	25 tons/in.^{2}	25 tons/in.^{2}
„ Strain „ . .	19·2 „	35·7 „
„ Shear „ . .	12·5 „	25 „
Const. Strain Energy Theory .	15·4 „	21·1 „

81. Representation of the above Theories.—For purposes of comparing the various theories with each other and with the results of experiment, and for purposes of calculation, it is convenient to have a graphical means of expression.

Let p_1 and p_2 be the principal stresses in a two-dimensional stress system when the elastic limit is reached.

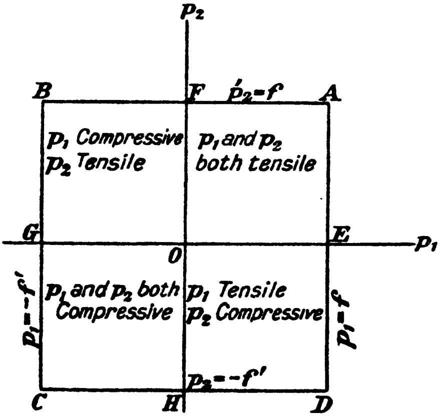

Fig. 84.—Representation of Maximum Stress Theory.

Take two axes at right angles (Fig. 84) along which we shall measure p_1 and p_2. Taking a tension as positive, if p_1 and p_2 are both tensile stresses, the point $(p_1 p_2)$ will be in the N.E. quadrant of the diagram; if they are both compressions $(p_1 p_2)$ will be in the S.W. quadrant; if one is tensile and the other compressive, $(p_1 p_2)$ will be in either the N.W. or S.E. quadrant.

(i) **Maximum Stress Theory.**—Let f and f' be the elastic limits of the material in simple tension and simple compression respectively. In the diagram draw the lines AB and AD, parallel to the co-ordinate axes, at distances from them equal to f; also draw the lines BC and CD parallel to, and at distances $-f'$ from, the axes. Then the material will reach its elastic limit when the point (p_1, p_2) passes outside the square $ABCD$. If $f' = f$ the centre of the square will be at the origin.

(ii) **Maximum Strain Theory.**—The strains arising from the stresses p_1 and p_2 are

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE}$$

and

$$e_2 = \frac{p_2}{E} - \frac{p_1}{mE}$$

The elastic limit will be reached when either of these is equal to f/E, i.e. when

$$p_1 - \frac{p_2}{m} = f$$

or

$$p_2 - \frac{p_1}{m} = f$$

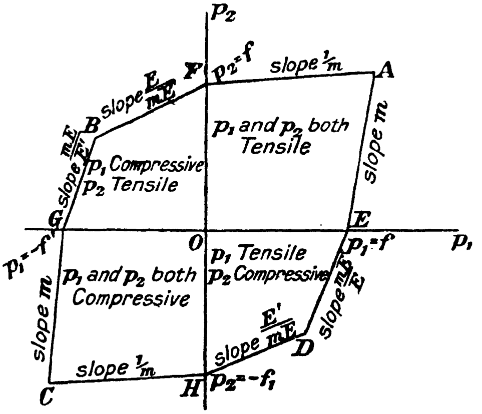
p_2
F
$p_2 = f$
slope $\frac{1}{m}$
A
slope $\frac{E}{mE'}$
B
slope $\frac{mE}{E'}$
p_1 and p_2 both Tensile
slope m
p_1 Compressive
p_2 Tensile
G
E
O
p_1
$p_1 = -f'$
$p_1 = f$
p_1 Tensile
p_2 Compressive
slope $\frac{mE}{E'}$
p_1 and p_2 both Compressive
slope m
E'
slope $\frac{E'}{mE}$
D
slope $\frac{1}{m}$
C
H
$p_2 = -f_1$

FIG. 85.—Representation of Maximum Strain Theory (f and E unequal in tension and compression).

If p_1 and p_2 are both positive (i.e. tensile) the boundary representing the elastic limit will be the lines EA and FA (Fig. 85) representing these equations.

If p_2 be negative (i.e. compressive) let f' and E' be the values of f and E in compression. The elastic limit will be reached when

$$\frac{p_1}{E}-\frac{p_2}{mE'}=\frac{f}{E}\quad\text{or}\quad\frac{p_2}{E'}-\frac{p_1}{mE}=-\frac{f'}{E'}.$$

That is when $p_1-\frac{E}{E'}\frac{p_2}{m}=f$ or $p_2-\frac{E'}{E}\frac{p_1}{m}=-f'$.

The boundary is now given by the lines ED and HD which represent these equations.

Similarly the boundaries in the other quadrants are as shown.

If, as in the case of most metals, $E'=E$ and $f'=f$, the figure $AEDHCGBFA$ becomes a parallelogram, as shown in Fig. 86, symmetrically placed with regard to the axes.

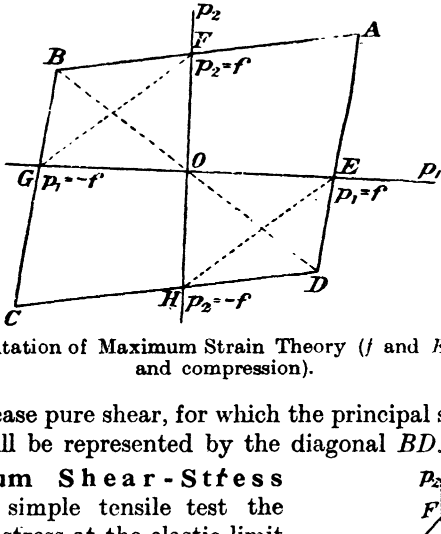

FIG. 86.—Representation of Maximum Strain Theory (f and E equal for tension and compression).

In the latter case pure shear, for which the principal stresses are equal and opposite, will be represented by the diagonal BD.

(iii) **Maximum Shear-Stress Theory.**—In a simple tensile test the maximum shear stress at the elastic limit is $\frac{1}{2}f$. In a general two-dimensional system, when the principal stresses are p_1, p_2 and 0, the greatest shearing stress is $\frac{1}{2}(p_1 \sim p_2)$ if p_1 and p_2 are of opposite signs; if p_1 and p_2 have the same sign the greatest shearing stress is $\frac{1}{2}p_1$ or $\frac{1}{2}p_2$, according to which is the greater. In the former case, then, the elastic limit will be passed when $p_1-p_2=f$ or $p_2-p_1=f$, i.e. when the point (p_1, p_2) lies outside the space between the parallel lines FG and EH in Fig. 87. In the second case, when p_1 and p_2 have the same sign, the point (p_1, p_2) will be outside the spaces $FAEO$ or $OGCH$ when the material has passed the elastic limit. Hence, in general, (p_1, p_2) must lie within the figure $GFAEHCG$. Thus it will be seen that, for like stresses, this theory gives the same boundary as the maximum principal stress theory.

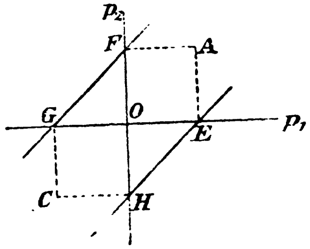

FIG. 87.—Representation of Maximum Shear-Stress Theory (f the same for tension and compression).

(iv) **Mohr's Theory.**—Guest gives as the algebraic expression of this law

$$p_1 - p_2 + \lambda(p_1 + p_2) = \text{constant},$$

λ depending on the material. This does not readily lend itself to representation in the above scheme owing to the absence of a value for λ. The same remarks apply to Becker's theory.

(v) **Haigh's Strain Energy Theory.**—The strain energy per unit volume in a general stress system is (§ 74)

$$\frac{1}{2E}\left\{p_1^2 + p_2^2 + p_3^2 - \frac{2}{m}(p_1p_2 + p_2p_3 + p_3p_1)\right\}$$

and the strain energy at the elastic limit in pure tension is $f^2/2E$. Hence we must have, in the two-dimensional case,

$$p_1^2 + p_2^2 - \frac{2}{m}p_1p_2 = f^2$$

This is the equation of an ellipse whose centre is at the origin, with axes inclined at 45° to the co-ordinate axes, as shown in Fig. 88. For an isotropic material, having the same elastic limit and modulus of elasticity for tension and compression, this ellipse is the boundary giving the elastic limit for any condition of stress.

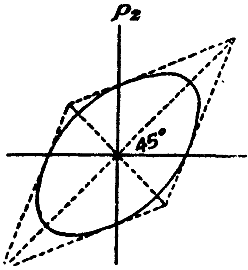

FIG. 88.—Representation of Maximum Strain-Energy Theory.

The ellipse is inscribed in the parallelogram given by the maximum strain theory, touching it at its points of intersection with the coordinate axes as shown by the dotted lines.

(vi) **Misés-Hencky Theory.**—In a three-dimensional system, this is expressed by

$$\tfrac{1}{2}(p_1 - p_2)^2 + \tfrac{1}{2}(p_2 - p_3)^2 + \tfrac{1}{2}(p_3 - p_1)^2 = f^2,$$

which, when $p_3 = 0$, reduces to

$$p_1^2 + p_2^2 - p_1p_2 = f^2.$$

This is an ellipse similar to the Haigh ellipse; they are identical if $m = 2$, the smallest possible value of m.

(vii) **Sunatani's Theory** (Technology Reports, Tôhoku University, Vol. III. i).—Sunatani claims that his theory, stated below, has been shown to agree with experiment for brittle and ductile materials, metals and non-metals, but it does not appear to have received much attention in this country. Briefly his theory is as follows:

If $p_1 > p_2 > p_3$, failure is by shear or tension according as

$$\tfrac{1}{2}\left\{p_1 - p_3 + (p_1 + p_3)\sin\phi\right\}\sec\phi > q_o$$

$$\text{or } \tfrac{1}{2}\left\{p_1 - p_3 + (p_1 + p_3)\cos\phi\right\}\sec\phi > f_o.$$

Sliding failure occurs: in simple tension, along a plane whose normal makes an angle $(\pi/4 - \phi/2)$ with the direction of the load; in simple compression the angle is $(\pi/4 + \phi/2)$; in simple torsion along helical surfaces inclined at $\phi/2$ and $(\pi/2 - \phi/2)$ to the torsion axis.

Tensile failure occurs: in simple tension, across a plane whose normal makes an angle $\phi/2$ with the direction of the load; in simple torsion across helical surfaces inclined at $(\pi/4 + \phi/2)$ and $(\pi/4 - \phi/2)$ to the torsion axis.

In the above formulæ

q_0 = shearing stress on plane on which there is no normal stress.
f_0 = tensile „ „ „ „ shear „

For brittle materials, $q_0 = \dfrac{1 - \sin\phi}{2\cos\phi} f_c$, and $f_0 = \dfrac{1 + \cos\phi}{2\cos\phi} f_t$

where f_c and f_t denote the axial stress in simple compression and tension respectively, at point of failure.

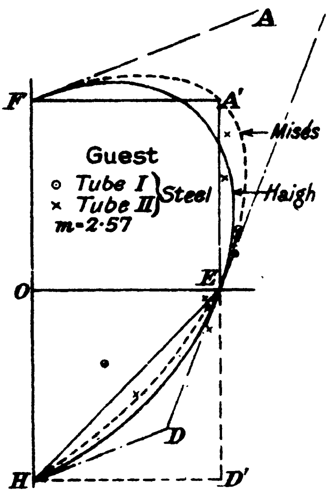

Fig. 89.

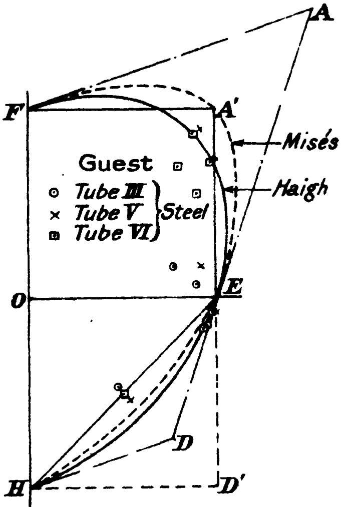

Fig. 90.

For ductile materials, $q_0 = \dfrac{1 + \sin\phi}{2\cos\phi} f_t$; f_0 cannot easily be found and is not wanted.

The angle ϕ is approximately 20° in all cases, for all materials.

82. Analysis of Experiments.—In recent years several investigators have experimented on the failure of materials under combined stresses, but from our present point of view none of these researches is completely satisfactory: often because of the uncertainty of the value of m, and sometimes because the yield point has been taken instead of the Elastic Limit. However, for the present, we must take the results as they stand and glean such facts as we can from them.

The first important experiments on the subject were published[*] by J. J. Guest in 1900, who conducted his researches with thin-walled tubes.

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[*] *Phil. Mag.*, 1900, ii.

The results of many of his experiments are represented in Figs. 89–92 in the manner described above, the principal stresses being plotted. Fig. 93 shows the results of some experiments by Hancock* and Turner.†

Experiments‡ by Lodé on thin tubes of nickel, iron, and copper under simple tension, internal pressure, and both combined, showed that the intermediate principal stress is not negligible, and agreed within 3 per cent. with the figures calculated by the Misés-Hencky theory, which receives further confirmation from experiments by Taylor and Quinney.§

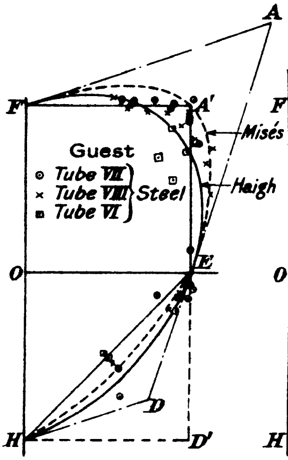
A
F
A′
Guest
⊙ Tube VII
× Tube VIII Steel
▪ Tube VI
Misés
Haigh
O
E
H
D
D′

FIG. 91.

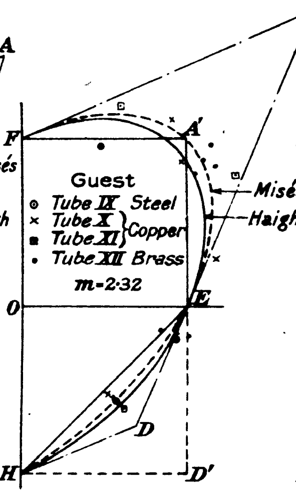
A
F
A′
Guest
⊙ Tube IX Steel
× Tube X
▪ Tube XI Copper
• Tube XII Brass
m=2·32
Misés
Haigh
O
E
H
D
D′

FIG. 92.

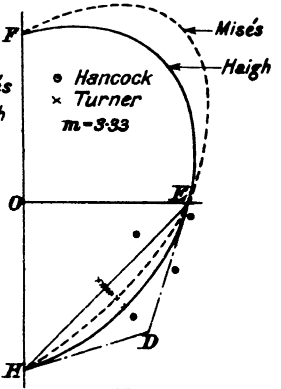
F
Misés
Haigh
• Hancock
× Turner
m=3·33
O
E
H
D

FIG. 93.

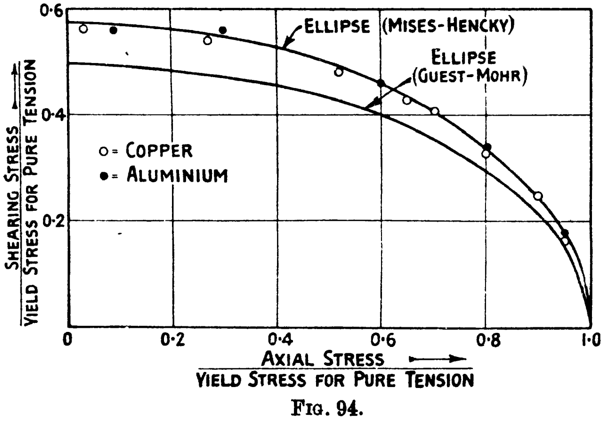

FIG. 94.

In these experiments thin tubes were subjected to tension combined with torsion, and the results are compared with theory in Fig. 94, which requires some explanation.

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* *Phil. Mag.*, 1906, i. † *Engineering*, February, 1909.

‡ *Zeit. Physik.*, Vol. 36, 1926. We quote here from a paper by Gough and Clenshaw, Inst. Marine Engineers, Nov. 1935. § *Phil. Trans. Roy. Soc., A.*, Nov., 1931.

Let p_x denote the tensile stress due to direct tension, and q the shear stress due to torsion. Then the principal stresses are $\frac{1}{2} p_x \pm \sqrt{\frac{1}{4} p_x^2 + q^2}$ and zero. Using these values, the relation between p_x and q at yield is given by (i) $p_x^2 + 4q^2 = f^2$ for the Guest theory, (ii) $p_x^2 + 3q^2 = f^2$ for the Misés-Hencky theory, and $p_x^2 + 8q^2/3 = f^2$ for the Haigh theory, taking $m = 3$. These all represent ellipses, which are plotted in Fig. 94.

The following figures* are selected as the most important of those obtained by Cook with steel. The table shows the percentage difference between the actual failing load and that calculated by the three most important theories.

Theory of Failure.	Type of Loading.	Steel A. 0·13% Carbon.	Steel B. 0·13% Carbon.	Steel C. 0·21% Carbon.	Mean.
Maximum shear stress	Torsion . . .	19	10	10	13
	Flexure . . .	16	11	12	13
	Internal pressure	15	3	6	8
Total strain energy (Haigh)	Torsion . . .	– 4	– 12	– 12	– 9
	Flexure . . .	16	11	12	13
	Internal pressure	– 6	– 17	– 16	– 13
Shear strain energy (Misés)	Torsion . . .	1	– 5	– 4	– 3
	Flexure . . .	16	11	12	13
	Internal pressure	0	– 11	– 8	– 6

In flexure there is only one principal stress, so that all the theories give the same result, and the large error suggests that the distribution of stress is an important factor. For torsion and internal pressure the balance of evidence here is certainly against the maximum shear stress theory and favours Misés-Hencky rather than Haigh.

Additional evidence is furnished by the experiments of Cook and Robertson† on thick-walled tubes subjected to internal pressure. It is shown below, in Chapter XXVIII, that if P is the limiting internal pressure, f the elastic limit of the material in simple tension, and k the ratio of the external to the internal diameters, P/f should be given by the graphs which have been plotted in Fig. 95, together with the experimental points, *I* denoting the maximum stress theory, *II* the shear theory, and *III* the total strain-energy theory.

The general conclusions to be drawn from these diagrams with regard to *ductile metals* may be summarized thus: (i) The maximum strain theory is useless. (ii) The maximum shear stress theory is a good approximation in two dimensional stress systems. (iii) The Haigh and Misés theories are better. For brittle materials the small amount of available evidence indicates that the maximum principal stress theory or Sunatani's theory should be used. (See also Fig. 357.)

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* *Phil. Trans. Roy. Soc., A.,* Vol. ccxxx, and B.A. Report on Stresses in Overstrained Materials, 1931.

† *Engineering*, Dec. 15, 1911.

Thus, at present, this most important problem remains unsolved. In our search for a solution I think it is important to bear in mind the following points: (i) The fracture of a brittle material and the yield of a ductile material are qualitatively different, and therefore the transition from one to the other will be accompanied by a change in the quantitative law governing the events. (ii) That a mild steel tension specimen breaks with a 45° cup and cone is sometimes adduced as an argument in favour of the maximum shear stress theory; the legitimacy of this is doubtful since the latter refers to the change of phase at the end of the first part of the experiment and the former to the change of phase at the end of the experiment. (iii) It is not necessary that the same law will be found to govern the case of like and unlike principal stresses: if p_1 be kept tensile whilst p_2 be changed in a series of experiments from tensile to compressive, this qualitative change may be accompanied by a change in the quantitative law. Similarly there may, without inconsistency, be a discontinuity in the formula when a third principal stress is introduced. (iv) Reference has been made above to the effects of stress distribution, and Scoble has suggested* a possible reinforcement of the outer layers. That there is a surface effect is by no means impossible. It must always be borne in mind that yield is a change of phase, in the sense that the transition from streamline fluid motion to turbulent motion, or the boiling of a liquid, represents a change of phase. In both cases the passage of the liquid through the critical point depends on the conditions obtaining at the surface of separation between two media. Thus there is nothing improbable in the notion that the yield point should be related to the stress gradient at the boundary, just as the initiation of turbulent fluid motion is related to the pressure gradient at the boundary.

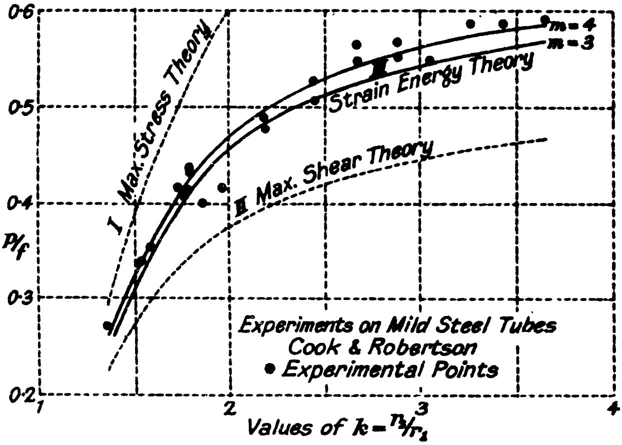

FIG. 95.

r_2 = outside radius, r_1 = inside radius.

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* *Engineering*, Vol. cxxiii, p. 65.

CHAPTER VII

THIN CYLINDRICAL AND SPHERICAL SHELLS UNDER INTERNAL PRESSURE

83. Introductory.—Easy examples of compound stresses are provided by thin cylindrical shells of circular section, or spherical shells, under internal fluid pressure. In such cases the pressure is uniform over the internal surface of the vessel, if we disregard the weight of the fluid. In general, the shell will experience two principal stresses in directions parallel to the tangent plane at any point, and these stresses will vary in their intensity from the inside to the outside of the shell. When the thickness of the shell is small compared with the diameter, for instance in the case of boilers, we may neglect this variation without sensible error. The shell will also suffer a third stress in the direction of the normal to the surface, which, however, is usually very small compared with either of the other principal stresses. When these conditions are not fulfilled the problem is more difficult and is postponed to a later chapter.

84. Thin Cylindrical Shell of Circular Section.—

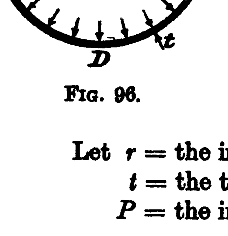

FIG. 96.

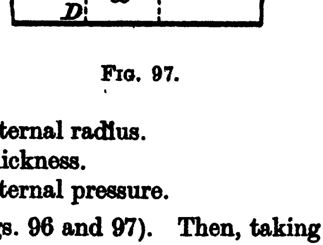

FIG. 97.

Let r = the internal radius.
t = the thickness.
P = the internal pressure.

Consider a length x of the shell (Figs. 96 and 97). Then, taking the two halves of this portion on each side of a diametral plane AB, the force tending to separate them is $2rxP$.

Let p_1 be the circumferential stress in the tube, across the diameter

AB. This stress is usually called the "hoop" stress. The resistance which the two halves *ACB* and *ADB* offer to separation is then $2p_1xt$.

Hence we must have

$$2p_1xt = 2rxP$$

$$\therefore \mathbf{p_1} = \frac{\mathbf{Pr}}{\mathbf{t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Again, unless the ends of the shell are stayed by longitudinal stays, the pressure on the ends will produce a tendency to tear across planes such as *ACBD*. Whatever be the shape of the ends, plane or dished, the force tending to produce rupture across the planes *ACBD* is $P \times \pi r^2$.

Let p_2 be the longitudinal stress in the shell, then the resistance to this type of fracture is $2\pi rt \times p_2$. Therefore we must have

$$2\pi rtp_2 = \pi r^2 P$$

$$\therefore \mathbf{p_2} = \frac{\mathbf{Pr}}{\mathbf{2t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

We thus see that the longitudinal stress is one-half of the hoop stress.

There is also a radial pressure which varies from *P* inside the shell to atmospheric pressure outside, but this is usually negligible. Formulæ (1) and (2) give the two major principal stresses, except very close to the ends, where the stresses depend on the nature of the ends : for instance, if these are flat circular plates the radial expansion of the shell is prevented and bending stresses are introduced.

Let *f* denote the yield point of the material, then (i) according to the maximum stress theory the permissible pressure is given by

$$p_1 = f$$

or

$$P = \frac{t}{r}f.$$

(ii) according to the strain-energy theory, we must have at yield

$$p_1^2 + p_2^2 - \frac{2}{m}p_1p_2 = f^2$$

or

$$\frac{P^2r^2}{t^2}\left(1 + \tfrac{1}{4} - \frac{2}{m} . 1 . \tfrac{1}{2}\right) = f^2$$

which gives

$$P = \frac{t}{r}f . \sqrt{\frac{4m}{5m - 4}}$$

If $m = \frac{10}{3}$, this gives $P = 1{\cdot}025\frac{tf}{r}$, or about $2\frac{1}{2}$ per cent. greater than the value given by the maximum stress theory.

85. Thin Spherical Shell under Internal Pressure.—From symmetry it is clear that the tensile stress across all diametral planes is the same.

Let r = the internal radius of the sphere.
t = the thickness.
P = the internal fluid pressure.
p = the tensile stress across any diametral section.

Then the total force tending to separate two halves of the sphere $= \pi r^2 P$. The resistance to fracture is $p \times 2\pi rt$. Hence we must have

$$p.2\pi rt = \pi r^2 P$$

$$\therefore \mathrm{p} = \frac{\mathrm{Pr}}{2\mathrm{t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At any point in the shell the principal stresses are $p_1 = p_2 = Pr/2t$, and the negligible radial stress. If $f =$ the yield point of the material, the internal pressure required to produce yield will be given by

$$p = f$$

or

$$P = \frac{2t}{r}f.$$

The maximum strain-energy theory requires

$$\frac{p^2 r^2}{4t^2}\left(1 + 1 - \frac{2}{m}\right) = f^2$$

or

$$P = \frac{2t}{r}.f\sqrt{\frac{m}{2(m-1)}}$$

If $m = \frac{10}{3}$ this makes $P = 0{\cdot}85\left(\frac{2t}{r}f\right)$, or about 15 per cent. less than the maximum stress theory.

86. Thin Cylindrical Shell with Hemispherical Ends.—Unless the thicknesses of the ends and sides are so proportioned that the radial expansion is naturally the same for both, bending stresses will be set up near the junction of the ends with the body of the shell.

Let $r =$ the internal radius of the cylinder, and of the hemispheres forming the ends.

$t_1 =$ the thickness of the cylindrical part.

$t_2 =$ the thickness of the ends.

$P =$ the internal fluid pressure.

Then the principal stresses in the cylinder are

$$\frac{Pr}{t_1} \text{ and } \frac{Pr}{2t_1},$$

and the circumferential strain, if the walls are free from restraint, will be

$$\frac{Pr}{Et_1} - \frac{Pr}{2mEt_1} = \frac{Pr}{Et_1}\left(\frac{2m-1}{2m}\right) \quad . \quad . \quad . \quad \text{(i)}$$

The principal stresses in the ends, if these are free from restraint, will both be $\frac{Pr}{2t_2}$, and the circumferential strain will be

$$\frac{Pr}{2Et_2} - \frac{Pr}{2mEt_2} = \frac{Pr}{Et_2}\left(\frac{m-1}{2m}\right) \quad . \quad . \quad . \quad \text{(ii)}$$

It will be seen from (i) and (ii) that if $t_1 = t_2$ the expansion of the spherical ends and the cylindrical shell will tend to differ, so that if they are forcibly made the same by a joint extra stresses must be incurred. To avoid these extra stresses the natural circumferential strains may be made equal. This requires

$$\frac{2m-1}{2mt_1} = \frac{m-1}{2mt_2}$$

or

$$\frac{t_2}{t_1} = \frac{m-1}{2m-1}$$

If $m = \frac{10}{3}$, this will make $\frac{t_2}{t_1} = \frac{7}{17}$.

But the weakest part of the shell will now be the ends: taking $m = \frac{10}{3}$, and adopting the strain-energy theory as the criterion, we shall have for the pressure required to produce yield in the sides (§ 84) $P = 1{\cdot}025\frac{t_1 f}{r}$; the pressure required to produce yield in the ends will be (§ 85) $P = 0{\cdot}85\frac{2t_2 f}{r} = 0{\cdot}7\frac{t_1 f}{r}$, which is about 30 per cent. less than that required to produce yield in the sides.

Example 1.—The air vessel of a torpedo is $17\frac{3}{4}''$ external diameter and $\frac{3}{8}''$ thick, the length being 5 ft. Find the external diameter and length when charged to 1,500 lbs./in.^{2} (Mech. Sc. Trip., 1916.)

We shall take $E = 30 \times 10^6$ lbs./in.^{2}, and $m = \frac{10}{3}$.

r = the internal radius = $8{\cdot}5''$.
t = the thickness = $0{\cdot}375''$.
P = the internal pressure = 1,500 lbs./in.^{2}

The hoop stress is

$$p_1 = \frac{Pr}{t} = \frac{1{,}500 \times 8{\cdot}5}{0{\cdot}375} = 34{,}000 \text{ lbs./in.}^2$$

The longitudinal stress $= p_2 = 17{,}000$ lbs./in.^{2}
Neglect the radial stress.
The circumferential strain is

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} = \frac{p_1}{E}\left(1 - \frac{1}{2m}\right) = \frac{34{,}000}{30 \times 10^6} \times \frac{17}{20}$$
$$= 0{\cdot}963 \times 10^{-3}$$

The longitudinal strain is

$$e_2 = \frac{p_2}{E} - \frac{p_1}{mE} = \frac{p_1}{E}\left(\tfrac{1}{2} - \frac{1}{m}\right) = \frac{4}{17}e_1 = 0{\cdot}226 \times 10^{-3}.$$

The increase in external diameter

$$= 17{\cdot}75'' \times 0{\cdot}963 \times 10^{-3} = 0{\cdot}0171''.$$

The increase in length

$$= 60'' \times 0{\cdot}226 \times 10^{-3} = 0{\cdot}0136''.$$

Example 2.—A thin cylindrical shell is subjected to internal fluid pressure, the ends being closed (*a*) by two watertight pistons attached to a common piston rod, (*b*) by flanged ends. Find the increase in internal diameter in the two cases, having given: Internal diameter = $8''$; pressure = 500 lbs./in.^{2}; Poisson's ratio = $1/3{\cdot}5$; thickness = $0{\cdot}2''$; $E = 30 \times 10^6$ lbs./in.^{2} (R.N.C., Greenwich, 1922.)

We have $r = 4''; \ P = 500 \text{ lbs./in.}^2; \ t = 0{\cdot}2''.$

In both cases the hoop stress is $p_1 = \frac{Pr}{t} = \frac{500 \times 4}{0{\cdot}2} = 10{,}000 \text{ lbs./in.}^2$

(*a*) There is no longitudinal stress. The hoop strain

$$= \frac{p_1}{E} = \frac{10{,}000}{30 \times 10^6} = \frac{10^{-3}}{3}.$$

The increase of internal diameter is $8 \times \frac{10^{-3}}{3} = 0{\cdot}00267''$.

(*b*) The longitudinal stress is 5,000 lbs./in.²

The hoop strain $= \frac{p_1}{E}\left(1 - \frac{1}{2m}\right) = \frac{6}{7}\frac{p_1}{E}.$

The increase of internal diameter $= \frac{6}{7} \times 0{\cdot}00267 = 0{\cdot}00229''$.

87. Thin Tube under External Pressure.—When a thin circular tube is subjected to external pressure it collapses with longitudinal corrugations when the pressure exceeds a certain value. The problem of finding the pressure required to bring about this collapse depends for its solution upon what is known in mathematical elasticity as the "General Theory of Thin Shells." This theory is elaborate and difficult, and so far has not yielded any other results of great practical importance, whilst the more approximate theory of cylindrical shells, which can be developed on the lines of Chapter XXXI, cannot be made to yield the correct answer. For these reasons we have decided to omit the theory here, and, for convenience of reference in practical cases, give only the results, which are due to R. V. Southwell.*

The formula for the collapsing pressure is

$$p = \frac{Et}{r}\left[(n^2 - 1)\frac{m^2}{12(m^2 - 1)}\left(\frac{t}{r}\right)^2 + \frac{\pi^4}{n^4(n^2 - 1)}\left(\frac{r}{l}\right)^4\right] \quad . \quad (4)$$

where

$r =$ the mean radius of the tube,
$t =$ the thickness of the tube,
$1/m =$ Poisson's ratio,
$l =$ the free length of tube,
$n =$ the number of corrugations.

and the ends of the tube are compelled to retain their circularity, but are not otherwise restrained.

When the tube is very long compared with the diameter, we get

$$p = \frac{m^2Et^3}{12(m^2 - 1)r^3}(n^2 - 1);$$

the smallest permissible value of n is 2, giving

$$p = \frac{m^2Et^3}{4(m^2 - 1)r^3}.$$

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* *Phil. Trans. Royal. Soc.* (*Ser. A*), 1913; *Phil. Mag.*, 1913, Vol. 25. See also Love's *Theory of Elasticity*, 3rd Ed. For experimental investigation see article by Prof. G. Cook. *Phil. Mag.*, Oct., 1925; the results show good agreement with the theory, so far as collapsing load is concerned, but not so good for the number of corrugations formed.

In this case the section takes an elliptic form. When the tube is short, $n = 3$ may produce collapse at a lower pressure than $n = 2$, the section becoming a three-lobed curve of the form $r = a + b \cos 3\theta$ in polar co-ordinates. For a shorter length still the collapsing pressure is given by taking $n = 4$, and so on. Thus, in the case of steel, taking $E = 30 \times 10^6$ lbs./in.^{2}, $t/r = 1/50$, and $m = \frac{10}{3}$, the collapsing pressure is given by the discontinuous curve shown in Fig. 98; for a ratio of l/r corresponding with A the tube might collapse with either two or three lobes; at B it might collapse with either three or four lobes, and so on.

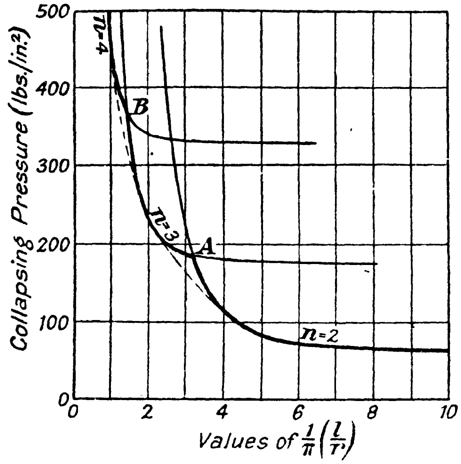

FIG. 98.

EXAMPLES VII

1. A pipe 4″ inside diameter and $\frac{3}{16}$″ thick is made of material whose ultimate strength in tension is 28 tons/in.^{2} What is the maximum allowable inside pressure * if the factor of safety be 4? Assume a uniform distribution of stress over the cross section. (Special Exam., Cambridge, 1907.)

2. A long boiler tube has to stand an internal test pressure of 500 lbs./in.^{2}, without the mean hoop stress exceeding 8 tons/in.^{2} The internal diameter of the tube is 2″, and the weight 490 lbs./ft.^{3} Find the weight per foot run. (R.N.E.C., Keyham, 1918.)

3. A tube 3″ internal diameter is $\frac{3}{16}$″ thick in the wall. What internal pressure can be applied before the stress in the metal reaches 6 tons/in.^{2}? (R.N.E.C., Keyham, 1920.)

4. Two boiler plates $\frac{3}{8}$″ thick are connected by a longitudinal double-riveted butt joint with two cover plates $\frac{1}{4}$″ thick. The rivets used are $\frac{3}{4}$″ diameter and their pitch is $3\frac{1}{4}$″. The boiler is 4′ internal diameter and has an internal pressure of 120 lbs./in.^{2} Find the shearing stress in the rivets and the tensile stresses in the boiler plates and the cover plates. (Special Exam., Cambridge, 1911.)

5. A steam boiler 6′ 0″ diameter is constructed of steel having an ultimate

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* Estimate on the maximum stress theory.

strength of 27 tons/in.^{2} The shell is $\frac{5}{8}''$ thick, and the steam pressure 200 lbs./in.^{2} If the joints have an efficiency of 85 per cent., find the factor of safety. (Special Exam., Cambridge, 1914.)

6. A long straight cylinder $6''$ bore and $\frac{1}{8}''$ thick, is made of a steel which is found to yield under a simple tensile stress of 45,000 lbs./in.^{2} Water pressure is applied inside the tube until it yields. Calculate what the pressure will be in lbs./in.^{2} if (i) shear stress, (ii) strain is the circumstance which decides the yield. Take $m = \dfrac{10}{3}$ and $E = 30 \times 10^6$ lbs./in.^{2} (Intercoll. Exam., Cambridge, 1908.)

7. In Question 6 what will be the pressure if (i) stress, (ii) strain energy is the deciding circumstance ?

8. A long steel tube, $3''$ internal diameter and $\frac{1}{16}''$ thick, is plugged at the ends and subjected to internal fluid pressure such that the maximum stress in the tube is 8 tons/in.^{2} Assuming $m = 10/3$, and $E = 12{,}000$ tons/in.^{2}, find the percentage increase in the capacity of the tube. (R.N.C., Greenwich, 1921.)

9. A cylinder of concrete $10''$ diameter is surrounded by a steel tube $\frac{1}{2}''$ thick. The concrete is subjected to a longitudinal thrust of 500 lbs./in.^{2} By equating the change in diameter of the cylinder to the change in diameter of the tube, show that the normal pressure exerted on the concrete by the tube is approximately 110 lbs./in.^{2} E for concrete $= 2 \times 10^6$ lbs./in.^{2}; E for steel $= 30 \times 10^6$ lbs./in.^{2}; $m = \dfrac{10}{3}$ for both.* (Intercoll. Exam., Cambridge, 1912.)

10. A copper pipe $6''$ internal diameter and $\frac{1}{8}''$ thick is closely wound with a single layer of steel wire of diameter $0{\cdot}072''$, the initial tension of the wire being 2·5 lbs. If the pipe is subjected to an internal pressure of 400 lbs./in.^{2} find the stress in the copper and in the wire (*a*) when the temperature is the same as when the tube was wound, (*b*) when the temperature throughout is raised 200° C. E for steel $= 30 \times 10^6$ lbs./in.^{2}, E for copper $= 15 \times 10^6$ lbs./in.^{2}, coefficient of linear expansion for steel $= 11 \times 10^{-6}$, for copper 18×10^{-6} per 1° C. (Mech. Sc. Trip., 1912.)

11. A cylindrical pipe $2'\ 0''$ diameter and $25'\ 0''$ long is made of steel $\frac{1}{2}''$ thick. The ends are closed by plates which are bolted to flanges on the cylindrical shell, and are also tied together by longitudinal steel rods. The rods are tightened up till they begin to yield, which happens when the total tension in them is 60,000 lbs. Show that if the pipe be then subjected to hydraulic pressure of 500 lbs./in.^{2}, it will stretch longitudinally by about $0{\cdot}026''$. Take $m = 3{\cdot}5$, and $E = 30 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip., 1914.)

12. A thin spherical copper shell of internal diameter $12''$ and thickness $\frac{1}{16}''$ is just full of water at atmospheric pressure. Find how much the internal pressure will be increased if $1\frac{1}{2}$ cubic ins. of water are pumped in. Take $m = 3{\cdot}5$ for copper and $K = 46 \times 10^6$ lbs./ft.^{2} for water. (Intercoll. Exam., Cambridge, 1923.)

13. A spherical shell of $24''$ diameter is made of steel $\frac{1}{4}''$ thick. It is closed when just full of water at 15° C., and the temperature is then raised to 35° C. For this range of temperature, water at atmospheric pressure increases 0·0059 per unit volume. Find the stress induced in the steel. The modulus of cubic compressibility of water is 46×10^6 lbs./ft.^{2}, E for steel is 30×10^6 lbs./in.^{2}, and the coefficient of linear expansion of steel is 12×10^{-6} per 1° C., and Poisson's ratio $= 0{\cdot}3$. (Intercoll. Exam., Cambridge, 1921.)

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* It can be shown that, when a solid circular cylinder is acted on by uniform radial pressure, the hoop stress and radial are constant and equal to the applied pressure.

CHAPTER VIII

THE TORSION OF CIRCULAR SHAFTS

88. Introductory.—We shall now consider the stress in a uniform straight rod of circular cross section which is acted on by terminal couples in planes at right angles to the axis of the rod. The method of investigation which we shall follow is that of assuming that the rod is strained in a particular manner, and deducing the force system which must be applied to produce this strain. We shall assume (i) that the torsion is uniform along the rod, i.e. that all normal cross-sections which are the same distance apart suffer equal relative rotation, (ii) that the cross sections remain plane, and (iii) that radii remain straight; from this we shall deduce that the external force system which is necessary to produce this state of strain consists only of equal and opposite couples applied to the ends of the rod in planes perpendicular to the axis. Conversely, if these couples are applied to the rod the strain will be that assumed at the beginning. This is more satisfactory than trying to find the strain produced by applying given couples, for the process must involve either lengthy mathematical analysis or making certain deductions from symmetry * which are untrue when applied to shafts of square (e.g.) section, although there is no obvious reason why these deductions from symmetry should be true for circular sections and untrue for square sections. As usual we assume that the material is homogeneous and isotropic.

89. Relations between Twisting Moment, Twist and Shear Stress.—Let l = the length of a circular rod AB (Figs. 99 and 100). Let the rod receive uniform twist so that the end B is rotated through an angle θ radians relative to the end A, which we can suppose fixed. Now imagine the shaft made up of a number of concentric tubes, and let AB be a line, on one of these tubes, which is parallel to the axis of the rod before strain. After strain the line AB becomes a helix AB' making a small angle φ with AB.

Then, since θ and φ are both small, we have

$$l\varphi = BB' = r\theta$$

$$\therefore \varphi = \frac{r}{l}\theta$$

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* Cf. *Dictionary of Applied Physics*. The argument in § 89 is practically that of Cauchy and is given in Searle's *Experimental Elasticity* and Kelvin and Tait's *National Philosophy*, Part ii.

Now consider a portion of the tube AB of length x, as shown on the right in Fig. 100. Let δr be the thickness of the tube. When the rod is twisted, parallel lines, such as PQ and RS, will be strained into the positions PQ' and SR', making angles φ with their original positions. In other words, the element $PQRS$ undergoes a shear φ in the plane $PQRS$; the thickness δr, the lengths of the sides PS and QR, and the length x all remain constant. There is no distortion in radial planes or in planes perpendicular to the axis, so that the shear φ is the total strain undergone by the element.

Such a strain requires for its production only shearing stresses $q = C\varphi$, acting on the ends SP and QR in planes perpendicular to the axis, together with the equal complementary shear stress along QP and SR in radial planes. No other stresses are necessary.*

The stresses along QP and SR are provided by the action of the similar elements contiguous with QP and SR, so that only the stresses along SP and QR have to be provided by the action of neighbouring normal slices of the tube. The total force acting along SP and QR is $q.SP.\delta r$, and the moment of this about the axis is $rq.SP.\delta r$. For the whole cross section of the tube the moment will be $rq.2\pi r.\delta r = 2\pi r^2 q\delta r$. A couple of this moment is the whole action which has to be transmitted over normal sections of the tube to produce the assumed strain.

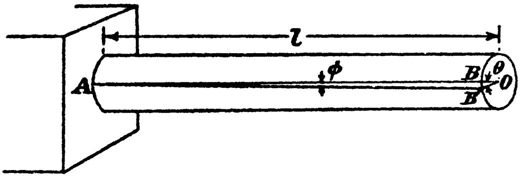

FIG. 99.

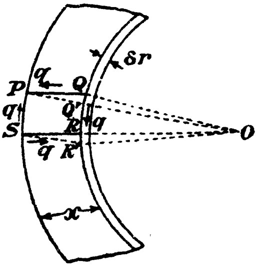

FIG. 100.

Since we have seen that no other stresses are involved, i.e. the assumed strain does not involve any action across the cylindrical boundaries of the tube, we can add together the couples for all the tubes comprising the rod.

Let T = the total twisting couple transmitted along the rod.
r_1 = the outside radius of a cross section.
r_2 = the inside „ „ „

Then we have $$T = \int_{r_2}^{r_1} 2\,\pi r^2 q dr.$$

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* The implication is made that $PQ' = PQ$, and that in consequence the planes through PS and QR, perpendicular to the axis, draw nearer together, so that the shaft is shortened. If this shortening be prevented, tensile stresses parallel to the axis will be induced, which in non-circular shafts may become important. (See *Phil. Mag.*, 1914, p. 778; and *Strength of Materials* (vol. i), Timoshenko.)

But we have shown that

$$q = C\varphi = Cr\theta/l, \text{ hence}$$

$$T = \int_{r_2}^{r_1} \frac{2\pi C\theta}{l} r^3 dr$$

$$= \frac{C\theta}{l} \cdot \frac{\pi(r_1^4 - r_2^4)}{2}$$

The quantity $\pi(r_1^4 - r_2^4)/2$ is the polar moment of inertia of the cross section, and is usually denoted by J. We can thus write

$$T = CJ\theta/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$J = \frac{\pi}{2}(r_1^4 - r_2^4) = \frac{\pi}{32}(d_1^4 - d_2^4) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if d_1 and d_2 denote the external and internal diameters of the shaft.

We have thus established that a strain consisting of only a uniform twist, *in the case of a circular rod,* requires for its production only the twisting couple given by (1); conversely, a twisting couple of this magnitude will produce a uniform twist of θ/l radians per unit length, θ/l being given by (1).

For a solid shaft it is only necessary to make r_2 or d_2 zero.

At any radius r the shearing stress is given by

$$q = Cr\theta/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The maximum will occur when $r = r_1$; denoting this by q_m we have

$$q_m = Cr_1\theta/l = Cd_1\theta/2l \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (1) and (4), eliminating $C\theta/l$, we have

$$q_m = \frac{r_1 T}{J} = \frac{d_1 T}{2J} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Equations (1) and (3) can conveniently be combined in the formula

$$\frac{\mathbf{q}}{\mathbf{r}} = \frac{\mathbf{T}}{\mathbf{J}} = \frac{\mathbf{C\theta}}{\mathbf{l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

90. Principal Stresses in a Twisted Shaft.—At any point in a twisted circular rod the stresses consist of equal shearing stresses q in planes normal to the axis in directions at right angles to the axis, and in radial planes in directions parallel to the axis, as shown in Fig. 101. Therefore, the principal planes make angles of 45° with the axis, the principal stresses being $\pm q$; thus, at P (Fig. 101) the principal planes cut the cylindrical surface in the lines AC and BD, the diagonals of the square $ABCD$.

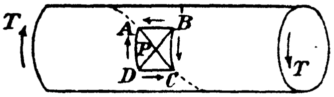

FIG. 101.

If f denote the yield point of the material, the strain-energy theory will give as the equation for the greatest permissible shearing stress

$$q^2 + q^2 + \frac{2}{m}q^2 = f^2$$

whence

$$q = f\sqrt{\frac{m}{2(m+1)}}$$

With $m = \frac{10}{3}$, $q = 0{\cdot}62\,f$. The maximum shear stress theory would give $q = 0{\cdot}5\,f$.

Referring to Fig. 101, the stress across the line BD will be tensile, while that across AC will be compressive. It will be seen that, if a thin tubular shaft be cut spirally, the spiral lying along AC as indicated by the dotted line, it would still transmit the torque T in the direction shown, but it would not do so if the spiral cut ran in the direction BD, since tensile stresses could not be carried across the gap. In this case it would transmit a torque acting in the opposite direction to that shown.

Example 1.—What turning-moment, in ton-ins., applied to a hollow circular shaft of 10″ outside diameter and 7″ inside diameter will produce a maximum shearing stress of 5 tons/in.^{2} in the material ? (Special Exam., Cambridge, 1907.)

Let T = the turning-moment.

We have $r_1 = 5''$, $r_2 = 3{\cdot}5''$, and

$$J = \frac{\pi(5^4 - 3{\cdot}5^4)}{2} = (625 - 150)\frac{\pi}{2} = 745 \text{ ins.}^4$$

Also $q_m = 5$ tons/in.^{2}

From (5) we have

$$T = \frac{Jq_m}{r_1} = \frac{745 \text{ ins.}^4 \times 5 \text{ tons/in.}^2}{5 \text{ ins.}} = 745 \text{ tons. ins.}$$

Example 2.—A steamship propeller shaft has external and internal diameters of 10″ and 6″. What H.P. can be transmitted at 110 r.p.m. with a maximum shearing stress of 5 tons/in.^{2}, and what will then be the twist (degrees) in a 30 ft. length of the shaft ? $C = 12 \times 10^6$ lbs./in.^{2} (Special Exam., Cambridge, 1919.)

$$r_1 = 5'',\ r_2 = 3'',\ l = 30 \text{ ft.}$$

$$J = \frac{\pi(625 - 81)}{2} = 854 \text{ ins.}^4$$

$$q_m = 5 \text{ tons/in.}^2 = 5 \times 2{,}240 \text{ lbs./in.}^2$$

Hence

$$T = \frac{854 \times 5 \times 2{,}240}{5} \text{ lbs. ins.}$$

$$= 854 \times 2{,}240 \text{ lbs. ins.}$$

At 110 r.p.m., the H.P. is

$$\frac{2\pi \times 110 \times 854 \times 2{,}240}{33{,}000 \times 12} = 3{,}340.$$

From (1) we have

$$\theta = \frac{lT}{CJ} = \frac{30 \text{ ft.} \times 854 \times 2{,}240 \text{ lb. ins.}}{12 \times 10^6 \text{ lbs./in.}^2 \times 854 \text{ ins.}^4} \text{ radians}$$

$$= \frac{30 \times 12 \times 2{,}240}{12 \times 10^6} \times \frac{180}{\pi} \text{ degrees}$$

$$= 3{\cdot}85^\circ.$$

Example 3.—A solid circular shaft of 9″ diameter is to be replaced by a hollow shaft, the ratio of the external to internal diameters being 2 to 1. Find the size of the hollow shaft if the maximum shear stress is to be the same as for the solid shaft. What percentage economy in weight will this change effect ? (Intercoll. Exam., Cambridge, 1919.)

Let r = the inside radius of the new shaft
then $2r$ = ,, outside ,, ,, ,,

$$J \text{ for the new shaft } = \frac{\pi}{2}(16r^4 - r^4) = 7{\cdot}5\pi r^4.$$

$$J \text{ ,, old ,, } = \frac{\pi}{2} \times 4{\cdot}5^4 = 204{\cdot}5\pi \text{ ins.}^4$$

If T be the applied torque, the maximum shear stress for the old shaft is

$$\frac{4{\cdot}5T}{204{\cdot}5\pi},$$

and that for the new one is

$$\frac{2rT}{7{\cdot}5\pi r^4} = \frac{T}{3{\cdot}75\pi r^3}$$

These have to be equal, therefore

$$\frac{4{\cdot}5}{204{\cdot}5} = \frac{1}{3{\cdot}75r^3}$$

$$\therefore\ r^3 = \frac{204{\cdot}5}{4{\cdot}5 \times 3{\cdot}75} = 12{\cdot}1 \text{ ins.}^3$$

$$r = 2{\cdot}3''.$$

Hence the internal diameter will be 4·6″ and the external 9·2″.

To find the saving in weight we have :

$$\frac{\text{area of new cross section}}{\text{area of old cross section}} = \frac{9{\cdot}2^2 - 4{\cdot}6^2}{9^2} = 0{\cdot}785.$$

Thus the saving in weight will be 21·5 per cent.

Example 4.—The propeller shaft of a steamship has to transmit 10,000 H.P. at 240 r.p.m. The shaft has an internal diameter of 6″. Calculate the minimum permissible external diameter if the shear stress in the shaft is to be limited to 10 tons/in. (Mech. Sc. Trip., 1923.)

$$T = \text{the twisting moment} = \frac{10{,}000 \times 33{,}000}{240 \times 2\pi \times 2{,}240} \text{ tons. ft.}$$
$$= 97{\cdot}5 \text{ tons. ft.}$$
$$= 1{,}170 \text{ tons. ins.}$$

$$d_2 = 6''$$
$$q_m = 10 \text{ tons/in.}^2$$
$$J = \frac{\pi}{32}(d_1^4 - 1{,}296) \text{ ins.}^4$$

From (5) we have

$$d_1 = \frac{2Jq_m}{T} = \frac{2 \times \pi(d_1^4 - 1{,}296) \times 10}{32 \times 1{,}170}$$

Hence

$$d_1^4 - 1{,}296 = \frac{32 \times 117}{2\pi} d_1$$

or

$$d_1^4 - 596d_1 - 1{,}296 = 0.$$

Solving this by trial we find $d_1 = 9{\cdot}04''$.

91. Torsion Combined with Thrust or Tension.*—When a shaft is subjected to longitudinal thrust, or tension, as well as twisting, the direct stress must be calculated in addition to the shear stress, and from these the principal stresses can be estimated.

Let P = the longitudinal pull.
p = the corresponding tensile stress.

Then, with the notation of § 89,

$$p = \frac{P}{\pi(r_1^2 - r_2^2)} = \frac{4P}{\pi(d_1^2 - d_2^2)}$$

where for a solid shaft $r_2 = 0$.

The maximum shear stress due to torsion is given by (5), and the principal stresses are found from the formula

$$\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 + 4q_m^2}.$$

The maximum shear stress in the shaft is $\frac{1}{2}\sqrt{p^2 + 4q^2}$.

Example 1.—A steel shaft, 8″ external diameter and 3″ internal, is subjected to a turning moment of 10 tons. ft., and a thrust of 5 tons. Find the shear stress due to the turning moment alone; also the percentage increase when the thrust is taken into account. (R.N.C., Greenwich, 1921.)

We have

$r_1 = 4''$; $r_2 = 1{\cdot}5''$
S = area of cross section $= \pi \times 13{\cdot}75 = 43{\cdot}2$ in.^{2}
p = 5 tons.

$$\therefore\ p = \frac{-5\text{ tons}}{43{\cdot}2\text{ in.}^2} = -\ 0{\cdot}116\text{ tons/in.}^2$$

$$J = \frac{\pi}{2}(256 - 5{\cdot}06) = 393\text{ ins.}^4$$

$\therefore\ T$ = 10 tons. ft.

$$\therefore\ q_m = \text{shear stress due to torque only} = \frac{4\text{ ins.} \times 10\text{ tons. ft.}}{393\text{ in.}^4}$$
$$= 1{\cdot}22\text{ tons/in.}^2$$

The maximum shear stress due to the combined load

$$= \tfrac{1}{2}\sqrt{p^2 + 4q^2}$$
$$= \tfrac{1}{2}\sqrt{0{\cdot}1345 + 5{\cdot}952}\text{ tons/in.}^2$$
$$= 1{\cdot}235\text{ tons/in.}^2,$$

so that the increase due to the thrust is about 1·2 per cent.

Example 2.—A thin steel tube of 1″ diameter and $\frac{1}{16}$″ thickness has an axial pull of 1 ton, and an axial torque of 0·094 tons. ins. applied to it. Find the magnitude and direction of the principal stresses at any point. (Mech. Sc. Trip., 1912.)

It will be easier, and sufficiently accurate, to neglect the variation in the shear stress from the inside to the outside of the tube.

Let q = the mean shear stress due to torsion (tons/in.^{2}).
r = ,, radius = $\frac{15}{32}$″.
t = the thickness = $\frac{1}{16}$″.

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* For torsion combined with bending, see p. 368. For the stability of a shaft under end-thrust, see p. 375.

Then the moment of the total resistance to shear

$$= 2\pi r^2 qt$$
$$= 2\pi \times (\tfrac{15}{32})^2 \times \tfrac{1}{16} q$$
$$= \frac{225\pi q}{8192} \text{ tons. ins.}$$

Hence

$$\frac{225\pi q}{8192} = 0{\cdot}2$$

$$\therefore\ q = \frac{0{\cdot}094 \times 8192}{225\pi} = 1{\cdot}09 \text{ tons/in.}^2$$

The area of the cross section, approximately,

$$= 2\pi rt = 2\pi \times \tfrac{15}{32} \times \tfrac{1}{16} = \frac{15\pi}{256} \text{ ins.}^2$$

Hence the tensile stress $= 1 \div \dfrac{15\pi}{256} = \dfrac{256}{15\pi}$ tons/in.^{2}

or $p = 5{\cdot}45$ tons/in.^{2}

The principal stresses are (cf. § 91)

$$\tfrac{1}{2}\{p \pm \sqrt{p^2 + 4q^2}\} = \tfrac{1}{2}\{ 5{\cdot}45 \pm \sqrt{29{\cdot}7 + 4{\cdot}75}\}$$
$$= \tfrac{1}{2}\{ 5{\cdot}45 \pm 5{\cdot}87\}$$
$$p_1 = -\ 0{\cdot}21 \text{ tons/in.}^2,\ p_2 = 5{\cdot}66 \text{ tons/in.}^2,$$

the positive sign denoting tension.

The planes across which they act make angles θ and $\theta + \dfrac{\pi}{2}$ with the axis, where

$$\tan 2\theta = \frac{2q}{p} = \frac{2{\cdot}18}{5{\cdot}45} = 0{\cdot}4$$

whence $\theta = 10° \, 54'$. The planes are shown in Fig. 102.

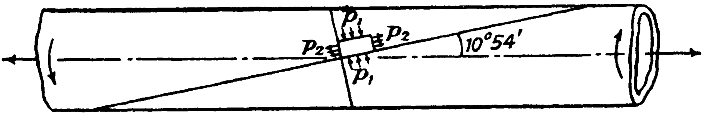

FIG. 102.

92. Strain Energy of Torsion.—If we consider one end of the shaft fixed whilst the couple applied at the other end is gradually increased from zero to its final value T, the angle turned through by this end will be proportional to T and reach the final value θ. Hence the total work done will be $\tfrac{1}{2}T\theta$. This will be the strain energy stored in the twisted shaft, and we have, if U denote the total strain energy,

$$U = \tfrac{1}{2}T\theta.$$

This can be expressed in various other forms: from (1) we have

$$\theta = \frac{lT}{CJ}$$

$$\therefore\ \mathrm{U} = \frac{l\mathrm{T}^2}{2\mathrm{CJ}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Again, from (4) we have

$$\theta = \frac{lq_m}{Cr_1},$$

so that, eliminating θ, we can write

$$U = \frac{lJq^2_m}{2Cr_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\text{A})$$

93. Keyways and Serrations.—The effect of a keyway is to reduce the strength of the shaft, and usually by a greater amount than would be indicated by supposing the outer boundary of the cross section of the shaft to be a circle reaching only to the bottom of the keyway. The chief factor in determining the weakening effect of a keyway is the radius of the corners at the bottom of the slot. This matter has been investigated by A. A. Griffith and G. I. Taylor * during the war, and the results are of considerable importance. The figures which are given below are the results of soap-film experiments † for a shaft 10″ external diameter and 5·8″ internal diameter, but can be applied to similar shafts of different absolute dimensions. In the experiments the keyway was 2·5″ wide by 1″ deep. For shafts of different proportions from these the figures we shall give will indicate the general tendency of things.‡

In the tables below, the term "strength ratio" means the ratio of the torque required to produce a given shear stress in the keyed or serrated shaft to that required to produce the same stress in an uncut shaft of the same *maximum* diameter.

TORSION OF KEYWAYED HOLLOW SHAFTS

Radius of bottom corners of keyway.	Ratio of radius to depth of keyway.	Strength Ratio.	Strength comp. with shaft having radius = radius to *bottom* of keyway.
0·1″	0·1	0·187	0·447
0·2″	0·2	0·297	0·710
0·4″	0·4	0·435	1·04
0·7″	0·7	0·502	1·20

TORSION OF SERRATED HOLLOW SHAFTS (see Fig. 103).

Radius of bottom of *V*.	Ratio of radius to depth of *V*.	Strength Ratios as above.	
0·10″	0·164	0·293	0·475
0·15″	0·247	0·377	0·610
0·20″	0·329	0·401	0·650
0·25″	0·410	0·425	0·689

The torsional weakness of keywayed hollow shafts is due partly—in fact, mainly—to a region of high local stress near the corners, and partly to the concentration of shear stress in the thin region between the bottom of the keyway and the inner wall of the shaft. To some extent the strength of the shaft may be increased by making the bore eccentric, but in most cases the ensuing want of balance would prohibit this. The presence of the high local stress is particularly important where the shaft is subjected to alternating stresses, for in such cases the ductility of the metal is of less account in the prevention of cracks. If the stress merely fluctuates between maximum and minimum values of the same sign it is probable that the strength of the shaft is not reduced so much. It must be remembered that the above figures are only accurate for isotropic materials within the elastic limit.

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* Advisory Committee for Aeronautics, Reports and Memoranda, No. 392, 1918. H.M. Stationery Office.

† Ditto, Nos. 333 and 399 explain the soap-film method of solving torsion problems; *Engineering*, Vol. 104 (1917), pp. 695, 699.

‡ The strength of keywayed solid shafts may be estimated by the method given on pp. 480–483. For an investigation by polarized light see *Photo-Elasticity*, Coker and Filon.

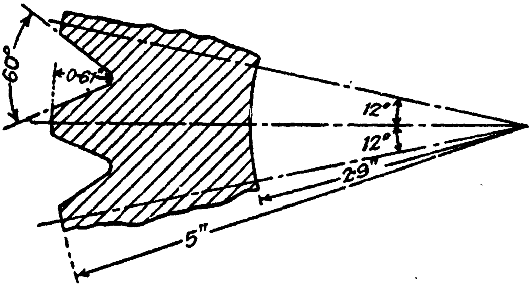


FIG. 103.

EXAMPLES VIII

1. Find the maximum shear stress in a propeller shaft 16" external, and 8" internal, diameter, when subjected to a twisting moment of 1,800 tons. ins. If $C = 5{,}200$ tons/in.^{2}, what is the angle of twist in a length of 20 diameters ? What diameter would be required for a solid shaft with the same maximum stress and twisting moment ? (R.N.C., Greenwich, 1921.)

2. A propeller shaft, 134 ft. long, transmits 15,000 H.P. at 80 r.p.m. The external diameter of the shaft is 22·45", and the internal diameter 9·45". Assuming that the maximum torque is 1·19 times the mean torque, find the maximum shear stress produced. Find also the relative angular movement of the ends of the shaft when transmitting the average torque. Take $C = 10{\cdot}5 \times 10^6$ lbs./in.^{2} (R.N.C., Greenwich, 1922.)

3. A hollow steel shaft has to transmit 8,000 H.P. at 110 r.p.m. Taking the maximum shear stress at 9,000 lbs./in.^{2}, and assuming $d_2 = \frac{3}{5}d_1$, find the necessary dimensions of the shaft. (R.N.E.C., Keyham, 1921.)

4. What diameter of solid shaft is required to transmit 80 H.P. at 60 r.p.m. if the maximum torque is 30 per cent. greater than the mean, and the limiting shear stress is 8,000 lbs./in.^{2} ? Assuming $C = 12 \times 10^6$ lbs./in.^{2}, calculate the angle of twist of the shaft on a length of 10 ft. (R.N.C., Greenwich, 1922.)

5. A steel tube, 8′ 0" long, 1·5" diameter, 0·025" thick, is twisted by a couple of 50 lbs. ft. Find the maximum shear stress, the maximum tensile stress, and the angle through which the tube twists. Take $C = 12 \times 10^6$ lbs./in.^{2} (Special Exam., Cambridge, 1912.)

6. Compare the weight of a solid shaft with that of a hollow one to transmit a given horse-power at a given speed with a given maximum shearing stress, the inside diameter of the hollow shaft being two-thirds of the outside diameter. (Intercoll. Exam., Cambridge, 1905.)

7. A hollow propeller shaft has an internal diameter of 6″. It is designed to transmit 8,000 H.P. at 150 r.p.m. Show that the external diameter which will limit the shear stress to 10,000 lbs./in.^{2} is given by the equation $d^4 - 1{,}710d = 1{,}296$. (Intercoll. Exam., Cambridge, 1912.)

8. A straight steel shaft $\frac{3}{4}''$ diameter and 30″ long has its ends rigidly fixed. The shaft is subjected to a twisting couple of 50 lbs. ft. applied at a section distant 10″ from one extremity. Determine (i) the couples required to hold the ends, (ii) the magnitude of the greatest shear stress set up in the shaft, (iii) the angular rotation of the section at which the couple is applied. Take $C = 12{\cdot}5 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip., 1911.)

9. A 1″ circular steel shaft is provided with enlarged portions *A* and *B* as shown in Fig. 104. On to this enlarged portion a steel tube $\frac{1}{20}''$ thick is shrunk. While the shrinking process is going on, the 1″ shaft is held twisted by a couple of magnitude 50 lbs. ft. When the tube is firmly set on the shaft this twisting couple is left on the shaft, the shaft and tube being made of the same material. (Mech. Sc. Trip., 1910.)

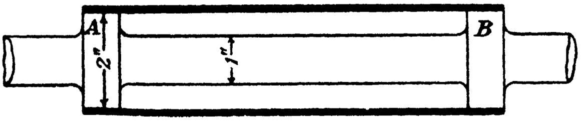


FIG. 104.

10. To secure torsional stiffness in a shaft transmitting power, the greatest angle of torsion allowed is frequently 1° for a length of 20 diameters. Prove that, for a shaft for which $C = 12 \times 10^6$ lbs./in.^{2}, this corresponds to an allowable shear stress of 5,236 lbs./in.^{2}

A hollow shaft, with outer diameter twice the inner, is to transmit 10,000 H.P. at 150 r.p.m. The greatest shear stress is to be 5,000 lbs./in.^{2} Find the diameters, and also the resilient energy stored in 120 ft. of shaft, when running as above. (Mech. Sc. Trip., 1919.)

11. A thin tube of mean diameter 1″ and thickness $\frac{1}{20}''$ is subjected to a pull of $\frac{3}{4}$ ton, and an axial twisting moment of $\frac{3}{8}$ ton-inches. Find the magnitude and direction of the principal stresses. (Intercoll. Exam., Cambridge, 1922.)

12. A 4″ diameter shaft is subjected to an axial thrust of 35 tons while transmitting 40 H.P. at 100 r.p.m., there being no appreciable bending. Calculate the maximum normal and tangential stresses in the shaft. (R.N.E.C., Keyham, 1922.)

13. A hollow shaft of mild steel, 10′ long, 4″ external and 2″ internal diameter, is put under torque. Given that the yield point of the metal in shear is 10 tons/in.^{2}, and assuming that, when the yield point of the outer skin is passed, the shear stress remains uniform at that intensity until the whole tube has yielded, find through what angle the shaft will twist before the material close to the inner surface is at the yield point. Find also the approximate torque when this occurs. Take $C = 12 \times 10^6$ lbs./in.^{2} (Birmingham University, 1911.)

14. The H.P. pinion of a set of reduction gearing runs at 2,900 r.p.m. and transmits 9,000 H.P. If the pinion is 40″ long, with a 10″ pitch circle diameter, estimate the relative torsional deflection of the two ends of the teeth in thousands of an inch. The internal diameter of the pinion is 3″, and for this calculation it may be assumed that the pinion is equivalent to a plain hollow shaft of external diameter 10″, and that the load is equally distributed along the length. Take $C = 5{,}000$ tons/in.^{2} (R.N.E.C., Keyham, 1926.)

CHAPTER IX

BENDING MOMENTS AND SHEARING FORCES DUE TO STEADY LOADS

94. Bending Moments and Shearing Forces Defined.—Let us consider first the simplest example of a beam, namely, one which is fixed rigidly at one end, as shown in Fig. 105. Such a beam is called a cantilever, a familiar example of which is an ordinary fishing-rod when held in one hand.

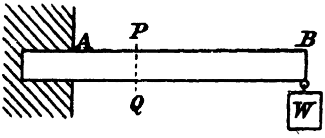


FIG. 105.

To concentrate ideas, imagine that the beam is a horizontal balk of timber with one end embedded in a vertical wall, and that a weight is hung on the other end. Now conceive the beam divided into two portions by a vertical plane PQ: each of the portions AP and PB must be in equilibrium by themselves. If we neglect the weight of the beam itself, any other forces acting on PB besides the weight W must arise from the action of the part AP across the plane PQ. And whatever action AP exerts on PB, PB will exert an equal and opposite reaction on AP.

Now, since AP and the weight W keep PB in equilibrium, the action of AP must be equal to the action of W. In the first place W tends to move the part PB bodily downwards with a force W, so that AP must exert an equal upward force on PB. Thus, there is a shearing action between the two portions of the beam, the total shearing force on the section PQ being equal to W.

Next, W tends to turn PB bodily round Q with a moment $W \times PB$, and AP must exert an equal and opposite moment on PB. Hence, besides the shearing force, a couple is transmitted across the section PQ; this couple is referred to as the *bending moment* on the section.

If the beam is inclined to the horizontal at an angle θ, as shown in Fig. 106, we can resolve the force due to W into two components $W \cos \theta$ and $W \sin \theta$. The former gives rise to a shearing force and bending moment on the section PQ, and the latter to a direct thrust.

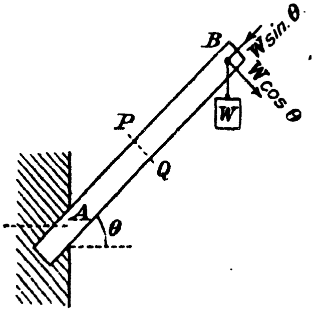


FIG. 106.

Similar considerations will apply whatever loads are applied to the cantilever, including the reaction of any support that might be placed at B.

We shall now make the following conventions and definitions :

(i) In practice beams may have any position relative to the horizontal, but, for convenience, and since the most familiar examples of beams, namely bridges, are usually horizontal, we shall always refer to a beam in a horizontal position, and we shall generally represent the forces which cause bending as weights.

We shall assume that these forces act in one vertical plane which we call the plane of bending.

(ii) We define the *axis* of a beam as the line on which lie the centres of area of all normal cross sections of the beam, and when we say the beam is horizontal we mean that its axis is horizontal.

(iii) The *shearing force* at any cross section of a beam is the algebraic sum of the components, perpendicular to the axis, of all the forces acting on the beam, on one side of this section.

That it is immaterial which side of the section we consider is obvious : the sum of the components of all the forces on one side of any section must be equal and opposite to the sum of all the components on the other side, since the beam as a whole is in equilibrium.

(iv) Shearing force will be considered positive when, with a horizontal beam, the right-hand portion tends to move upwards relative to the left-hand portion.

(v) The *bending moment* at any cross section of a beam is the algebraic sum of the moments, about a line in the section, through the axis of the beam and perpendicular to the plane of bending, of all the forces acting on the beam, on one side of this section. Again it is immaterial which side of the section we consider.

(vi) Bending moments will be considered positive for horizontal beams when they tend to make the beam concave upwards; such bending moments are sometimes called " sagging " bending moments, the opposite kind being called " hogging " bending moments.

Figures which show graphically the bending moment and shearing force for all cross sections along a beam are called bending moment diagrams and shearing force diagrams. The two quantities are plotted above the line when positive, below when negative. Before we can proceed to calculate the stresses and deformation of beams we must be able to find the bending moment and shearing force at any section, and we shall now show how this can be done.

95. Concentrated and Distributed Loads.—An example of what we mean by a concentrated load is afforded by a weight suspended from a bolt passing through the beam, as in Fig. 106. In making calculations such a load is assumed to be localized at a point, although, in reality, it is distributed over a length of beam equal to the diameter of the bolt, and so-called concentrated loads must in practice be distributed over a small length of beam. Examples of distributed loads are walls built on the top of girders, aqueducts carrying water, and loads due to fluid pressure, such as wind pressure on telegraph poles, or on the wings of aeroplanes.

96. Relation between Load, Shearing Force and Bending Moment.—That these three quantities are not independent is easily seen by considering the equilibrium of a small length of a loaded beam.

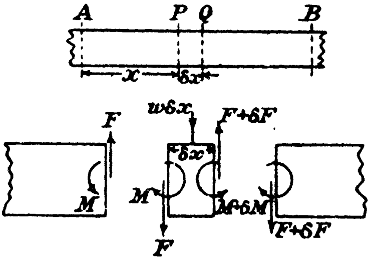


FIG. 107.

Let AB (Fig. 107) represent a portion of a loaded beam, and let PQ be an element of this beam; let $AP = x$ and $PQ = \delta x$, the positive direction of x being towards the right.

Let $w =$ the average load per unit length on PQ, so that the total load on PQ is $w.\delta x$.

Let M and F denote the bending moment and shearing force at P, and let $M + \delta M$, $F + \delta F$ denote the values of the same quantities at Q.

Now consider the portion PQ; for this element the directions of the arrows shown in Fig. 107 agree with the sign convention stated above: if the shearing force is positive at Q, the right-hand part QB is tending to move upwards, i.e. to pull PQ upwards, so that the force $F + \delta F$ acts upwards on PQ; likewise M and $M + \delta M$ are drawn in such directions that they tend to make PQ concave upwards.

Consider the equilibrium of PQ: resolving vertically we must have

$$F + \delta F = F + w\delta x.$$

$$\therefore \frac{\delta F}{\delta x} = w$$

or, in the limit, when PQ is infinitesimal,

$$\frac{dF}{dx} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore, by integration,

$$\int_A^B dF = \int_A^B w.dx$$

or

$$F_B - F_A = \int_A^B w.dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is, the increase of shearing force from A to B is given by the area of the curve representing the load distribution over AB, or the difference between F_B and F_A equals the total load on AB.

Again, taking moments about P, we must have

$$(F+\delta F)\delta x+(M+\delta M)-M-w.\delta x.\frac{\delta x}{2}=0,$$

or, to the first order of small quantities,

$$F\delta x+\delta M=0.$$

Therefore, in the limit, we have

$$\frac{\mathrm{dM}}{\mathrm{dx}}=-\mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating this we have

$$\int_A^B dM=-\int_A^B F.dx$$

or

$$M_B-M_A=-\int_A^B F.dx$$

that is

$$M_A-M_B=\int_A^B F.dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the *decrease* of positive bending moment from A to B is given by the area of the curve showing the change of the shearing force along the beam.

The two results (2) and (4) are extremely useful for finding the bending moments and shearing forces on beams with irregularly distributed loads; examples of their use will be found below.

Substituting for F from (2) in (4) we have

$$M_A-M_B=\int_A^B\int_A^B w.dx.dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

An important point which is shown by equation (3) is that the bending moment has a stationary value when the shearing force is zero. Differentiating (3) we have

$$\frac{d^2M}{dx^2}=-\frac{dF}{dx}=-w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which shows that M is mathematically a maximum, since d^2M/dx^2 is negative. The significance of the word "mathematically" will be understood after reading § 107.

CANTILEVERS

97. Cantilever with Concentrated Load (Fig. 108).—Let W be the load, acting at a distance l from the support, and neglect the weight of the beam itself.

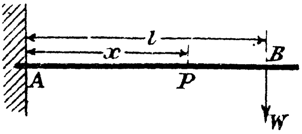


Fig. 108.

Let M and F denote the bending moment and shearing force on a section P distant x from the support.

Then, resolving vertically for BP we see that F is constant for all values of x, between 0 and l, and equal to W. Similarly, taking moments about P for PB, M is equal to $W(l-x)$, which increases uniformly from zero when $x = l$ to Wl when $x = 0$. With regard to sign: W tends to make the part PB move downwards relative to AP, and to make the beam concave downwards, so that both M and F must be considered negative according to the conventions laid down above, and are accordingly drawn downwards. The results are shown graphically in Fig. 109, which shows the bending moment and shearing force diagrams for the beam, drawn separately on AB as zero line. Expressed algebraically we have:

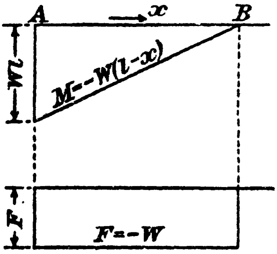


Fig. 109.

$$F = -W$$
$$M = -W(l-x).$$

The greatest value of the bending moment is given by

$$\mathbf{M_{max} = -W\mathit{l}}$$

98. Cantilever with Uniformly Distributed Load (Fig. 110).—

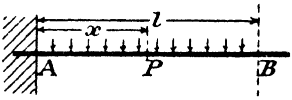


Fig. 110.

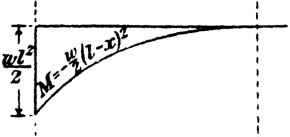


Fig. 111.

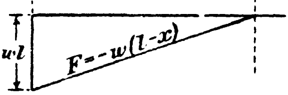


Fig. 112.

Let w = the load per unit length over the length $AB(=l)$ of the beam, acting downwards.

Let M and F denote the bending moment and shearing force at a section P distant x from the support. Then:

$F =$ the total force tending to move PB upwards relative to AP

$$= -w(l-x) \qquad (7)$$

and

$M =$ the moment about P of the load on PB, and it must be considered negative since the load will make the beam concave downwards.

The load on PB is $w(l-x)$, and its centre of gravity will be at the middle of PB, i.e. $\frac{1}{2}(l-x)$ from P. Hence

$$M = -w(l-x) \times \tfrac{1}{2}(l-x)$$

i.e.

$$M = -\frac{w}{2}(l-x)^2 \qquad (8)$$

The results (7) and (8) are shown graphically in Figs. 111 and 112, which are the bending moment and shearing force diagrams for the beam. The maximum bending moment is $wl^2/2$, and the greatest shearing force is wl, omitting the signs.

Equation (7) represents a straight line and the shearing force increases uniformly from B to A.

Equation (8) represents a parabola with its vertex at B, and its axis vertically downwards.

Allowance for the Weight of Beams.—We can allow for the weight of a beam by regarding the beam as weightless and loaded with a distributed load representing its weight.

99. Cantilever with Non-uniformly Distributed Load (Fig. 113).

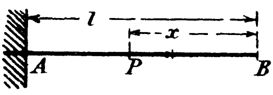


Fig. 113.

—Let w denote the load per unit length, w being some function of x.

Let M and F denote the bending moment and shearing force at a distance x from B, i.e. at P.

According to § 96, the change of shearing force from B to P is given by

$$F_P - F_B = \int_B^P w.dx = \text{area of load curve from } B \text{ to } P.$$

But, unless there is a localized load at B, F_B is zero, so that, remembering that the shearing force at P must be considered negative, we have

$$F_P = -\int_B^P w.dx = -\text{ area of load curve from } B \text{ to } P.$$

Thus the shearing force curve can be plotted.

In the same way we have

$$M_P - M_B = \int_B^P F.dx = \text{area of shearing force curve from } B \text{ to } P.$$

If there is no external couple applied at B, the bending moment there is zero, and M_P is given directly by the value of the integral or the area of the shearing force curve. In accordance with our sign convention, M is considered negative with downward vertical loads.

The method of carrying out this method, and of allowing for the scales of the drawing, will be made clear by Example 4 below.

100. Cantilever with any Manner of Loads.—Whatever may be the loads on a cantilever they can be resolved into the groups treated above, and each can be treated separately. The complete bending moment and shearing force diagrams can then be obtained by adding together those for the separate loads.

Example 1.—The hand lever of a brake is 30″ long and the greatest pull which may be expected on the end is 80 lbs. What is the bending moment 20″ from the point of application of the pull, and the maximum bending moment ?

(i) B.M. = 80 × 20 = 1,600 lb. ins.
(ii) B.M. = 80 × 30 = 2,400 lb. ins.

Example 2.—A cantilever 5 ft. long carries a uniformly distributed load of 10 lbs. per inch over the outer 4 feet, and a concentrated load of 250 lbs. at its middle point; it is required to draw the shearing force and bending moment diagrams. Referring to Fig. 114, BC is that part of the beam which carries the distributed load, and D is the point of attachment of the 250 lb. load.

The shearing force due to the distributed load increases uniformly from zero at C to 10 × 48 = 480 lbs. at B, and keeps that value from B to A; this is shown by the lines marked (1) in the S.F. diagram. The shearing force due to the concentrated load is 250 lbs. from D to A, as shown by the line (2) in the S.F. diagram. Adding the two together we get the total shearing force shown by the lines (3).

The bending moment due to the distributed load increases parabolically from zero at C to

$$\frac{10 \text{ lb./ins.} \times 48^2 \text{ in.}^2}{2} = 11{,}520 \text{ lb. ins. at } B.$$

(See § 98.)

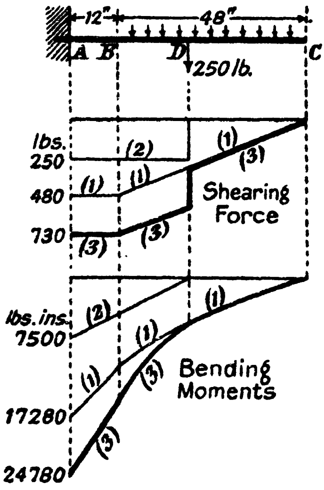


Fig. 114.

The total load on BC is 480 lbs., with its centre of gravity 36″ from A; therefore the B.M. at A due to this is

480 lbs. × 36 ins. = 17,280 lbs. ins.

From B to A the bending moment increases uniformly, hence we have the graph (1) in the B.M. diagram.

The bending moment due to the concentrated load increases uniformly from zero at D to $250 \times 30 = 7{,}500$ lb. ins. at A, as shown by the line (2) in the B.M. diagram. Combining the two we have the total B.M. shown by the graph (3).

Example 3.—A uniform rod of length l rotates as a conical pendulum round a vertical axis through one end, and is inclined at an angle θ to the vertical. If the weight of the rod is w per unit length, and the angular velocity ω, find the bending moment at the top end if the rod can be assumed to act as a rigid cantilever.

Consider an element dx of the rod, distant x from the top. The weight of the element is $w.dx$, and the centrifugal force acting on it is

$$\frac{wdx}{g}\omega^2 .\, x \sin \theta .$$

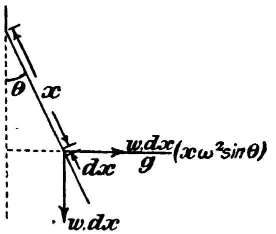


FIG. 115.

The bending moment at the top, due to these forces, is

$$\frac{w\omega^2 \sin \theta .\, xdx}{g} x \cos \theta - wdx .\, x \sin \theta$$

$$= \left(\frac{w\omega^2}{g} \sin \theta \cos \theta .\, x^2 - w \sin \theta .\, x\right)dx$$

tending to make the rod concave outwards. The total bending moment is

$$w \int_0^l \left(\frac{\omega^2}{2g} \sin 2\, \theta .\, x^2 - \sin \theta .\, x\right)dx$$

$$= w\left(\frac{\omega^2 l^3}{6g} \sin 2\theta - \frac{l^2}{2} \sin \theta\right)$$

$$= \frac{wl^2}{2}\left(\frac{l\omega^2}{3g} \sin 2\theta - \sin \theta\right).$$

Example 4.—The pressure distribution along the blade of an aeroplane propeller is given by the curve w in Fig. 116. in lbs./ins. The radius of the propeller from the axis to the tip of the blade is 4′ 2″, and the boss is 10″ in diameter. It is required to draw curves of shearing force and bending moment from the tip to the boss, neglecting the twist of the blade.

In the original diagram each square had 1″ sides, and *the figures below refer to the original drawing.*

To draw the shearing force curve we must (§ 99) integrate the load curve, i.e. plot its area beginning at the blade-tip, so obtaining the curve F. The scales of the load are

1″ = 5″ of blade length
1″ = 5 lbs./inch load.
∴ 1 in.^{2} area of load curve = 25 lbs.

The shearing force curve F is plotted to a scale of 1″ = 5 in.^{2} of load curve = 125 lbs.

The bending moment curve is then found by integrating the curve F, which gives the curve M.

The scales of the curve F are

1″ = 5″ of blade length
1″ = 125 lbs.
∴ 1 in.^{2} = 5 ins. × 125 lbs. = 625 lb. ins.

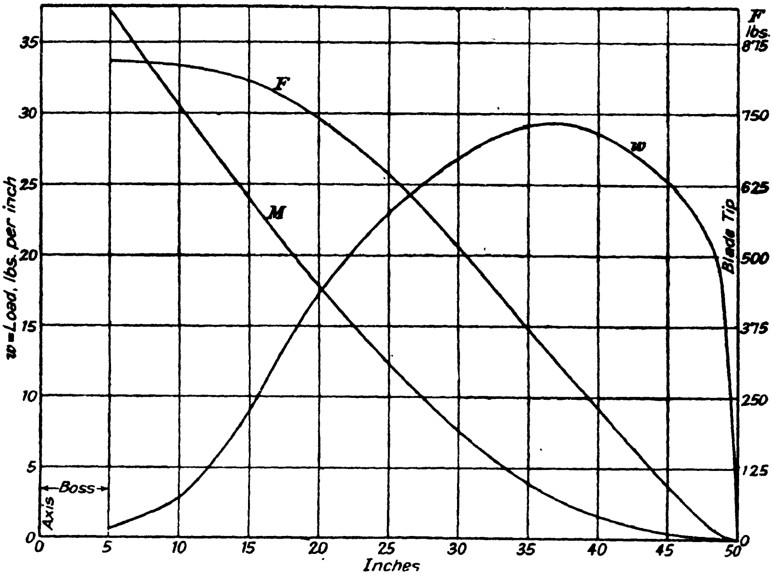


FIG. 116.

The curve M is plotted to a scale of $1'' = 5$ in.^{2} of the shearing force curve $= 625 \times 5$ lb. ins., i.e.

$$1'' = 3{,}125 \text{ lb. ins.}$$

The maximum bending moment is

$$7{\cdot}45 \times 3{,}125 = 23{,}300 \text{ lbs. ins.}$$

BEAMS FREELY SUPPORTED AT EACH END

By "freely supported" we mean that the supports are of such a nature that they do not apply any resistance to flexure; for instance, knife-edges or frictionless pins perpendicular to the plane of bending. The general remarks concerning bending moments and shearing forces, which we made in reference to cantilevers (§ 94), apply equally to beams supported at each end, or to any kind of beam.

101. Freely Supported Beam with Concentrated Load (Fig. 117).—A beam is freely supported at its ends A and B, and carries a load W at a distance a from A. The length of the beam AB is l. We require to draw the graphs of bending moment and shearing force along the beam.

Let R_a and R_b be the reactions at A and B, then, by the ordinary principles of statics, we must have

$$R_a + R_b = W$$
$$lR_b = aW.$$

From these equations we have

$$R_a = \frac{bW}{l}, \quad R_b = \frac{aW}{l}.$$

Let x be the distance of any section from A, and let M and F denote the bending moment and shearing force for this section.

For the left-hand portion, AP, of the beam we have $(x < a)$

$$-F^{*} = \quad R_a = \frac{bW}{l}$$

$$M^{*} = x.R_a = \frac{xbW}{l} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The minus sign is attached to F since the reaction at A tends to make

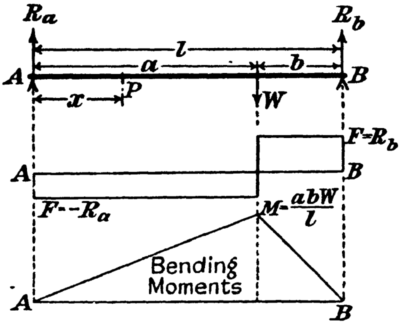


FIG. 117.

the *left*-hand portion of the beam move upwards relative to the right, so that, according to the convention of § 94, the shearing force must be considered negative.

For the portion of the beam to the right of $W (a < x < l)$, we have

$$F = R_b = \frac{aW}{l}.$$

$$M = R_b(l - x) = \frac{aW}{l}(l - x) \quad . \quad . \quad . \quad \text{(ii)}$$

In both cases M is positive, since the action of R_a and R_b is such as to make the beam concave upwards. When $x = a$, i.e. where the load is supported, either (i) or (ii) give

$$M = \frac{abW}{l} = \frac{abW}{a + b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which is greatest, for a given load, when $a = b$, i.e. when the load is at the centre of the beam.

The shearing force and bending moment diagrams are shown in Fig. 117; it must be noted that the bending moment is zero at each support, as must always be the case at a freely supported end.

---

* Alternatively, for the right-hand portion, PB, we have

$$F = R_b - W = \frac{aW}{l} - W = -\frac{bW}{l}$$

$$M = R_b(l - x) - W(a - x) = \frac{xbW}{l}$$

102. Freely Supported Beam with Uniformly Distributed Load (Fig. 118).—Let l be the length of the beam, and let the load be w per unit length; for the rest the notation is as before.

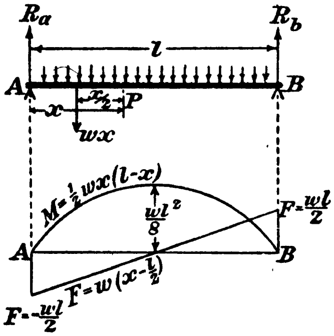


FIG. 118.

Evidently for equilibrium we must have

$$R_a = R_b = \frac{wl}{2}$$

The total load on $AP = wx$.

The shearing force at $P =$ total upward force on PB, or total downward force on AP. Hence

$$F = wx - R_a$$

$$= w\left(x - \frac{l}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

At A, $x = 0$ and $F = -\frac{wl}{2}$; at B, $x = l$ and $F = \frac{wl}{2}$. Now, the centre of gravity of the load on AP is at a distance $\frac{x}{2}$ from P, and the bending moment at. $P =$ the moment about P of all the forces on either side of P, reckoned positive if they tend to make the beam concave upwards; hence

$$M = xR_a - wx.\frac{x}{2}$$

$$= \frac{wlx}{2} - \frac{wx^2}{2}$$

$$M = \frac{w}{2}(lx - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is a maximum when $x = \frac{l}{2}$, i.e. at the centre of the beam. Thus

$$M_{max} = \frac{wl^2}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Equation (10) represents a straight line; equation (11) represents a parabola with its vertex upwards, and its axis vertical and passing through the centre of the beam. The two graphs are shown in Fig. 118.

103. Freely Supported Beam with Non-uniformly Distributed Load (Fig. 119).

—Let the load distribution be given by the curve *ACB*, and let the rest of the notation be as before.

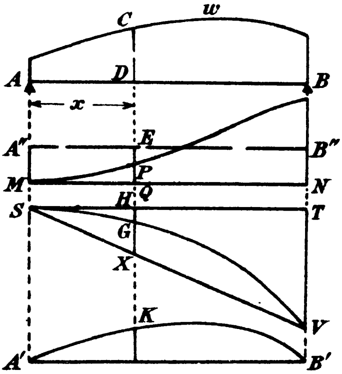


FIG. 119.

The general integrals of equations (1) and (3) of § 96 are

$$F = \int w.dx + F_0.$$

$$M = -\iint w.dx.dx - F_0 x + M_0,$$

where F_0 and M_0 denote the values of F and M at $x = 0$. In the case under consideration M is zero at $x = 0$, so that

$$M_0 = 0.$$

To carry out these operations graphically proceed as follows: Taking a line MN as base, plot the curve ML, its ordinates being given by

$$PQ = \int_0^x w.dx = \text{area } ACD.$$

This graph does not immediately represent F on account of the unknown F_0, which will be found presently.

Next, on ST as base, draw the graph SV, its ordinates being given by

$$HG = -\int_0^x PQ.dx = -\iint w.dx.dx.$$

$$= -\text{ area } MPQ.$$

To obtain the value of M, we must add $-F_0x$. Clearly we can effect this as follows: Join SV, then XG is the value of M, and the slope of the straight line SV represents the value of F_0, i.e.

$$F_0 = \frac{TV}{ST} \text{ and is negative.}$$

If we wish, we can re-plot the ordinates GX on a horizontal base $A'B'$, producing the bending moment diagram $A'KB'$.

We can now draw the zero line for the shearing force curve: Make $MA'' = -F_0$ (i.e. positive), then $A''B''$ is the zero line for the shearing force curve ML, and

$$F_A = F_0 = A''M \text{ (negative)},$$

and the shearing force at any point D is given by EP.

104. Another Graphical Method of Drawing Bending-Moment Diagrams.—The following graphical method is sometimes useful for dealing with a series of concentrated loads or an unevenly distributed load. In Fig. 120 five loads are shown acting at C_1, C_2 . . . First, draw a force diagram $abcdef$, beginning at the top with the force on the right. Then, taking a pole O on the right of this line, draw the funicular polygon $P_bP_1P_2P_3P_4P_5P_aP_b$, and so determine the reactions R_a and R_b in the usual way: fr and ra. Then, with P_aP_b as base the bending moment at any section is given by the height of the funicular polygon. The proof of this statement is as follows:

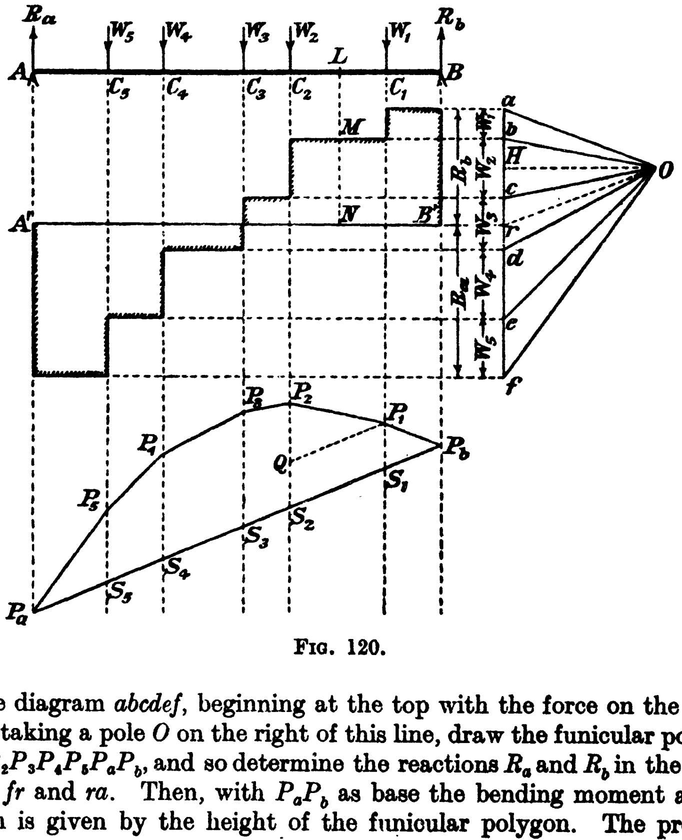


FIG. 120.

Draw OH horizontal. The triangle $P_bP_1S_1$ is similar to the triangle Oar since the sides of the two triangles are parallel.

$$\therefore \frac{P_1S_1}{P_bS_1} = \frac{ar}{rO} = \frac{R_b}{rO}$$

$$\therefore P_1S_1 = \frac{P_bS_1 \times R_b}{rO} = \frac{BC_1 \times R_b}{OH} = \frac{M_1}{OH}$$

where M_1 is the bending moment at C_1.

Again, the triangle P_1P_2Q, where P_1Q is drawn parallel to P_bP_a, is similar to the triangle Obr.

$$\therefore \frac{P_2Q}{QP_1} = \frac{br}{rO} = \frac{R_b - W_1}{rO}$$

$$\therefore P_2Q = (R_b - W_1)\frac{QP_1}{rO} = (R_b - W_1)\frac{C_1C_2}{OH}$$

Then

$$\begin{aligned} P_2S_2 &= P_2Q + P_1S_1 \\ &= \frac{(R_b - W_1)C_1C_2 + R_b.(BC_1)}{OH} \\ &= \frac{R_b(BC_2) - W_1.(CC_1)}{OH} \\ &= \frac{M_2}{OH} \end{aligned}$$

where M_2 is the bending moment at C_2. Thus

$$M_1 = OH \times P_1S_1$$
$$M_2 = OH \times P_2S_2$$

and so on for the bending moments at C_3, C_4, C_5. Also, we know that between these points the B.M. diagram is straight, so that the funicular polygon gives the bending moment at any section.

SCALES.—If the beam is drawn to a scale of 1″ to s″, and the force diagram $abc..f$ to a scale 1″ $= w$ lbs., the scale of the funicular polygon or bending moments will be $1 = w.OH$ lbs. × s ins. $= ws.OH.$ lbs. ins., where OH is measured in inches.

Hence, if we wish the resulting B.M. diagram to be to a scale 1″ $= m$ lbs. (or tons)-ins., we must make

$$OH = \frac{m}{ws} \text{ inches.}$$

If desired the B.M. diagram may be redrawn on a horizontal base.*

The shearing force diagram is drawn directly by projecting across horizontally from the line of forces, as shown in Fig. 120. Thus

$$MN = ra - ab - bc = R_b - W_1 - W_2$$
$$= \text{the shearing force at } L.$$

To apply the method to distributed loads, the curve of loading must be divided into a number of vertical strips, and the distributed load is replaced by a series of concentrated loads given by the areas of the strips. When the funicular polygon for these loads has been drawn as above, a fair curve should be drawn touching this inside: this curve will be the B.M. diagram.

---

* In the case of a beam built-in at A and free at B (i.e. a cantilever), O should be taken on the same horizontal level as a. The $S.F.$ and $B.M.$ diagrams will terminate on the vertical through C_1, and the horizontal line through P_1, will be the base of the $B.M.$ diagram.

105. Freely Supported Beam with Couples applied to Both Ends (Fig. 121).— Let couples M_a and M_b be applied to the ends A and B of a beam AB, the ends of which are otherwise freely pivoted.

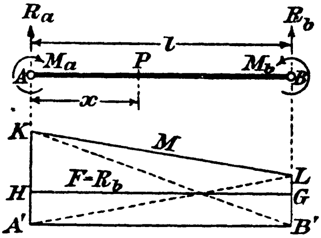


Fig. 121.

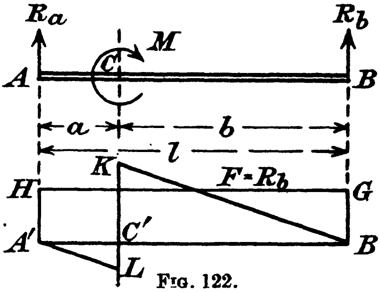


Fig. 122.

The conditions imagined are those of a beam pivoted on frictionless pins at the ends, and couples applied by some external agency, such as cranks. For equilibrium we must have, taking moments about A,

$$lR_b + M_b = M_a.$$

$$\therefore R_b = \frac{M_a - M_b}{l} \quad \ldots\ldots\ldots \quad (13)$$

We must also have $R_a + R_b = O$, so that $R_a = -R_b$.

The shearing force at all sections is the same and equal to R_b, represented by the line HG in Fig. 121.

The bending moment at any section P is

$$M = (l - x)R_b + M_b$$

$$= \frac{l - x}{l}(M_a - M_b) + M_b$$

or

$$M = \frac{l - x}{l} M_a + \frac{x}{l} M_b \quad \ldots\ldots\ldots \quad (14)$$

The bending moment diagram is the figure $A'KLB'$.

106. Freely Supported Beam with Couple applied Between the Supports (Fig. 122).—Let a couple M be applied at the point C to the beam AB, the ends of which are freely pivoted.

From the conditions of equilibrium it is evident that

$$R_b = -R_a = M/l.$$

For a point distant x from A, the bending moment is

and

$$\left.\begin{aligned} (l-x)R_b &= \frac{l-x}{l}M, \text{ if } a<x<l \\ (l-x)R_b - M &= -\frac{x}{l}M, \text{ if } 0<x<a \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

Thus the shearing force diagram is $A'HGB'$, whilst the bending moment is given by the discontinuous graph $A'LKB'$. The bending moment just to the right of C is bM/l, and just to the left of C it is $-aM/l$.

The bending-moment and shearing diagrams for any straight beam, loaded in any manner and having only two supports, can be obtained by regarding the load system as a combination of two or more of the cases already treated.* We shall now consider two examples.

107. Beam Freely Supported at each End, carrying a Uniformly Distributed Load, acted on by Couples at both Ends.—In Fig. 123, AB is a beam resting on two supports A and B, carrying a uniformly distributed load w per unit length, and the ends A and B are acted on by couples M_a and M_b. The rest of the notation is shown in the figure.

The reactions R_a and R_b can be found directly by taking moments about A and B in turn, or by combining the results of §§ 102 and 106.

$$R_a = \frac{wl}{2} - \frac{M_a - M_b}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$R_b = \frac{wl}{2} + \frac{M_a - M_b}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

These give the shearing forces at the ends, and so the shearing force diagram is the line NQ which has been drawn on a base $A'B'$.

To draw the bending moment diagram we regard AB as the combination of a freely supported beam carrying a distributed load, and a beam with terminal couples M_a and M_b. The former gives rise (§ 102) to a parabolic bending moment curve $A''RB''$ of height $wl^2/8$.

The latter (§ 106) gives rise to the straight line bending moment HK, where $A''H = M_a$ and $B''K = M_b$. Three cases are shown in the figure according to the signs of M_a and M_b.

The complete B.M. diagram is obtained by adding (algebraically) the ordinates of the two curves $A''RB''$ and HK. This is most readily done by replotting $A''RB''$ on HK as base, so that $R'T' = RT$. Then the ordinates of the curve $HR'K$, measured from $A''B''$, give the bending moment at any section in AB.

---

* This sentence covers the case of beams which are constrained at the ends, provided we know the constraining couples. The determination of these requires the application of principles which are developed below, and forms the subject of Chapter XXV.

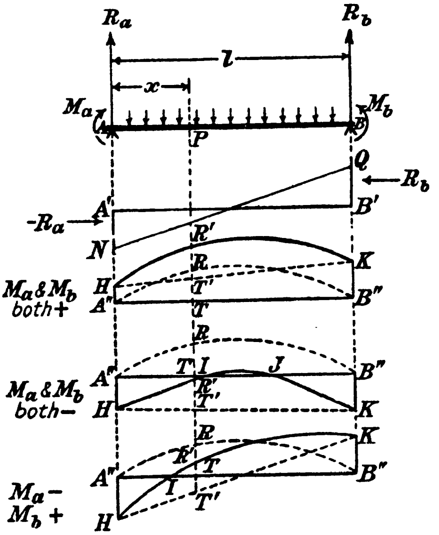
We can obtain an expression for the bending moment at any point P in AB as follows:

$$R'T' = RT = \frac{w}{2}(lx - x^2) \ldots \text{by equation 11, p. 124.}$$

$$TT' = \frac{l-x}{l}M_a + \frac{x}{l}M_b \ldots \text{by § 106.}$$

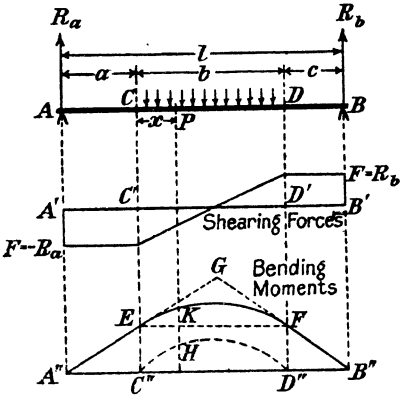
$$\therefore M = TR' = \frac{l-x}{l}M_a + \frac{x}{l}M_b + \frac{w}{2}(lx - x^2) \quad . \quad . \quad (18)$$

This could also be obtained by the usual method of taking moments about P for PB. From this we can deduce the position and value of the "maximum" bending moment.



R_a
R_b
l
x
M_a
M_b
P
Q
R_b
$-R_a$
A'
B'
N
R'
R
K
M_a & M_b both +
H
T'
T
A''
B''
R
T
I
J
M_a & M_b both −
A''
B''
R'
T'
H
K
R
K
R'
T
A''
B''
M_a −
M_b +
I
T'
H

FIG. 123.



R_a
R_b
l
a
b
c
C
D
A
B
x
P
$F=R_b$
C'
D'
A'
B'
Shearing Forces
$F=R_a$
G
Bending Moments
E
K
F
H
A''
C''
D''
B''

FIG. 124.

If M is a maximum we must have

$$\frac{dM}{dx} = -\frac{M_a}{l} + \frac{M_b}{l} + \frac{w}{2}(l - 2x) = 0$$

which gives

$$x = \frac{l}{2} - \frac{M_a - M_b}{wl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Substituting this value of x in (18) gives

$$M_{max} = \frac{wl^2}{8} + \frac{M_a + M_b}{2} + \frac{w}{2}\left(\frac{M_a - M_b}{wl}\right)^2 \quad . \quad . \quad . \quad (20)$$

We can now point out the significance of the word "mathematically" at the end of § 96, and why we have put *maximum* in inverted commas a few lines above. Equation (19) gives the value of x for which the curve of M reaches its highest point, and (20) gives the corresponding value of M, which, mathematically speaking, is its maximum value. But it does not follow that M_{max} given by (20) is numerically greater than M_a or M_b, and it will probably be the numerically greatest of the three which is required in considering the strength.

Points of Inflexion.—When M_a and M_b are both negative there will usually be two points in the beam AB, where the bending moment is zero, and, as we shall see later, where the curvature of the deformed beam changes sign; these points, I and J, are called the points of inflexion or points of contraflexure. Their position is found from the condition $M = 0$, i.e.

$$\frac{l-x}{l}M_a + \frac{x}{l}M_b + \frac{w}{2}(lx - x^2) = 0,$$

or

$$x^2 - x\left\{l - \frac{2(M_a - M_b)}{wl}\right\} - \frac{2M_a}{w} = 0 \quad . \quad . \quad . \quad . \quad (21)$$

The roots of this are

$$\frac{l}{2} - \frac{M_a - M_b}{wl} \pm \sqrt{\frac{l^2}{4} + \left(\frac{M_a + M_b}{w}\right) + \left(\frac{M_a - M_b}{wl}\right)^2} \quad . \quad (22)$$

The length IJ is equal to the difference between these roots, so that

$$\lambda = IJ = 2\sqrt{\frac{l^2}{4} + \left(\frac{M_a + M_b}{w}\right) + \left(\frac{M_a - M_b}{wl}\right)^2} \quad . \quad (23)$$

The distances of I and J each side of the centre of the beam AB can readily be shown to be

$$\frac{\lambda}{2} \pm \frac{M_a - M_b}{wl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

When M_a and M_b are of opposite signs one of the roots of (21) will be greater than l, so that there is only one point of inflexion, as shown at the bottom of Fig. 123.

We have entered into this problem in some detail as the results are of importance in connection with beams resting on several supports, and beams fixed in direction at the ends.

108. Freely Supported Beam with Uniformly Distributed Load over Part of the Length.—The beam AB (Fig. 124) carries a uniformly distributed load w per unit length over the portion CD; it is required to draw the shearing force and bending moment diagrams.

This problem can be treated in a similar way to the last, the portion CD being regarded as a beam with a distributed load and terminal couples equal to the bending moments at C and D.

Taking moments about A and B we find

$$R_a = \frac{bw}{l}\left(c + \frac{b}{2}\right) \text{ and } R_b = \frac{bw}{l}\left(a + \frac{b}{2}\right)$$

Then the bending moments at C and D are

$$\left.\begin{aligned} M_C &= aR_a = \frac{abw}{l}\left(c + \frac{b}{2}\right) \\ M_D &= cR_b = \frac{bcw}{l}\left(a + \frac{b}{2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The bending moment diagrams for the portions AC and BD are given by the lines $A''E$ and FB''. The B.M. diagram for CD by itself would be a parabola of height $wb^2/8$, as shown dotted; that due to the couples M_C and M_D is EF; the complete diagram is obtained by plotting the parabola $C''HD''$ again on EF as base obtaining the curve EKF..

Measuring x from C, the bending moment at P is easily shown, in the manner of § 107, to be

$$M = \frac{b-x}{b}M_C + \frac{x}{b}M_D + \frac{w}{2}(bx - x^2) \quad . \quad . \quad . \quad \text{(ii)}$$

From this we can show that the B.M. diagram might be obtained by drawing a parabola through E and F to touch $A''E$ and $B''F$:

We have from (ii)

$$\begin{aligned} \frac{dM}{dx} &= \frac{M_D - M_C}{b} + \frac{w}{2}(b - 2x) \\ &= \frac{b(c-a)w}{2l} + \frac{w}{2}(b - 2x), \text{ from (i)} \end{aligned}$$

At C, $x = 0$, and we have

$$\frac{dM}{dx} = \left\{\frac{b(c-a)}{2l} + \frac{b}{2}\right\}w = \frac{bw}{l}\left(c + \frac{b}{2}\right) = R_a.$$

But R_a is the slope of the line $A''E$, so that the curve (eq. ii) touches $A''E$ at E *; similarly, we can show it touches $B''F$ at F.*

The shearing force diagram presents no difficulties, and is shown in the figure on the line $A'B'$ as base,

108a. Useful General Method for Drawing Bending Moment Diagrams. The following set of rules has been devised to facilitate the drawing of bending moment diagrams when the load system is somewhat complicated.

(1) If any loads be applied to the beam in the way shown in Fig. 125, they should be replaced as follows: instead of W acting down CD, place a load W acting down AB, together with a couple of moment aW, having its axis at B, where AB cuts the neutral axis† of the beam.

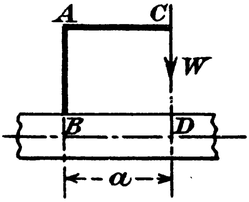


FIG. 125.

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* This can also be proved thus: since there is no concentrated load at C or D the shearing force, and therefore $\frac{dM}{dx}$ is continuous as we pass along the beam. Thus the slope of the curve of bending moments is continuous at C and D.

† The neutral axis is here understood to mean the line through the centroids of all cross-sections. It is fully explained in Chapter XI.

(2) By the ordinary rules of statics, find the values of the reactions on the supports, *ignoring all couples applied to the beam,* both those specified explicitly, and those introduced by (1) above.

(3) Draw the shearing force diagram: commencing at the right-hand end of the beam, draw a line which descends as downward forces are met and ascends when upward forces are met: the ascent or descent will be abrupt for localized forces, gradual for distributed loads (Fig. 126).

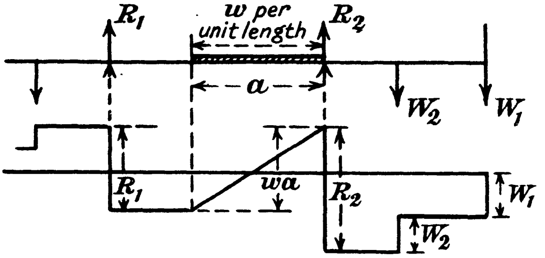


FIG. 126.

(4) Correct the shearing force diagram for the effect of the couples ignored in (3), by raising *the portion between the two supports* an amount equal to the total clockwise couple divided by the distance between the supports.

(5) Draw the bending-moment diagram thus: commencing at the right-hand end of the beam plot a graph representing the total area of the shearing force diagram up to any point. Areas below the zero line are to be taken negative, those above the zero line positive. At any point in the beam where there is a clockwise couple applied, a sudden drop, equal to the moment of this couple, must be made in the bending-moment diagram: after this drop continue as before. For anticlockwise couples make a corresponding abrupt rise in the bending-moment diagram.

Example 1.—A traction engine (Fig. 127) has a wheel base of 9 ft., and the front and back axle loads are 3 and 5 tons respectively; it is crossing a bridge with a span of 20 ft. and the front wheels have just reached the centre of the span. Draw diagrams of bending moment and shearing force on the bridge. (Special Exam., Cambridge, 1919.)

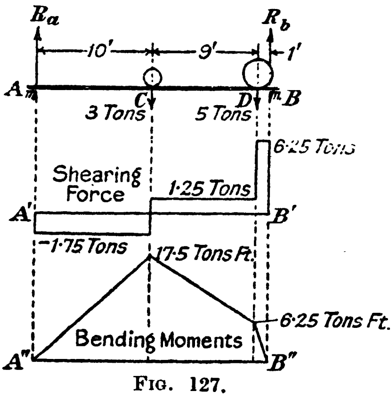


FIG. 127.

First we find R_a and R_b:

$$20 \text{ ft.} \times R_b = 3 \times 10 + 5 \times 19 = 125 \text{ tons. ft.}$$
$$\therefore R_b = 6{\cdot}25 \text{ tons.}$$
$$R_a = 8 - 6{\cdot}25 = 1{\cdot}75 \text{ tons.}$$

The shearing force between D and B is constant and $= 6{\cdot}25$ tons.

To the left of D the shearing force $= 6{\cdot}25 - 5 = 1{\cdot}25$ tons, and it has this value all along DC.

From C to A the shearing force $= 1{\cdot}25 - 3 = -1{\cdot}75$ tons $= -R_a$.
The bending moment at C $= 10$ ft. $\times R_a = 17{\cdot}5$ tons. ft.
„ „ „ „ D $= 1$ ft. $\times R_b = 6{\cdot}25$ tons. ft.

Since the forces are all localized at points the B.M. diagram will consist of straight lines as shown in Fig. 125.

The same result could be obtained by drawing the bending moment diagrams for the two loads separately and adding them together.

Example 2.—*AB is a vertical post of a crane (see Fig.* **128**). *The sockets at A and B offer no constraint against flexure. The horizontal arm CD is hinged to AB at C and supported by the strut FE which is freely hinged at its two extremities to AB and CD. Construct the bending moment diagrams for AB and CD.* (Intercoll. Exam., Cambridge, 1909.)

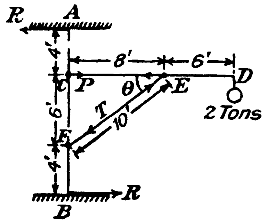


FIG. 128.

It is clear from considering the equilibrium of the whole crane that the horizontal reactions at A and B must be equal and opposite, and that the couple due to them must equal the moment of the 2-tons weight. Let R be the magnitude of the horizontal reactions at A and B, then

$$14 \text{ ft.} \times R = 14 \text{ ft.} \times 2 \text{ tons.}$$
$$\therefore R = 2 \text{ tons.}$$

Let P = the pull in CE, and T = the thrust in FE. Then taking moments about C for the rod CD we have

$$8 \text{ ft.} \times T \sin\theta = 14 \times 2 \text{ tons ft.}$$
$$\therefore T = \frac{7}{2} \operatorname{cosec}\theta \text{ tons} = \frac{7}{2} \times \frac{10}{6} = \frac{35}{6} \text{ tons.}$$

Resolving horizontally for AB we have

$$P = T\cos\theta = \frac{7}{2}\cot\theta = \frac{7}{2} \times \frac{8}{6} = \frac{14}{3} \text{ tons.}$$

The vertical reaction at $E = T\sin\theta = \frac{7}{2}$ tons.

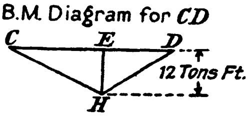


FIG. 129.

We can now draw the bending moment diagrams for AB and CD, considering only the forces at right angles to each beam; let us take CD first. CD is a beam freely supported at C and E and loaded at D. The B.M. at $E = 6$ ft. $\times$ 2 tons $= 12$ tons-ft., to which value it rises uniformly from zero at D; from E to C the B.M. decreases uniformly (§ 97) to zero. The diagram is shown in Fig. 129.

AB is supported at A and B and loaded with equal and opposite loads at C and F.

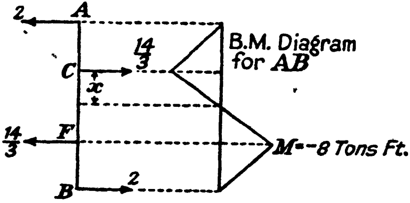


FIG 130.

The bending moment at $C = 2$ tons $\times$ 4 ft. $= 8$ tons. ft.
 „ „ „ „ $F = -2$ tons $\times$ 4 ft. $= -8$ tons ft.
At any point x between C and F, the B.M. is

$$M = 2 \text{ tons } (x + 4) \text{ ft.} - \frac{14}{3} \text{ tons } x \text{ ft.} = 8 - 1\,\frac{8}{3}x \text{ tons. ft.}$$

The B.M. diagram is therefore as shown in Fig. 130, positive bending moments being those which make the beam concave to the left, and plotted to the left in the figure.

Example 4.—The girder shown in Fig. 131 is supported by a wall at A, and by a stanchion at B, the points of support being 12 ft. apart. Between A and B there is a uniform floor load of 30 tons carried directly by the girder; between B and D there is a uniform floor load of 18 tons, carried by floor beams which rest on the girder at B and D. There is also a concentrated load of 24 tons at a point C, 9 ft. from A. Draw the diagrams of shearing force and bending moment, marking the values of the principal ordinates. Also find the position of the point at which the flexure of the beam is reversed. (Mech. Sc. Trip., 1914.)

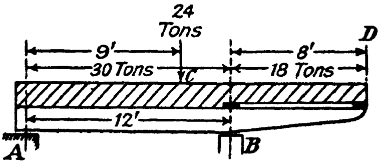


FIG. 131.

The load distributed on $AB = \dfrac{30 \text{ tons}}{12 \text{ ft.}} = 2{\cdot}5$ tons/foot.

The load on BD gives concentrated loads of 9 tons each at B and D.

Thus the load system for the beam is as shown at the top of Fig. 132.

The B.M. at $B = -9$ tons $\times$ 8 ft. $= -72$ tons. ft., and the diagram for the part BD is the straight line $D'M$.

The B.M. diagram for AB will be the sum of those due to—

(i) A couple $= -72$ tons. ft. at B. This gives the straight line (§ 105) MA'. At a section distant x from A, the B.M.

$$= -\frac{x}{12} \times 72 = -6x.$$

(ii) A concentrated load of 24 tons at C. This gives the two straight lines $B'L$ and LA' (§ 101) where $LC' = \dfrac{24 \times 9 \times 3}{12} = 54$ tons. ft.

At a section distant x from A, the B.M. $= \dfrac{x}{9} \times 54 = 6x$ if

$$0 < x > 9 \text{ ft.}$$

Thus over AC the B.M.'s due to the overhang and the concentrated load cancel each other.

If 9 ft. $< x <$ 12 ft., the B.M. due to the concentrated load

$$= \frac{12 - x}{3} \times 54 = 18(12 - x).$$

(iii) A distributed load of 2·5 tons/ft. This gives the parabola $A'PB'$ of maximum height $\dfrac{2{\cdot}5 \times 144}{8} = 45$ tons. ft.

For any section between A and B the corresponding B.M. is (§ 102) $1{\cdot}25\,(12x - x^2)$.

The resultant B.M. diagram for AC, then, will be part of this parabola, and for any section between A and C, where

$$0 < x < 9 \text{ ft.}, \quad M = 1{\cdot}25(12x - x^2) \text{ tons. ft.}$$

The maximum is when $x = 6$ ft., and then $M = 45$ tons. ft.

At C, $x = 9$ ft., $M = 1{\cdot}25\ (108 - 81) = 33{\cdot}75$ tons. ft.

For CB, 9 ft. $< x <$ 12 ft.,

$$\begin{aligned} M &= -6x + 18(12 - x) + 1{\cdot}25(12x - x^2) \\ &= 216 - 9x - 1{\cdot}25x^2 \text{ tons. ft.} \end{aligned}$$

which is zero when

$$1{\cdot}25x^2 + 9x - 216 = 0$$

the positive root of which is

$$x = 10{\cdot}05 \text{ ft.}$$

This gives the point of inflexion J.

The complete B.M. diagram is shown by the thick curve.

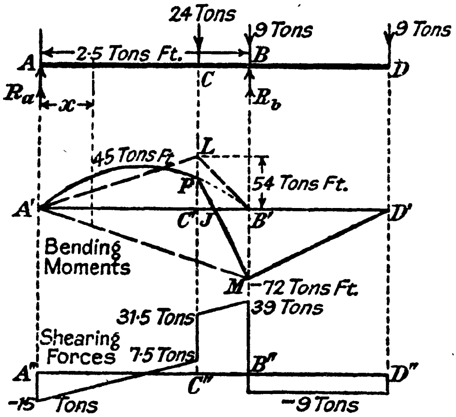


FIG. 132.

We will now consider the shearing force diagram.

Taking moments about A gives $R_b = 57$ tons, and since the total load is 72 tons, $R_a = 15$ tons.

The shearing force from D to B is constant and $= -9$ tons.

To the left of B, the shearing force $= -9 - 9 + 57 = 39$ tons.

The load on $BC = 7{\cdot}5$ tons, therefore the shear to the right of $C = 39 - 7{\cdot}5 = 31{\cdot}5$ tons;

To the left of C the shear $= 31{\cdot}5 - 24 = 7{\cdot}5$ tons.

The load on $CA = 22{\cdot}5$ tons, therefore the shear to the right of $A = 7{\cdot}5 - 22{\cdot}5 = -15$ tons.

The shear is shown in the diagram on $A''\ D''$ as base.

Example 4.—Fig. 133 shows diagrammatically the arrangement of the top front spar of a certain aeroplane. The spar is supported by pin-joints at A and B. The part AB carries a distributed load of 100 lbs. per foot, and the overhang BD carries a distributed load which decreases in intensity uniformly, from 100 lbs./ft. at B to 70 lbs./ft. at D. The load acts upwards and the reaction at B is taken by a wire BC. The pin B to which the wire is fixed has its axis 1″ below the axis of the spar. The pin at A, by which the spar is attached to the rest of the structure, has its axis ½″ below the axis of the spar. It is required to draw the B.M. diagram for the spar.

The load distribution is shown graphically by the line LMN with the spar axis as base.

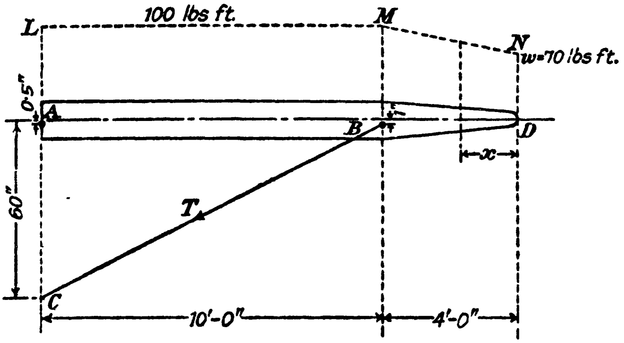


FIG. 133.

Let w, F, and M denote the load intensity, the shear, and the bending moment at a section in BD distant x ft. from D, the units being pounds and feet. Then

$$w = 70 + \frac{30x}{4} \text{ lbs./ft.}$$

$$\therefore F = \int_0^x w dx = 70x + \frac{15x^2}{4} \text{ lbs.} \quad . \quad . \quad . \quad \text{(i)}$$

$$\text{and } M = \int_0^x F dx = 35x^2 + \frac{5x^3}{4} \text{ lbs. ft.} \quad . \quad . \quad . \quad \text{(ii)}$$

When $x = 4'$, that is at B,
$F = 340$ lbs., which gives the total load on BD.
$M = 640$ lbs. ft., which is the moment of this load about B.

Hence the centre of force of this load is $\frac{640 \text{ lbs. ft.}}{340 \text{ lbs.}} = 1{\cdot}88$ ft. from B.

Let R_b = the vertical reaction at B. Then taking moments about A:

$$10 \text{ ft.} \times R_b = 5 \times 1{,}000 + 11{\cdot}88 \times 340 \text{ lbs. ft.}$$
$$= 9{,}040 \text{ lbs. ft.}$$
$$\therefore R_b = 904 \text{ lbs.}$$

Let T = the tension in BC. Then

$$T = \frac{BC}{CA} \times 904 \text{ lbs.}$$

The horizontal component of this is

$$\frac{AB}{BC} \times T = 904 \text{ lbs.} \times \frac{AB}{CA} = 1{,}808 \text{ lbs.}$$

This produces a bending moment at B, acting on AB, equal to 1,808 lbs. × $\frac{1}{12}$ ft. = 151 lbs. ft., in the opposite direction to the moment due to the load on the overhang.

Thus we shall have a discontinuity at B in the B.M. diagram: just to the right of B the bending moment is 640 lbs. ft. on account of the overhang, and in passing to the left of B this is decreased by 151 lbs. ft., making there a total of 489 lbs. ft. This tends to make the spar AB concave upwards.

The horizontal component of the pull in the wire must be balanced by an equal thrust at A, which will also give rise to a bending moment of 1,808 lbs. × $\frac{1}{24}$ ft. = 75 lbs. ft., tending to make AB concave downwards.

The distributed load on AB gives a parabolic bending moment curve of height

$$\frac{100 \times 10^2}{8} = 1{,}250 \text{ lbs. ft.},$$

making the beam AB concave downwards.

Thus the B.M. diagram is as shown in Fig. 134. The sloping dotted line is the B.M. due to the couples 489 and -75 lbs. ft. applied to the ends of AB (see § 106); the dotted curve is the parabola for the distributed load on AB by itself. The thick black curve is the resulting bending moment diagram.

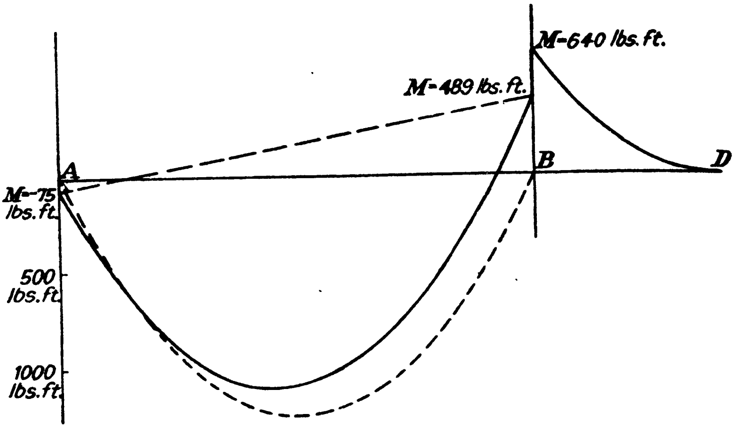


Fig. 134.

EXAMPLES IX

Draw the bending moment diagram and shearing force diagrams for the following beams.

1. A cantilever carrying a load of 10 tons at a distance of 15 ft. from the supported end.

2. A cantilever carrying a load of 10 tons uniformly distributed over the inner 15 ft. of its length.

3. A cantilever carrying a load of 8 tons, 5 ft. from the supported end, and a load of 0·5 ton per foot over its whole length, which is 12 ft.

4. A beam 20 ft. long freely supported at each end and carrying a load of 20 tons at its middle point.

5. The same beam when the load is 5 ft. from one end.

6. A beam 20 ft. long, freely supported at each end, and loaded with 100 lbs. per foot.

7. A beam 16 ft. long is freely supported at each end and loaded with 500 lbs. at a point 4 ft. from one support, the weight of the beam being 50 lbs. per foot.

8. A horizontal cantilever 7 ft. long carries a distributed load of 125 lbs. per ft., which extends over the middle 5 ft. of the beam and acts downwards. There are also concentrated loads of 400 lbs. downwards at a point 3 ft. from the support, and 200 lbs. upwards at the outer end. Draw the diagrams of bending moment and shearing force.

9. Draw to scale Shearing Force and Bending Moment diagrams for a horizontal beam of 10 ft. length supported at two points 1 ft. and 9 ft. from one end, and loaded as follows, distances being all measured from the same end :—1 ton at 0, 2 tons at 3 ft., 2 tons at 7 ft., and 1 ton at 10 ft. (Special Exam., Cambridge, 1907.)

10. Fig. 135 shows two of the girders *AC* and *BC* strengthening a pair of lock-gates. If the load on each girder amounts to 1,500 lbs. per ft. run, find the bending moment at the middle of each girder. (Intercoll. Exam., Cambridge, 1912.)

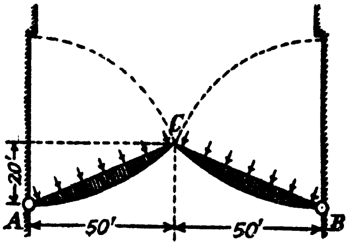


FIG. 135.

11. A cylindrical mast 8" diameter and 60 ft. high, is pivoted at its base and stayed in a vertical position by means of four light ropes (*N*, *S*, *E*, and *W* in plan) attached to the pole by a ring 40 ft. above the pivot. Sketch the bending moment diagram for the pole, for the loads due to a wind producing a pressure of 30 lbs. per ft.^{2} of projected area. It may be assumed that the flexure of the pole is negligible. (R.N.E.C., Keyham, 1918.)

12. A beam 60 ft. long is supported horizontally by vertical reactions at one end *A*, and at a point *B*, 40 ft. from *A*. There is a uniformly distributed load of 0·5 ton per foot over the whole beam, a concentrated load of 12 tons at the middle point of *AB*, and another concentrated load of 8 tons at the end of the overhanging portion. Draw accurately the Bending-Moment and Shearing-Force diagrams for the beam. (Intercoll. Exam., Cambridge, 1914.)

13. Fig. 136 shows a girder *ABCDE* bearing on a wall for a length *BC* and prevented from overturning by a holding-down bolt at *A*. The packing under *BC* is so arranged that the pressure over the bearing is uniformly distributed and the 3 ton load may also be taken as a uniformly distributed load. Neglect the weight of the beam and draw its B.M. and S.F. diagrams. (Intercoll. Exam., Cambridge, 1912.)

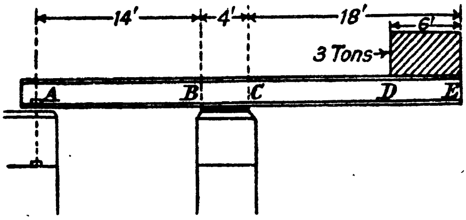


FIG. 136.

14. *ABC* is a straight horizontal beam resting upon two supports *A* and *B* 80 ft. apart, the overhanging portion *BC* being 40 ft. long. The girder carries a load of 1 ton per foot run distributed over its whole length, a concentrated load of 10 tons at *C*, and a concentrated load of 20 tons at the middle point of *AB*. Use graphical methods to construct the B.M. diagram for the beam to the following scales : 1" to represent 400 tons. ft., 1" to represent 10 ft. (Intercoll. Exam., Cambridge, 1913.)

15. Fig. 137 gives the positions and magnitudes of the loads on a girder due to a locomotive at rest. Treating the girder as supported at the points

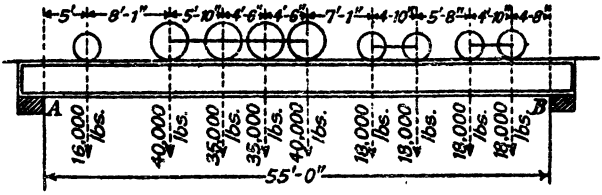


FIG. 137.

A and *B*, draw the bending moment and shearing force diagram. (Intercoll. Exam., Cambridge, 1909.)

16. *ABC* is a vertical post projecting 12 ft. above a concrete emplacement. *CD* is a horizontal bar fixed by a pin-joint to the top *C*. The outer end of *CD* is connected to the post by a sloping rod *DB*, pin-jointed at both ends. *CD* is 6 ft. long and the angle $CBD = 30°$. A load of 1 ton is suspended from *D* Draw S.F. and B.M. diagrams for the post.

17. Draw the bending moment and shearing force diagrams for the beam shown in Fig. 138. The beam is supported horizontally by the strut *DE*, hinged at one end to a wall, and at the other end to the projection *CD* which

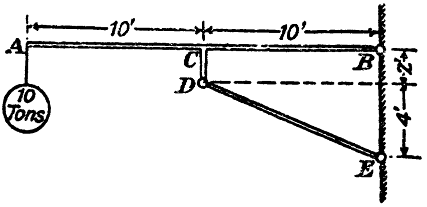


FIG. 138.

is firmly fixed at right angles to *AB*. The beam is freely hinged to the wall at *B*. The weight of the beam and strut can be neglected. (Mech. Sc. Trip., 1910.)

18. A timber dam is made of planking backed by vertical piles in the manner shown in Fig. 139. The piles are encastered at the section *A* where they enter the ground and they are supported by horizontal struts whose centre lines are 30 ft. above *A*. The piles are spaced 4 ft. apart between centres and the depth of water against the dam is 30 ft.

Assuming that the thrust in the strut is two-sevenths the total water pressure resisted by each pile, sketch the form of the Bending Moment and Shearing Force diagram for a pile. Determine the magnitude of the bending moment at *A* and the position of the section which is free from bending moment. (Intercoll. Exam., Cambridge, 1913.)

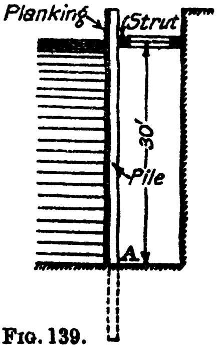


FIG. 139.

19. A cantilever of length l carries a distributed load which varies uniformly from w_1 per unit length at the inner end to w_2 at the outer end. Calculate the bending moment at the support.

20. The pressure distribution along the overhanging portion of the top wing of a certain aeroplane is given by the following table:—

Distance from support (ft.)	0	1	2	3	4	4·5	4·8	5
Load, lbs./inch. . .	9·85	9·45	8·75	7·9	6·7	5·7	4·2	0

Draw curves of bending moment and shearing force from the end of the wing to the support.

21. A beam rests on supports 13 ft. apart. The load increases at a uniform rate from zero at one end to a maximum at the other. The total load is 10 tons. Draw the Shearing-Force and Bending-Moment diagrams. (R.N.E.C., Keyham, 1921.)

22. A light shaft, running at 300 r.p.m. in two spherical bearings 10 ft. apart in a ship, is fitted with a flywheel, overhung 2 ft. from one bearing. The flywheel weighs 200 lbs. and its radius of gyration is 1 ft. If the vessel turns at the rate of 1° per second, draw Shearing-Force and Bending-Moment diagrams for the shaft in a vertical plane. Indicate the directions of rotation which are assumed. (R.N.E.C., Keyham, 1922.)

23. The load distribution (full lines) and upward water thrust (dotted lines) for a ship are given in Fig. 140, the numbers indicating tons per foot run. Draw the B.M. diagram for the ship to the following scales: 20 ft. to 1″ and 400 tons. ft. to 1″. (Mech. Sc. Trip., 1911.)

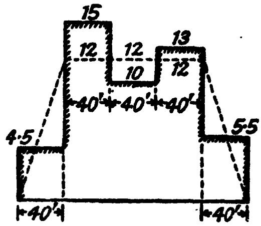


FIG. 140.

24. A beam of length l, supported at each end, carries a distributed load given by $w = w_0 \sin \frac{\pi x}{l}$, where x is measured from one of the supports. Find an expression for the bending moment at any section.

25. To obtain the form of the longitudinal bending moment and shear force diagrams for a floating dock when in light condition, consider a rectangular box length 12 ft. under total loading distributed along its lengths as follows: $\frac{1}{4}$ lb. per ft. run for 4 ft. each side of mid-length, and $\frac{1}{12}$ lb. per ft. for the two remaining end lengths (each 2 ft.). Draw the B.M. and S.F. diagrams for the hypothetical model if it were floating in water. (R.N.E.C., Keyham, 1928.)

26. A beam 50 ft. long resting on supports 10 ft. from each end carries a uniformly distributed load of 1 ton per foot run. At mid-length it is subjected to a couple of 200 tons ft., acting about an axis perpendicular to a plane containing the centre lines of beam and supports. Plot curves of shear force and bending moment. (R.N.E.C., Keyham, 1927.)

27. A cantilever is loaded as follows: 10 ft. from the fixed and a load of 10 tons; for the next 10 ft. a load of 2 tons per foot; at 20 ft. from the fixed end a positive couple of 400 tons. ft. Draw the shear force and bending moment diagrams. (R.N.E.C., Keyham, 1926.)

CHAPTER X

BENDING MOMENTS AND SHEARING FORCES DUE TO TRAVELLING LOADS

109. Introductory.—In the design of bridge girders it is frequently necessary to know the maximum bending moment and shearing force which each section will have to bear when a travelling load, such as a train, passes from one end of the bridge to the other. The diagrams which we have considered in Chapter IX show the simultaneous values of the bending moment, or shearing force, for all sections of the beam with the loads in one fixed position; we shall now see how to draw a diagram which shows the greatest value of these quantities for all positions of the loads. These diagrams are called maximum bending moment, or maximum shearing force, diagrams.

110. A Single Concentrated Load Crossing a Beam.—Let a single load W travel along a beam AB (Fig. 141) which is freely supported at each end. Let y be the distance of the load from A, and let P be any section, distant x from A.

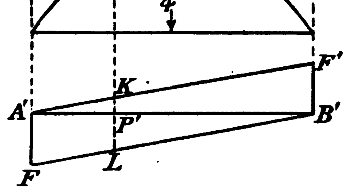


FIG. 141.

The reaction at A, $R_a = W\frac{l-y}{l}$.

Let M = the bending moment at P, then, if $x < y < l$, from (ii), p. 123

$$M = W.\frac{l-y}{l}.x \qquad \text{(i)}$$

From this we see that M increases as y decreases, i.e. as W gets nearer to P. That is, whether the load is coming on or going off the beam from the end B, the bending moment is greater the nearer W is to P. Similarly, if the load is coming on or going off from the end A, the bending moment is greater the nearer W is to P. Thus the bending moment at P will be a maximum when W is at P.

Let $\overline{M}$ denote the maximum B.M. at P, then, putting $y = x$ in (i) we have

$$\overline{M} = Wx\left(1 - \frac{x}{l}\right) \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The greatest value of $\overline{M}$ is when $x = \frac{l}{2}$, i.e. for the central section, and it then equals $Wl/4$.

Equation (1) represents a parabola, with its axis vertical and passing through the middle of the beam. Hence the maximum B.M. diagram is the parabola shown; it is the same as the ordinary B.M. diagram for a distributed load $2W/l$ per unit length, for this gives a parabola of height

$$\frac{2W}{l}\left(\frac{l^2}{8}\right) = \frac{Wl}{4}.$$

Let us now consider the shearing force. Referring to Fig. 141, the shearing force at P is given by

$$F = R_b - W = W\left(\frac{y}{l} - 1\right) \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

when $x < y < l$. Hence, when the load is to the right of P the shearing force is negative and increases numerically as y increases, i.e. the nearer W is to B.

When $y < x$, the shearing force at P is

$$F' = R_b = \frac{y}{l}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

which is positive and increases with y, i.e. the nearer W is to P.

Hence, if we consider the load coming on at B the shearing force at any section P is negative and decreases numerically as the load approaches the section; as the load crosses the section the shearing force changes sign and decreases as the load recedes towards A. Thus the shearing force, like the bending moment, is greatest when the load is at the section under consideration.

Putting $y = x$ in (iii) and (iv) we have, for any section,

$$\text{Maximum negative shearing force} = -\,W\left(1 - \frac{x}{l}\right)$$

$$\text{,,} \quad \text{positive} \quad \text{,,} \quad \text{,,} \quad = \quad W\frac{x}{l}.$$

These are represented by the two straight lines $B'LF$ and $A'KF'$: the maximum positive shearing force for the section P is KP', and the maximum negative shearing force is $P'L$. For purposes of stress calculation the sign of shearing force is not important; it is the greatest numerical value, positive or negative, that is required.

111. Uniformly Distributed Travelling Load of Sufficient Length to Cover the Whole Span.—The bending moment on every section will have its maximum value when the whole span is loaded, for any addition to the load on the beam must increase the bending moment at all sections. Thus the maximum B.M. diagram is identical with the ordinary B.M. diagram in this case, that is, it is a parabola of height $wl^2/8$, in our usual notation (p. 124).

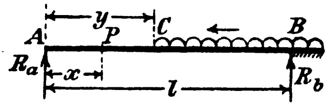


FIG. 142.

Now, consider the shearing forces. In Fig. 142 suppose that the load is advancing from the right and that the front of it has reached C where $AC = y$. Let P be any section of the beam, where $AP = x$. Then by taking moments about A and B in turn in the usual way we find

$$R_a = \frac{w(l-y)^2}{2l} \text{ and } R_b = \frac{w(l^2-y^2)}{2l}$$

Then, provided $x \not> y$, the shearing force at P is

$$F = -R_a = -\frac{w(l-y)^2}{2l}$$

This is negative and increases numerically as y decreases until $y = x$. When this happens we have

$$F = -\frac{w(l-x)^2}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

When the front of the load crosses P, we have $x > y$, and

$$\begin{aligned} F &= -R_a + w(x-y) \\ &= -\frac{w(l-y)^2}{2l} + w(x-y) \\ &= -\frac{w}{2l}(l^2+y^2-2lx) \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)} \end{aligned}$$

which decreases numerically as y decreases. Thus the greatest negative value of F is when the part of the beam PC is covered by the load, and the maximum value is given by (i).

After the front of the load has passed P we see from (ii) that F changes sign when

$$l^2 + y^2 = 2lx,$$

that is, when $y = \sqrt{l(2x-l)}$. Thus F does not change sign unless $x > l/2$.

Now, consider the case when the rear end of the load has passed B, as in Fig. 143. We have now

$$R_b = \frac{wy^2}{2l}$$

If $x < y$, the shearing force at P is given by

$$F = R_b - w(y - x)$$

$$= w\left(\frac{y^2}{2l} - y + x\right)$$

which increases as y decreases, until $y = x$, and then

$$F = \frac{wx^2}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

which is positive.

When $x > y$, we have

$$F = R_b = \frac{wy^2}{2l}$$

which decreases with y. Thus F has its greatest positive value when $y = x$, i.e. when AP is covered by the load, and it is given by (iii). We see that the maximum shearing force diagram is as shown in Fig. 143, which is to be interpreted thus: as the load advances from B the shearing force at P is negative and increases numerically to the value $P'L$, which it reaches when the front of the load is at P. As the load passes over P the shearing force gradually changes to the positive value $P'K$, as the rear of the load passes P, and then decreases. The curves $A'KF$ and $B'LF'$ give the maximum positive and negative shearing forces for any section of the beam.

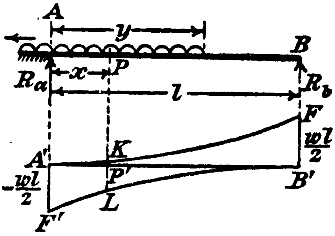


FIG. 143.

112. Two Concentrated Loads.—An example of this case is afforded by any ordinary four-wheeled vehicle crossing a bridge. In these cases the following analysis may be employed, but when there are more than two concentrated loads the method of the next article is more suitable.

Let W_1 and W_2 be the loads, and let a be their constant distance apart. Let P be any section of the beam at a distance x from A (Fig. 144). Now, from the considerations of the last article, it is evident that as the loads advance towards P from either end the bending moment at P increases until one of the loads is over the section. Therefore the B.M. at P will be greatest when (a) W_1 is at P, or (b) when W_2 is at P, or (c) when P is between W_1 and W_2; these three positions are shown in Fig. 144.

Let M_a, M_b, M_c denote the B.M. at P in the three cases.

In case (a) we have

$$R_b = \frac{W_1x + W_2(x - a)}{l}$$

$$\therefore M_a = (l - x)R_b = \frac{l - x}{l}\{(W_1 + W_2)x - W_2a\} \quad . \quad \text{(i)}$$

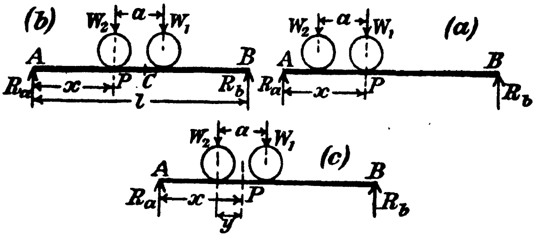


Fig. 144.

In case (*b*) we have

$$R_a = \frac{W_2(l-x) + W_1(l-x-a)}{l}$$

$$\therefore M_b = xR_a = \frac{x}{l}\{(W_1 + W_2)(l-x) - W_1 a\} \quad . \quad . \quad . \quad \text{(ii)}$$

In case (*c*) we have

$$R_b = \frac{W_2(x-y) + W_1(x+a-y)}{l}$$

$$\therefore M_c = (l-x)R_b - (a-y)W_1$$

$$= \frac{l-x}{l}\{(W_1 + W_2)(x-y) + W_1 a\} - (a-y)W_1 \quad . \ \text{(iii)}$$

Now it can be easily verified from these equations that we can write M_c in either of the two forms:

$$M_c = M_a - \frac{a-y}{l}\{xW_1 - (l-x)W_2\}$$

or

$$M_c = M_b + \frac{y}{l}\{xW_1 - (l-x)W_2\}.$$

Therefore, if

$$xW_1 > (l-x)W_2$$

we have $M_a > M_c > M_b$, but if

$$xW_1 < (l-x)W_2$$

then

$$M_b > M_c > M_a.$$

Hence, in either case M_c lies between M_a and M_b, i.e., **the B.M. at P is greatest when either W₁ or W₂ is at P.**

From (i) and (ii) we find

$$M_a - M_b = \frac{a}{l}\{xW_1 - (l-x)W_2\}$$

Hence $M_a > M_b$ if $\frac{x}{l-x} > \frac{W_2}{W_1}$ and conversely. Therefore, if we divide the beam at C so that $AC/CB = W_2/W_1$, any section in AC has its greatest bending moment when W_2 is at the section, and any section in CB has its greatest bending moment when W_1 is at the section.

Thus, for any section where $x > AC$, the maximum B.M. is given by (i), and when $x < AC$ by (ii). These equations represent two parabolas with their axes vertical.

Taking equation (i) we see that the parabola cuts the axis of x when $x = l$ and $x = \frac{W_2 a}{W_1 + W_2}$. Its maximum height is when

$$x = \frac{l}{2} + \frac{a}{2}\frac{W_2}{W_1 + W_2}.$$

Substituting this value in (i) gives

$$M_a = \frac{\{l(W_1 + W_2) - aW_2\}^2}{4l(W_1 + W_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The parabola given by (ii) cuts the axis when $x = 0$ and

$$x = l - \frac{W_1 a}{W_1 + W_2}.$$

Its maximum height occurs when $x = \frac{l}{2} - \frac{a}{2}\frac{W_1}{W_1 + W_2}$. Substituting this in (ii) gives

$$M_b = \frac{\{l(W_1 + W_2) - aW_1\}^2}{4l(W_1 + W_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $W_1 > W_2$, M_a will be $> M_b$, i.e. the maximum B.M. will occur under W_1, and (2) gives the greatest bending moment on the beam.

The two parabolas are shown in Fig. 145 by the curves (i) and (ii), and the maximum B.M. diagram for the beam is the discontinuous curve ADB.

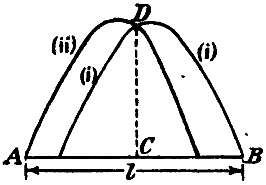


FIG. 145.

We have assumed in the above analysis that if either load goes off the bridge it continues to act, which is of course ridiculous. Let W_1 be the greater load, then, according to the above, the greatest bending moment will be at P (Fig. 144) when W_1 is at P and

$$x = \frac{l}{2} + \frac{a}{2} \cdot \frac{W_2}{W_1 + W_2}$$

or

$$x - \frac{aW_2}{W_1 + W_2} = l - x$$

i.e. when W_1 and the c.g. of the two loads are equidistant from the ends of the bridge.

If $x < a$ this will make W_2 come off the bridge, and (2) will no longer give the maximum bending moment. With W_2 off the bridge, the greatest B.M. which W_1 can produce is $W_1 l/4$. We must therefore inquire whether

$$\frac{\{l(W_1 + W_2) - aW_2\}^2}{4l(W_1 + W_2)} > \frac{W_1 l}{4}.$$

Let $W_1/W_2 = \lambda$, and $a/l = k$. Then the inequality becomes

$$\frac{\{(1+\lambda)-k\}^2}{4(1+\lambda)} > \frac{\lambda}{4}$$

i.e.

$$k^2 - 2k(1+\lambda) + (1+\lambda) > 0$$

i.e.

$$\left[k - \left\{(\lambda+1) + \sqrt{\lambda^2+\lambda}\right\}\right]\left[k - \left\{(\lambda+1) - \sqrt{\lambda^2+\lambda}\right\}\right] > 0$$

If

$$k < (\lambda+1) - \sqrt{\lambda^2+\lambda}$$

both factors will be negative and the inequality will be satisfied. Hence this is the condition that (2) gives the maximum bending moment on the beam when $W_1 > W_2$. The condition * is:

$$\frac{a}{l} < \frac{W_1}{W_2} + 1 - \sqrt{\left(\frac{W_1}{W_2}\right)^2 + \frac{W_1}{W_2}} \quad . \quad . \quad . \quad . \quad (4)$$

We must now consider the shearing forces. When both loads are to the right of any given section P, the shearing force at $P = -R_a$, which increases numerically as the loads are moved to the left until W_2 crosses the section. The shearing force then becomes $-R_a + W_2$ which increases until W_1 crosses the section, when it changes sign and becomes $W_1 + W_2 - R_a$. The greatest negative shearing force will therefore be when either W_1 or W_2 is at P.

With W_2 just to the right of P the shearing force is given by

$$-F = R_a = \frac{l-x}{l}(W_1 + W_2) - \frac{a}{l}W_1 \quad . \quad . \quad . \quad (5)$$

With W_1 just to the right of P we have

$$\begin{aligned} -F &= R_a - W_2 \\ &= \frac{(l-x)(W_1+W_2) + aW_2}{l} - W_2 \\ &= \frac{l-x}{l}(W_1+W_2) - \frac{l-a}{l}W_2 \quad . \quad . \quad . \quad (6) \end{aligned}$$

The value of $-F$ given by (5) will be greater than that given by (6) if

$$\frac{a}{l}W_1 < \frac{l-a}{l}W_2$$

i.e. if

$$a(W_1 + W_2) < lW_2$$

or

$$\frac{a}{l} < \frac{W_2}{W_1 + W_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If this is not true the greatest value of $-F$ will be that given by (6). In both cases the above equations only hold good as long as both loads are on the beam, i.e. between $x = 0$ and $x = l - a$ for equation (5), and between the limits $x = a$ and $x = l$ for equation (6).

To make the argument more definite let us limit ourselves to the case

---

* The proof of this is due to J. W. Landon.

when condition (7) is satisfied, so that $-F$ is given by (5), over the range $x = 0$ to $x = l - a$. When $(l - a) < x < l$, W_1 is off the girder and we have

$$-F = R_a = \frac{l - x}{l} W_2 \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Thus the diagram of maximum negative shearing force is as shown by the lines *EFB* in Fig. 146, the line *EF* being given by (5), and the line *FB* by (iv).

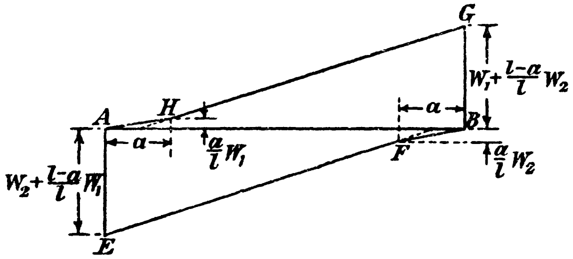


FIG. 146.

Similarly, the maximum positive shearing force is given by the lines *AHG*.

The case when $a/l > W_2/(W_1 + W_2)$ can be dealt with in the same way.

We can now summarize our results as follows ($W_1 > W_2$) :—

(*a*) If we divide the beam at *C* so that $AC/CB = W_2/W_1$ for any section in *AC* the greatest B.M. occurs when W_2 is at the section and is given by equation (ii) above ; for any section in *CB* it occurs when W_1 is at the section and is given by equation (i). This is always provided the condition (4) is fulfilled, which is usually the case.

(*b*) The maximum positive and negative shearing forces on any section are given by the lines *AHG* and *BFE* of Fig. 146, provided condition (7) is fulfilled. If this condition is not fulfilled the negative shearing force is given by (6) instead of (5).

(*c*) If condition (4) is not fulfilled the maximum B.M. is $W_1 l/4$.

113. Several Concentrated Loads.—In Fig. 147 let $W_1, W_2 \ldots W_n$

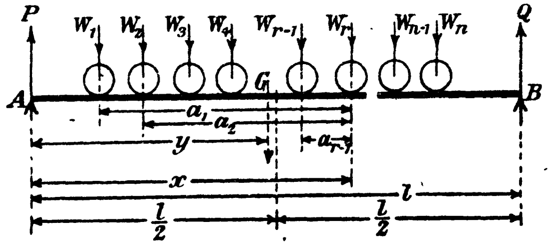


FIG. 147.

be a series of concentrated loads crossing the beam *AB* from left to right. It is evident from the latter part of § 112 that we cannot assume that the absolute maximum bending moment will occur when all the loads are on the beam, and on account of this we are not able to give general formulæ as simple as those for two travelling loads.

A little consideration will show that the maximum bending moment will occur under one of the loads. Let us, therefore, examine the B.M. at the rth load.

Let M_r = the bending moment under W_r, when that load is at a distance x from A. Then, referring to Fig. 147, we have

$$M_r = Px - (W_1a_1 + W_2a_2 \dots + W_{r-1}a_{r-1}).$$

This will be a maximum when

$$\frac{dM_r}{dx} = 0$$

that is, when

$$x\frac{dP}{dx} + P = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

provided the value of x given by this equation does not entail any of the loads being off the beam.

Now suppose all the loads advance a distance δx to the right. The reaction P is given by

$$lP = \sum_1^n (l - x + a_r)W_r$$

a being reckoned negative for loads between W_r and Q.

Therefore

$$l.\delta P = -\sum_1^n .W_r . \delta x$$

Hence, in the limit,

$$l\frac{dP}{dx} = -\sum_1^n W_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Let y = the distance of the c.g. of *all* the loads from A. Then, taking moments about B,

$$lP = (l - y)\sum_1^n W_r . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

But, from (i) and (ii) we have

$$lP = -lx\frac{dP}{dx} = x.\sum_1^n .W_r . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Comparing (iii) with (iv) we see that we must have

$$l - y = x$$

or

$$\frac{l}{2} - y = x - \frac{l}{2}.$$

Hence M_r is a maximum when W_r and the c.g. of all the loads are equidistant from the supports, or from the centre of the beam, provided this does not entail any of the loads going off the bridge; if it does, we must examine the matter further.

This result is of considerable help in examining the maximum bending moment, but for the rest we must be guided by general considerations, such as the fact that the greatest bending moment will usually occur in the middle portion of the beam.

Example.—A series of loads as shown in Fig. 148 passes over a bridge of 45 ft. span from left to right; it is required to find the maximum bending moment which the bridge will have to bear.

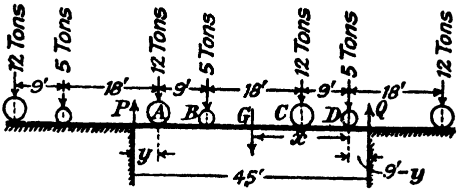


FIG. 148.

First find the horizontal position of the c.g. of all the loads on the bridge: it will be seen that four is the greatest number of loads which can be on the bridge simultaneously, as shown in Fig. 148.

Let $\bar{x}$ be the distance from D of the c.g. of the loads A, B, C, D which are on the bridge. Then

$$\bar{x} = \frac{12 \times 36 + 5 \times 27 + 12 \times 9}{34} = 19{\cdot}9 \text{ ft.}$$

The maximum bending moment will occur under one of the loads, so let us take each load in turn and see what is the greatest bending moment which occurs under each. Maximum bending moment under load A:—

This will occur when A and G are equidistant from the supports, provided one of the loads does not go off the bridge. Referring to the diagram we see that this requires:

$$y = (9 - y) + \bar{x} = 28{\cdot}9 - y$$
$$\therefore\ y = 14{\cdot}45 \text{ ft.}$$

But this will make D go off the bridge. Hence the above breaks down; with all the loads on the bridge the bending moment under A increases continuously as D gets nearer to the right-hand end. When D is only just on the bridge we have

$$45P = 19{\cdot}9 \text{ ft.} \times 34 \text{ tons} = 675 \text{ tons. ft.}$$
$$\therefore\ P = 15 \text{ tons.}$$

Then the bending-moment under $A = 9P = \underline{135 \text{ tons. ft.}}$

Next consider the bending-moment under the load B. It will have its greatest value when

$$y + 9 = (9 - y) + \bar{x} = 28{\cdot}9 - y$$
$$\therefore\ y = 9{\cdot}95 \text{ ft.}$$

This again makes D go off the bridge.

When D is only just on the bridge we have seen that $P = 15$ tons, so that then the bending moment under B is

$$15 \times 18 - 12 \times 9 = \underline{162 \text{ tons. ft.}}$$

Next consider the load C. The maximum bending moment under it will occur when

$$y + 27 = 9 - y + \bar{x} = 28{\cdot}9.$$
$$y = 0{\cdot}95 \text{ ft.}$$

We then find

$$P = 21{\cdot}1 \text{ tons}, \quad Q = 12{\cdot}9 \text{ tons}.$$

The bending moment under C is then

$$(18 - y)Q - 9 \times 5 = \underline{175 \text{ tons. ft.}}$$

Proceeding in the same way we find that the bending moment under the load D is greatest when $y = -3{\cdot}55$ ft., but its value is less than the greatest bending moment under C.

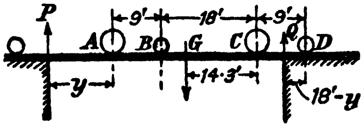


Fig. 149.

We must next consider the bending moments when D is off the bridge, as in Fig. 149.

The c.g. of A, B, and C is found to be 14·3′ from C.

Proceeding exactly as above we find that

the greatest bending moment under A = 168 tons. ft.

" " " " " B = 166·5 "

When we seek the value of y which will make the bending moment under C a maximum we find $y = 2{\cdot}65'$, which brings D back on to the bridge again; this might be inferred from the above results.

Finally consider the bending moments before A has come on to the bridge, as in Fig. 150.

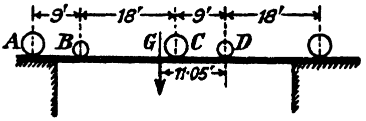


Fig. 150.

The only load which need be considered is C, and we find in the same way as before that the greatest bending-moment under it is 180 tons. ft. Comparing this with the figures underlined above, it will be seen that this is greatest of all the bending moments, and it should be noticed that it occurs when one of the loads, namely A, is off the bridge.

INFLUENCE LINES

114. Influence Lines.—A curve which shows the value of the bending-moment at a given section of a beam, for all positions of a travelling load is called the bending-moment *influence line* for that section; similarly a curve which shows the shearing force at the section for all positions of the load is called the shearing force influence line for the section. The distinction between influence lines and maximum bending-moment (or shearing force) diagrams must be carefully noted: for a given load there will be only one maximum bending-moment diagram for the beam, but an infinite number of B.M. influence lines, one for every section of the beam.

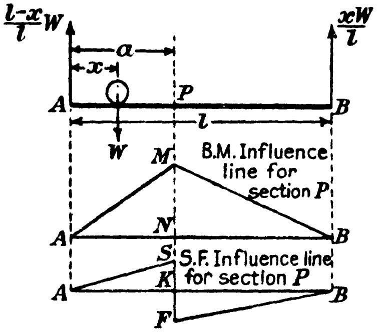
115. Single Concentrated Load.—In Fig. 151 let a load W be at a distance x from A, and let us find the bending-moment and shearing force influence lines for a section P distant a $(> x)$ from A.

The reaction at A is $\frac{l - x}{l} W$, and that at B is $\frac{xW}{l}$. Hence, the bending-moment at P is

$$M = (l - a)\frac{xW}{l},$$

which increases uniformly from zero when W is at A to the value $a(l-a)W/l$ when W is at P.

Thus the B.M. influence line for the section P, for all positions of W between A and P, is the straight line AM, where $MN = a(l-a)W/l$.



$\frac{l-x}{l}W$
$\frac{xW}{l}$
a
x
P
A
B
l
W
M
B.M. Influence line for section P
N
A
B
S
S.F. Influence line for section P
K
A
B
F

FIG. 151.

Similarly, for positions of W between P and B, the B.M. influence line is MB, and AMB is the B.M. influence line for the section P. Hence the bending moment at any section is greatest when the load is at that section.

Again, the positive shearing force at P is $\frac{xW}{l}$, which increases uniformly from zero to $\frac{aW}{l}$ as W advances from A to P; this is shown by the straight line AS. When $x > a$, the shearing force at P is

$$\frac{xW}{l} - W = W\left(\frac{x}{l} - 1\right),$$

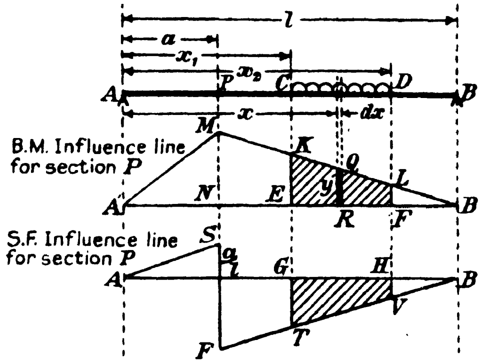
which is negative and decreases from $\frac{aW}{l} - W$ when W is at P to zero when W is at B: as the load passes P the shearing force changes abruptly by an amount W. Thus the shearing force influence line for the section P is given by $ASFB$, where $KS = aW/l$, and $SF = W$.

116. Uniformly Distributed Load.—In Fig. 152 let CD be a uniformly distributed load, w per unit length, which is advancing across the beam AB, and consider any section P of the beam. The B.M. influence line for unit concentrated load for the section P will be AMB, where $MN = a(l-a)/l$, according to § 115. Then, if we have a unit load at a distance x from A, the bending moment at P will be the corresponding ordinate $QR = y$ of the influence line; if the load be wdx instead of unity the bending moment at P will be $QR \times wdx = wydx$. Hence the bending moment at P due to the load CD in the position shown will be

$$\int_{x_1}^{x_2} wydx = w\int_{x_1}^{x_2} ydx = w \times \text{area } EKLF.$$

When the length CD is equal to or greater than AB the bending moment at P will evidently be greatest when the load covers the span, and is then given by $w \times$ area AMB.

In the same way, if $ASFB$ be the shearing force influence line for section P for unit concentrated load, the shearing force at P due to the load CD is given by $w \times$ area $GHVT$. It will be greatest when either C or D is at P, according to the position of P.


l
a
x_1
x_2
P
C
D
A
B
x
dx
M
B.M. Influence line for section P
K
Q
y
L
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E
R
F
S.F. Influence line for section P
S
$\frac{a}{l}$
G
H
V
T
F

FIG. 152.

EXAMPLES X

1. Two concentrated loads of 10 and 20 tons advance along a girder 20 ft. span, the distance between the loads being 8 ft. Find the position of the section which has to support the greatest bending moment, and calculate the value of this bending moment. (Intercoll. Exam., Cambridge, 1913.)

2. A traction engine advances across a bridge from left to right. The loads on the front and rear axles are 6 and 9 tons and the wheelbase is 10 ft. The span of the bridge is 40 ft. Construct the maximum B.M. diagram for the bridge. (Intercoll. Exam., Cambridge, 1908.)

3. Two concentrated loads of 12 tons and 8 tons, 8 ft. apart, advance across a horizontal girder of 45 ft. span. Draw to scale the maximum bending-moment diagram for this arrangement. (Intercoll. Exam., Cambridge, 1911.)

4. A braced girder of 200 ft. span, divided into ten equal panels, supports a rolling load of 1 ton per foot run which may extend over the whole length of the girder. Show that the maximum positive and negative shearing forces, due to the rolling load, in the $(r+1)$th panel from the end, are $10r^2/9$ and $10(9-r)^2/9$. (Mech. Sc. Trip., 1912.)

5. The loads on the front and rear axles of a motor lorry are 7,100 lbs. and 17,700 lbs. respectively, and the distance between them is 13 ft. The lorry advances over a girder having a clear span of 75 ft. Calculate the greatest bending moment set up and show that the equivalent uniformly distributed dead load is almost exactly 20 tons. (Mech. Sc. Trip., 1915.)

6. A load of 10 tons, followed by another load of 5 tons at a distance of 10 ft., advances across a girder of 100 ft. span. Obtain an expression for the maximum bending moment at a section of the girder distant x feet from an abutment. (Mech. Sc. Trip., 1924.)

CHAPTER XI

LONGITUDINAL STRESSES IN BEAMS

117. Physical Discussion.—We have already seen that, in general, the action between contiguous parts of a beam consists of a bending moment and a shearing force; we have also seen how to estimate the magnitudes of these actions. The next step towards calculating the strength of beams is the determination of the consequent stresses. This problem, in its most general form, is one of considerable complexity, and no general solution has yet been found. Fortunately for engineers, the one solution which is easily obtained, for certain simple conditions, is, in most cases, a sufficiently close approximation for all practical purposes when more complicated conditions prevail.

As a simple instance, think of a cantilever with a concentrated load at its free end, and imagine the beam to consist of a number of longitudinal filaments, like a bundle of wires. In general, some of these filaments will be extended and some contracted, resulting in a direct tensile or compressive stress; some of the filaments will be unstrained.

Since the filaments suffer longitudinal strain they must also undergo lateral strain: those which extend longitudinally will contract laterally and conversely. It follows from this that the shape of the cross-sections of the beam must change. In Fig. 153 *ABCD* represents the original section of the beam, consisting of, say, sixteen little filaments of square section; when the beam is bent the upper filaments will have their cross-section reduced, so that the edge *AB* will be contracted, whilst the sections of the lower filaments will be enlarged so that the edge *CD*

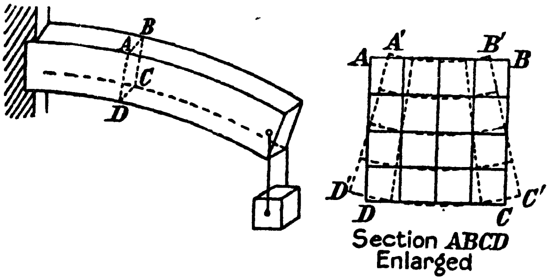

FIG. 153.

is stretched. At the same time we may expect that the edges AD and BC will not change in length very much. Thus the cross-section will be changed into a shape something like $A'B'C'D'$; the network of straight lines outlining the filaments in the unstrained section will now become a network of curves.

The cross-section also undergoes further deformation when there is shear, as there is in the case of the cantilever.

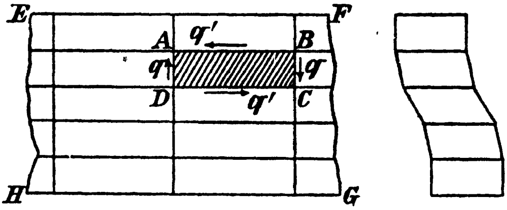

FIG. 154.

Let us suppose now that each longitudinal fibre is divided into short lengths, as shown in Fig. 154, so that we now think of the beam as built up of a number of small rectangular blocks such as $ABCD$. Then the ends of the block, BC and AD, will experience a shear stress q; therefore the upper and lower faces AB and CD will experience a complementary shear stress (§ 48) $q' = q$, and the block will undergo shear strain. Now, if the upper and lower surfaces EF and GH are not acted upon by any applied forces, it is clear that the shear stress must be zero at these surfaces. Thus we see that the shear stress, and therefore the shear strain, must diminish from the interior towards the upper and lower faces of the beam.

If we take a single column of our imaginary blocks extending from top to bottom of the beam, each will be distorted by shear, the distortion decreasing towards the top and bottom, as shown on the right in Fig. 154.

If the shear stress is constant across the width of each layer of filaments this distortion would mean that every cross section which was plane in the unstrained beam would be strained into a cylindrical surface. Usually, however, the shear stress varies across the beam and this leads to a curving of the lines which were originally at right angles to the plane of Fig. 154, so that the cross section becomes curved in both vertical and horizontal directions.

The matter of the distortion of normal plane sections of the beam was overlooked by the earlier investigators, who assumed that these sections remain normal plane sections after bending. This hypothesis is usually known as the Bernoulli-Euler theory, and St. Venant showed that it can only be true under very special circumstances, namely, when the bending moment does not vary along the beam.

When the load is distributed over the upper surface of the beam, compressive stresses in a vertical direction must be called into play, and these must gradually die away towards the lower surface if this be free from applied forces. Arguing from this, we can show, without much difficulty, that it is to be expected that there will exist shearing stresses on vertical planes parallel to the plane of bending, accompanied by their complementary shear stresses, and also direct stresses perpendicular to the plane of bending.

Hence it appears that, in the general case, the problem of determining the stresses and strains in a beam, for any given system of loads, is one of considerable complexity, and this indeed is true. Up to the present time the methods of mathematical analysis have not placed us in a position to solve directly the problem of finding the stresses, given the loads. The method employed by St. Venant and later investigators is to assume the character of some of the stresses or strains, to discover the system of loading which will agree with the assumptions, and to deduce the remaining stresses and strains. Proceeding in this way it can be shown * that, when a beam is bent by equal and opposite couples applied to its ends, the only stresses are the longitudinal stresses in filaments parallel to the axis, that in this case cross-sections of the beam remain plane and normal to the axis, and that the curvature is constant and proportional to the applied couples : this is the only case which lends itself to simple rigorous treatment. In the case of concentrated loads † there are in addition vertical shearing stresses on the cross sections, accompanied by shear between horizontal layers, and cross sections no longer remain plane. When the load is distributed ‡ along the beam all stress components are present, and the curvature is not proportional to the bending moment. Fortunately for engineers, in nearly all practical cases, the results of the theory when the beam is acted on only by terminal couples can be applied to other more complicated cases with all the accuracy that is necessary.

118. The Theory of Uniform Bending.§—We shall now investigate the flexure of a uniform rod of isotropic material within the limits of linear elasticity, subject to the following assumptions :—

(α) All longitudinal filaments of the rod bend into circular arcs which lie in parallel planes and have a common axis perpendicular to these planes. This axis is called the axis of bending.

(β) All normal sections of the rod remain plane and at right angles to the longitudinal filaments, so that after bending their planes pass through the axis of the circular arcs into which the filaments are strained.

(γ) The longitudinal filaments are free from mutual actions and reactions, and experience only longitudinal stress.

(δ) Young's Modulus is the same in compression as in tension.

The plan we follow is to discover what system of applied forces and couples is compatible with these assumptions, which we have pointed out above are not true in the general case.

In Fig. 155, *ABFD* is a longitudinal section of the beam in a plane parallel to the plane of bending ; *OL* is the axis of the circles into which the longitudinal filaments bend; HCG is the *neutral filament*, i.e. the filament which does not change in length, cutting the plane of a normal section MN at C.

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* Love, *Theory of Elasticity*, 3rd Ed., p. 127.

† Ditto, Chapters XV and XVI.

‡ Ditto, Chapter XVI, p. 359. Michell, *Quart. J. of Math.*, Vol. 32.

§ We follow here with slight modifications the treatment adopted by G. F. C. Searle, *Experimental Elasticity*, Chapter II.

At C take axes Cy and Cz in the plane MN, Cy being in the plane of bending.

Let R = the radius of the strained neutral filament.

Let $PQ = \delta s$, be an element of a filament at distance y from the neutral filament, subtending an angle $\delta\theta$ at the axis OL.

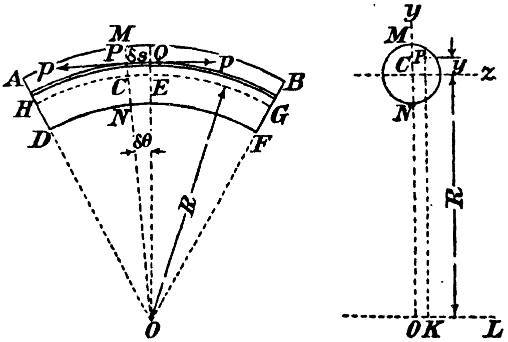

Fig. 155.

After strain, the length of PQ is $(R + y)\delta\theta$. Before strain the length was equal to that of CE, which is unaltered by the bending, i.e. it was $R\delta\theta$. Hence the strain in the filament PQ is

$$e = \frac{(R + y)\delta\theta - R\delta\theta}{R\delta\theta} = \frac{y}{R}$$

Let p be the longitudinal stress in PQ, then

$$p = Ee = \frac{Ey}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total force on the ends of the filament is $p.\delta S$, where δS is the area of the cross section of the filament. The moments of this about Cy and Cz are $z.p.\delta S$ and $y.p.\delta S$. Hence the resultant action[*] on the cross section will consist of

(i) a force $= \int p.dS = \frac{E}{R}\int y.dS$, normal to the section;

(ii) a couple about $Cy = \int pz.dS = \frac{E}{R}\int yz\,dS$;

(iii) a couple about $Cz = \int py.dS = \frac{E}{R}\int y^2.dS$,

the integrals being taken all over the cross section.

---

[*] The resultant of the forces on the ends of the element PQ is $p.\delta S.\delta\theta$, which must be balanced by radial actions between the filaments, but these forces are of a smaller order than the end forces $p.\delta S$ and we neglect them.

Since we have assumed that the bending all takes place in planes parallel to the longitudinal planes through Cy, the couple about Cy must vanish. Therefore we must have

$$\int yz.dS = 0$$

$\therefore$ *Cy must be a principal axis of inertia* [*] of the strained cross section. This is an important restriction.[†] We are then left with a force

$$P = \frac{E}{R} \int y.dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and a couple

$$\mathbf{M} = \frac{\mathbf{E}}{\mathbf{R}} \int \mathbf{y}^2.\mathbf{dS} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

about Cz, i.e. acting in the plane containing one set of corresponding principal axes of all cross sections.

We see then that the assumptions made above for a beam of constant cross section correspond with a constant normal force on the cross section and a bending moment which is constant along the beam. We shall confine our attention now to the case when the force P is zero, i.e. when

$$\int y.dS = 0$$

This integral represents the moment of the area of the strained section about Cz. Hence, in this case Cz must pass through the centre of the section, and, as it is at right angles to Cy, it must be the other principal axis of the section.

We are now left with the couple given by (3). The integral on the right-hand side of (3) represents the moment of inertia [‡] of the strained cross section about Cz. Denoting this by I, we have

$$\mathbf{M} = \frac{\mathbf{EI}}{\mathbf{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining this with (1) we have the relations

$$\frac{\mathbf{p}}{\mathbf{y}} = \frac{\mathbf{M}}{\mathbf{I}} = \frac{\mathbf{E}}{\mathbf{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These two equations, then, express the relations between the stress, the bending moment, the curvature and the dimensions of the cross section, when a straight beam of uniform section is bent by a constant bending moment, i.e. by equal and opposite couples applied to the ends, acting in planes containing one of the principal axes of inertia of the cross sections.

The conditions under which the formulæ (5) have been established must be carefully noted and remembered, but they can be applied with sufficient accuracy to nearly all the cases of bending of straight beams met with in engineering practice.

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[*] See § 131.

[†] See § 126.

[‡] See p. 176 *et seq.* for theory of moments of inertia.

It has been remarked above that Cy and Cz are the principal axes, and that I is the moment of inertia, of the *strained* cross section. In all practical cases in engineering, except where especially pointed out, the distortion of the cross section is so slight that the unstrained cross section is taken instead. The case when the plane of the applied bending moment does not contain the principal axes of the cross sections is dealt with in § 126.

119. Modulus of Section.—We see from (5) above that the stress at any point in the section is given by

$$p = \frac{My}{I}.$$

When y is zero, that is along Cz, the stress is zero, and this line is referred to as the *neutral axis* of the section.

For a given value of M the stress is a maximum when y is a maximum. Thus

$$p_{max} = \frac{My_{max}}{I} \qquad (6)$$

The quantity I/y_{max} is called the *Section Modulus*, and is usually denoted by Z, so that we can write

$$p_{max} = \frac{M}{Z} \qquad (7)$$

120. Application to Practical Cases of Bending.—As remarked above, in engineering practice the results of applying the above formulæ to cases when there is shearing as well as pure bending usually give all the accuracy that is required. This does not imply, of course, that the shear stresses need not be calculated; on the contrary, they usually should be calculated and we shall consider this question in a later chapter. In very many cases it happens that the greatest bending moment occurs at a mathematical maximum (see § 96 and § 107), when the shearing force is zero; we shall then expect the above formulæ to be a very close approximation to the truth.

The formulæ we have established will also usually give sufficiently accurate results when applied to beams whose cross sections change slowly along the beam, but they cannot be taken to give even approximate values of the stresses where there is an abrupt change of section, or in the immediate neighbourhood of supports or concentrated loads. But even in these cases, on account of the principle of St. Venant, we take the formulæ we have obtained as giving the stresses on sections which are at a greater distance from the point of application of the concentrated force than two or three times the greatest dimension of the cross section of the beam.

121. Moment of Resistance of Section.—The bending moment which can be applied to a beam of a particular cross section, without the maximum stress exceeding a certain value, is called the moment of resistance of that section for the prescribed stress.

122. Beams having Initial Curvature.—When the beam is initially curved we can apply the above formulæ in certain cases with sufficient accuracy for practical purposes. As a working rule it may be taken that the formulæ we have established may be applied provided the largest dimension of the cross section is small compared with the least radius of curvature of the central line. For instance, we can apply the formulæ to large sectioned beams of small curvature, such as the kind of built-up steel beam which might be used in an arched girder, or to a slight beam of relatively large curvature, such as a spiral spring, but we must not apply the above results to a large sectioned beam of relatively large curvature such as a crane hook. Such cases as this require special treatment and are considered below in Chapter XXV. In the cases where the formulæ are applicable, if, at any section, R_0 denote the initial radius of curvature and R the radius of curvature of the strained beam, it can be shown in the manner of § 118 that instead of $e = \frac{y}{R}$ we have

$$e = y\left(\frac{1}{R} - \frac{1}{R_0}\right),$$

and instead of (4) we have

$$\frac{\mathrm{M}}{\mathrm{EI}} = \frac{1}{\mathrm{R}} - \frac{1}{\mathrm{R_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

123. Beams made of Materials having Different Strengths in Tension and Compression.—Let y_t and y_c denote the distances of the extreme tension and compression filaments from the neutral axis, then the maximum tensile and compressive stresses in the beam are given by

$$p_t = \frac{My_t}{I} \text{ and } p_c = \frac{My_c}{I}.$$

If the beam has a section such that the neutral axis is at the middle of the depth, y_c and y_t will be equal, so that p_t and p_c will be equal also. In the case of a material like cast-iron, which is five or six times as strong in compression as it is in tension, this result is undesirable, and we should try to design the section in such a way that

$$\frac{y_t}{y_c} = \frac{p_t}{p_c}$$

where p_t and p_c now denote the maximum stresses which are to be allowed in tension and compression. For instance, if $p_c = 5p_t$ we should, on this basis, make $y_c = 5y_t$. In order to bring this about in a flanged girder we must have the tension flange very large in comparison with the compression flange. This has the disadvantage that the two cool at very different rates after casting, and large initial stresses are set up in the material. In practice, therefore, it is not usual to make y_c more than twice or thrice y_t, the tension flange being made wide and comparatively thin.

Example 1.—What load can a beam 12 ft. long carry at its centre, if the cross section is a hollow square 12″ × 12″ outside and 1·5″ thick, the permissible longitudinal stress being 5 tons/in.^{2}?

We must first find * the moment of inertia of the cross section about its neutral axis. The inside is a square 9″ × 9″. Then

$$I = \frac{12 \times 12^3}{12} - \frac{9 \times 9^3}{12}$$
$$= 1{,}728 - 547$$
$$= 1{,}181 \text{ ins.}^4$$

The length of the beam is 144″; therefore if W tons be a concentrated load at the middle, the maximum bending moment (§ 101) is

$$M = \frac{W \text{ tons} \times 144 \text{ ins.}}{4} = 36W \text{ tons. ins.}$$

In the notation of § 118, $y = 6''$. Hence the maximum stress is, from equation (5),

$$p = \frac{36W \text{ tons in.} \times 6 \text{ ins.}}{1{,}181 \text{ ins.}^4} = \frac{36 \times 6W}{1{,}181} \text{ tons/in.}^2$$

If $p = 5$ tons/in.^{2} we must therefore have

$$W = \frac{5 \times 1{,}181}{36 \times 6} = 27{\cdot}3.$$

Example 2.—Estimate the section modulus and the maximum longitudinal stress in a built-up I-girder, with equal flanges carrying a load of 2 tons per ft. run, with a clear span of 60 ft. The web is of thickness ½″ and the depth between flanges 5 ft. Each flange consists of four ½″ plates 24″ wide, and is attached to the web by angle irons 4″ × 4″ × ½″. (Mech. Sc. Trip., 1919.) (See Fig. 156.)

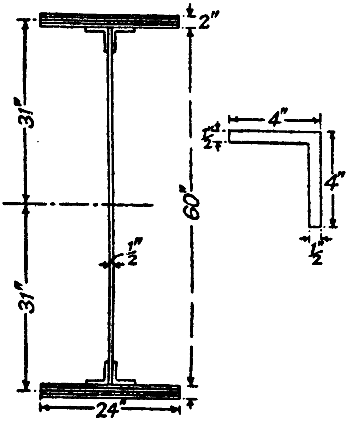

FIG. 156.

The moment of inertia of each flange $= 48 \times 31^2$
$= 46{,}100$ in.^{4}

The moment of inertia of the web

$$= \frac{1}{12} \times 0{\cdot}5 \times 60^3 = 9{,}000 \text{ in.}^4$$

The area of the horizontal part of each angle iron is 2 in.^{2}, and its centroid is 29¾″ from the neutral axis. Hence the moment of inertia of this is approximately

$$2 \times 29{\cdot}75^2 = 1{,}770 \text{ in.}^4$$

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* A table of moments of inertia is given on p. 180.

The area of the vertical part is 1·75 in.^{2}, and its centroid is 27·75″ from the neutral axis. Therefore the corresponding moment of inertia is approximately

$$1{\cdot}75 \times 27{\cdot}75^2 = 1{,}350 \text{ in.}^4$$

Then the moment of inertia of the whole section of the angle iron, about the neutral axis of the beam section, is 1,770 + 1,350 = 3,120 in.^{4}

The moment of inertia of the whole beam section is then

$$\begin{aligned} I &= 2 \times 46{,}100 + 9{,}000 + 4 \times 3{,}120 \\ &= 113{,}680 \text{ in.}^4 \\ \text{“}y\text{”} &= 32 \text{ ins.} \end{aligned}$$

∴ the section modulus is

$$Z = \frac{113{,}680 \text{ in.}^4}{32 \text{ ins.}} = 3{,}550 \text{ in.}^3$$

The bending moment at the middle of the span is, equation (12), p. 124,

$$M = \frac{2 \text{ tons/ft.} \times 60^2 \text{ ft.}^2}{8} = 900 \text{ tons. ft.} = 10{,}800 \text{ tons. ins.}$$

∴ the maximum longitudinal stress

$$= \frac{10{,}800 \text{ tons. ins.}}{3{,}550 \text{ in.}^3} = 3{\cdot}04 \text{ tons/in.}^2$$

Example 3.—A cast-iron girder has the dimensions shown in Fig. 157. Calculate the load per foot run which can be carried on a 15 ft. span, if the girder is simply supported at its ends, without the tensile stress exceeding 1 ton/in.^{2} What is the compressive stress then?

The first step is to find the centroid of the section.

Taking moments about the bottom edge, we see that the height of the centroid is

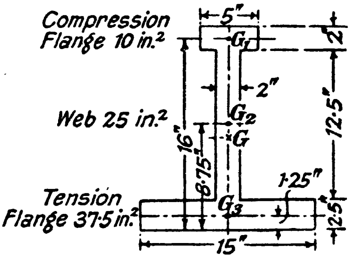

FIG. 157.

$$\frac{10 \times 16 + 25 \times 8{\cdot}75 + 37{\cdot}5 \times 1{\cdot}25}{10 + 25 + 37{\cdot}5} = 5{\cdot}88''$$

In Fig. 157 the centroids of each flange and the web are shown by the spots G_1, G_2, G_3, and that of the whole section by the G marked with a cross.

We now proceed to find the moment of inertia of the section about the neutral axis, i.e. a horizontal line through G:—

	Compression Flange.	Web.	Tension Flange.
M. of I. about horizontal axis through own G (in.^{4})	$\frac{5 \times 2^3}{12} = 3{\cdot}3$	$\frac{2 \times 12{\cdot}5^3}{12} = 325$	$\frac{15 \times 2{\cdot}5^3}{12} = 19{\cdot}4$
Area = S (in.^{2}) . . .	10	25	37·5
Distance of own G from neutral axis of section = h	10·12	2·87	4·63
Sh^2 (in.^{4})	1024	206	803
M. of I. about neutral axis of section (in.^{4}) . . .	1027·3	531	822·4

Hence the total moment of inertia of the whole section about the neutral axis

$$= 1027{\cdot}3 + 531 + 822{\cdot}4$$
$$= 2{,}381 \text{ in.}^4 \text{ (approx.)}.$$

The distance of the tension face from the neutral axis is 5·88″, hence if the bending moment be M tons. ins., the maximum tensile stress is

$$\frac{M \text{ tons. ins.} \times 5{\cdot}88 \text{ ins.}}{2{,}381 \text{ in.}^4} = \frac{5{\cdot}88M}{2{,}381} \text{ tons/in.}^2$$

Let w tons/ft. be the distributed load.
Then

$$M = \frac{w \text{ tons/ft.} \times 15^2 \text{ ft.}^2 \times 12 \text{ ins./ft.}}{8} = 338\, w \text{ tons ins.}$$

Hence the maximum tensile stress is

$$\frac{338w \times 5{\cdot}88}{2{,}381} \text{ ton/in.}^2$$

This is not to exceed 1 ton/in.², so the maximum load is

$$w = \frac{2{,}381}{338w \times 5{\cdot}88} = 1{\cdot}20 \text{ tons/ft.}$$

This gives $M = 405$ tons. ins.

The height of the compression face from the neutral axis = 11·12″, hence the maximum compressive stress

$$= \frac{405 \text{ tons. ins.} \times 11{\cdot}12 \text{ ins.}}{2{,}381 \text{ in.}^4} = 1{\cdot}89 \text{ tons/in.}^2$$

124. Reinforced Concrete.*—The following theory of the flexure of reinforced concrete beams is that most usually followed and has the advantage of simplicity. The reader who is interested in pursuing the matter should consult works devoted to the subject.† Concrete is weak in tension compared with its strength in compression, and the idea of ferro-concrete construction is to reinforce the tension side of a beam with steel rods or wires. Then, on account of the tensile weakness of concrete, and since cracks may develop in it, the stress calculations are worked out on the assumption that the steel takes all the tension and the concrete all the compression, which must be true at any rate after the tensile resistance of the concrete has broken down. The cross sections of the

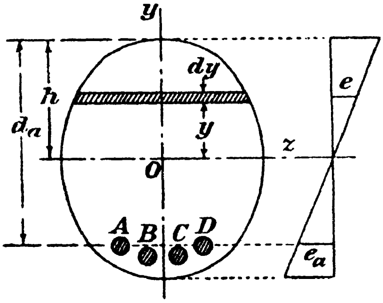

FIG. 158.

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* See also "Rapid Design of Ferro-Concrete Beams," by A. Esnouf and L. J. Coutanceau, *Engineering*, Feb. 10, 1922; and a method of determining the proportions of T-beams by G. I. Cope, *Engineering*, Feb. 25, 1921; see also March 18, 1921.

† For example, *Der Eisenbctonbau*, by Emil Mörsch. English Translation. *Concrete-Steel Construction*, by E. P. Goodrich (Constable).

reinforcement which were originally in the same plane as a given cross section of the concrete are assumed to remain so after bending; the concrete is also supposed to obey Hooke's Law, and, as before, we assume that normal cross sections remain normal,* and plane.

Let E_c and E_s denote the values of Young's Modulus for concrete and steel respectively.

Let h be distance of the neutral axis Oz from the extreme compression face of the beam (Fig. 158).

Let d_a be the depth, from the same face, of a reinforcing steel bar A of small cross section S_a. We shall neglect the variation of stress and strain over the area S_a.

Let S_c denote the area of the section of the concrete.

Let p be the compressive stress in the concrete at a distance y from Oz, and let p_a be the tensile stress in the steel rod A.

Let e and e_a denote the corresponding strains.

Then, if R be the radius into which the neutral filament is bent, at any section of the beam we have, in accordance with § 118,

$$p = \mathrm{E}_c . e = E_c \frac{y}{R} \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

and

$$p_a = E_s . e_a = E_s \frac{d_a - h}{R} \quad . \quad . \quad . \qquad \text{(ii)}$$

The total action across the section consists of a normal force

$$\int_o^h p.dS_c - \Sigma(p_a . S_a) \quad . \quad . \quad . \quad . \quad . \text{(iii)}$$

and a couple of moment

$$\int_o^h p.y.dS_c + \Sigma p_a . S_a . (d_a - h) \quad . \quad . \quad . \text{(iv)}$$

where dS_c is the area of the strip of concrete at height y, and the summation extends to all the reinforcing bars. In order that the action may be equivalent to a pure bending moment M, we must have the quantity (iii) equal to zero, and (iv) equal to M. Hence, substituting for p and p_a from (i) and (ii), we must have

$$\int_o^h y.dS_c = \frac{E_s}{E_c} . \Sigma(d_a - h).S_a \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and

$$M = \frac{E_c}{R} \int_o^h y^2.dS_c + \frac{E_s}{R} \Sigma S_a (d_a - h)^2 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Put into words, these equations are :—

The moment of the compression area of the concrete

$$= \frac{E_s}{E_c} \times \text{the moment of the total area of the steel,} \quad . \quad . \quad (9\text{A})$$

both taken about the neutral axis, and

---

* W. Hovgaard has shown that this is in accordance with the Principle of Minimum Strain Energy (*Proc. I.N.A.*, 1923).

$$MR = E_c \times \text{the moment of inertia of the compression area of the concrete}$$
$$+ E_s \times \text{the moment of inertia of the steel area,} \quad . \quad . \quad . \quad (10\text{A})$$

both taken about the neutral axis.

The former of these equations enables the position of the neutral axis to be found. The latter gives the relation between the bending moment and the radius R. If p_c denote the maximum compressive stress in the concrete, we have from (i)

$$R = \frac{E_c h}{p_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v})$$

eliminating R between this and (10) we have an equation to determine the maximum stress in the concrete. Then, from (ii) and (v) the stress in the bar A is given by

$$p_a = \frac{E_s}{E_c} \cdot \frac{d_a - h}{h} p_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

These are the general lines upon which to carry out stress calculations for reinforced concrete beams. The value of E_s/E_c usually lies between 10 and 15.

125. Reinforced Concrete Beam of Rectangular Section.—When the section of the beam is rectangular and all the steel bars are at the same depth, the above equations can be simplified considerably.

Let the dimensions be as shown in Fig. 159. Then (9) gives

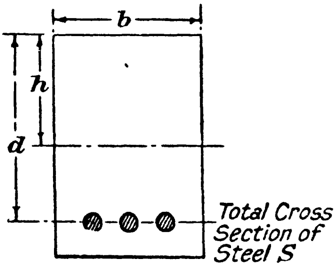

FIG. 159.

$$\frac{bh^2}{2} = \frac{E_s}{E_c}(d - h)S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vi})$$

or

$$h^2 + \frac{2E_sSh}{E_cb} - \frac{2E_sSd}{E_cb} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which is a quadratic for h, the positive root being the one required. Equation (10) becomes

$$MR = \tfrac{1}{3}E_ch^3b + E_s.S(d - h)^2 \quad . \quad . \quad . \quad . \quad (\text{vii})$$

which, combined with (v), gives p_c, and then (11) gives the tensile stress in the steel, p_a.

The equations in this form are applicable rather to calculating the strength of beams whose dimensions are known than to problems of design. In designing a beam it is obviously desirable that both the concrete and steel should reach their maximum allowable stresses at the same time. If these stresses are denoted by p_c and p_s, we have from (11):

$$\mathrm{p_s} = \frac{\mathrm{E_s}}{\mathrm{E_c}} \cdot \frac{\mathrm{d-h}}{\mathrm{h}}\,\mathrm{p_c} \qquad \ldots\ldots\ldots (13)$$

whence

$$\frac{\mathrm{h}}{\mathrm{d}} = \frac{\mathrm{p_c}}{\mathrm{p_c + p_s . E_c/E_s}} \qquad \ldots\ldots\ldots (14)$$

Eliminating R from (v) and (vii) gives

$$p_c = \frac{M}{\frac{1}{3}bh^2 + \frac{E_s}{E_c} \cdot \frac{S(d-h)^2}{h}} \qquad \ldots\ldots (\text{viii})$$

But we also have above

$$\frac{E_s}{E_c}(d-h)S = \tfrac{1}{2}bh^2,$$

hence (viii) becomes

$$\mathrm{p_c} = \frac{\mathrm{2M}}{\mathrm{bh}\left(\mathrm{d} - \frac{\mathrm{h}}{3}\right)} \qquad \ldots\ldots\ldots (15)$$

Then, from (13) and (15)

$$\mathrm{p_s} = \frac{\mathrm{E_s}}{\mathrm{E_c}} \cdot \frac{\mathrm{d-h}}{\mathrm{h}} \cdot \frac{\mathrm{2M}}{\mathrm{bh}\left(\mathrm{d} - \frac{\mathrm{h}}{3}\right)}$$

and the total tension in the steel, T, is

$$T = Sp_s = \frac{E_s}{E_c}S(d-h)\frac{2M}{bh^2\left(d - \frac{h}{3}\right)}$$

Hence, by (vi),

$$\mathrm{T} = \frac{\mathrm{M}}{\mathrm{d} - \frac{\mathrm{h}}{3}} \qquad \ldots\ldots\ldots (16)$$

Example 1.—Fig. 160 represents a reinforced concrete beam. If h be the depth of the neutral axis, and the working stress in the steel and concrete

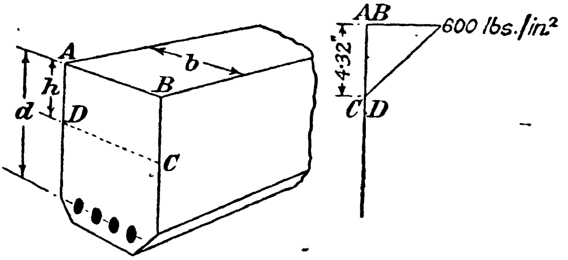

FIG. 160.

is to be limited to 16,000 and 600 lbs./in.² respectively, show that h/d should be 0·36; $E_s = 30 \times 10^6$ lbs./in.², and $E_c = 2 \times 10^6$ lbs./in.² The tensile strength of the concrete is to be neglected, and cross sections are supposed to remain plane.

If $b = 8''$ and $d = 12''$ calculate the sectional area of the reinforcing steel rods and the bending moment the beam can withstand without exceeding the above stresses. (Intercoll. Exam., Cambridge, 1911.)

From equation (13) of § 125 we have

$$16{,}000 = 15.\frac{d-h}{h}.600$$

or

$$16 = 9\left(\frac{d}{h} - 1\right)$$

$$\therefore \frac{d}{h} = \frac{16}{9} + 1 = \frac{25}{9}$$

$$\therefore \frac{h}{d} = \frac{9}{25} = 0{\cdot}36.$$

Hence, when $d = 12''$, we have $h = 4{\cdot}32''$.

Then, from (vi) of § 125, when $b = 8''$,

$$\frac{8 \times 4{\cdot}32^2}{2} = 15(12 - 4{\cdot}32)S$$

where S is the sectional area of the steel.

Hence

$$S = \frac{4 \times 4{\cdot}32^2}{15 \times 7{\cdot}68} = 0{\cdot}645 \text{ in.}^2$$

The total tension in the steel $= 16{,}000 \times 0{\cdot}645$
$= 10{,}300$ lbs.

The moment of this about the neutral axis is

$$10{,}300 \text{ lbs.} \times 7{\cdot}68 \text{ ins.} = 79{,}000 \text{ lbs. ins.}$$

The stress in the concrete is assumed to increase uniformly from zero on the neutral axis DC to a maximum of 600 lbs./in.^{2} on the line AB, as shown on the right of Fig. 160. Hence the total thrust on the section is

$$\tfrac{1}{2} \times 600 \text{ lbs./ins.}^2 \times 8 \text{ ins.} \times 4{\cdot}32 \text{ ins.} = 10{,}300 \text{ lbs.}$$

The centroid of the compression area is $\frac{2}{3} \times 4{\cdot}32 = 2{\cdot}88''$ above DC.

Hence the moment of the thrust $= 10{,}300 \times 2{\cdot}88 = 29{,}700$ lb. ins., and the total resisting moment of the section

$$= 29{,}700 + 79{,}000 = 108{,}700 \text{ lb. ins.}$$

As a check let us calculate M from equation (16) of § 125:

$$M = T\left(d - \frac{h}{3}\right) = 10{,}300(12 - 1{\cdot}44)$$

$$= 10{,}300 \times 10{\cdot}56 = 108{,}700 \text{ lb. ins.}$$

Example 2.—A rectangular sectioned ferro-concrete beam is 5″ wide, the reinforcement consisting of two steel rods $\frac{7}{16}''$ diameter with their centres $8\frac{1}{2}''$ below the compression face of the beam. Find the maximum bending moment that the beam will take, if the maximum compressive stress in the concrete is not to exceed 500 lbs./in.^{2} Find also the stress that this B.M. produces in the steel; E_s and E_c are as in Ex. 1. (Mech. Sc. Trip., 1912.)

We have:

$$b = 5'' ; \; d = 8{\cdot}5''$$

$$S = 2 \times \frac{\pi}{4} \times \left(\frac{7}{16}\right)^2 = 0{\cdot}302 \text{ in.}^2$$

The depth of the neutral axis is given by (12) of § 125:

$$h^2 + 30 \times 0\cdot302 \times \frac{h}{5} - 30 \times 0\cdot302 \times \frac{8\cdot5}{5} = 0$$

$$h^2 + 1\cdot812h - 15\cdot4 = 0$$

$$\begin{aligned} h &= -\,0\cdot906 \pm \tfrac{1}{2}\sqrt{3\cdot28 + 61\cdot6} \\ &= -\,0\cdot906 \pm 4\cdot026 \end{aligned}$$

Taking the positive value we have

$$h = 3\cdot12''$$

From (15) of § 125

$$\begin{aligned} M &= \tfrac{1}{2}p_c bh\left(d - \frac{h}{3}\right) \\ &= 250 \times 5 \times 3\cdot12 \times 7\cdot46 \\ &= 29{,}100 \text{ lb. ins.} \end{aligned}$$

Then from (13), the stress in the steel is

$$p_s = 15 \times \frac{5\cdot38}{3\cdot12} \times 500 = 12{,}900 \text{ lbs./in.}^2$$

Example 3.—Taking the safe stresses for concrete and steel to be 500 and 10,000 lbs./in.² respectively, and the ratio of the E's to be 10, determine d for the section shown in Fig. 161 in order that the neutral axis of the section may lie along AB. Find the diameter of the two steel rods, and the safe distributed load on the beam for a span of 20 ft. freely supported. (Mech. Sc. Trip., 1919.)

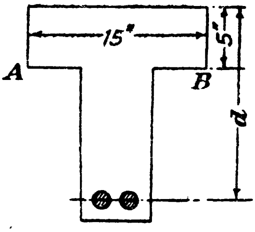

FIG. 161.

If AB is to be the neutral axis, by equation (9A) we must have the moment of the area of the flange about AB equal to 10 times the moment of the area of the steel about AB, that is

$$15 \times 5 \times 2\cdot5 = 10S(d - 5)$$

where S is the total area of the cross sections of both rods, i.e.

$$S(d - 5) = 18\cdot75 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

From (11) of § 124 we have

$$10{,}000 = 10 \times \frac{d-5}{5} \times 500$$

$$\therefore\ d = 15''$$

Hence, from (i)

$$S = 1\cdot875 \text{ in.}^2$$

∴ the area of each rod = 0·9375 in.², which requires a diameter of 1·09″.

The total tension in the two steel rods

$$= 1\cdot875 \times 10{,}000 = 18{,}750 \text{ lbs.}$$

The moment of this about the neutral axis

$$= 18{,}750 \times 10 = 187{,}500 \text{ lbs. ins.}$$

The total thrust in the concrete flange

$$= \tfrac{1}{2} \times 500 \times 15 \times 5 = 18{,}750 \text{ lbs.}$$

The moment about $AB = 18{,}750 \times (\tfrac{2}{3} \times 5) = 62{,}500$ lb. ins.

Hence the total B.M. which the beam can take

$$= 187{,}500 + 62{,}500 = 250{,}000 \text{ lb. ins.}$$

If w lbs./inch = the safe distributed load, we must have

$$\frac{w \times 240^2}{8} = 250{,}000$$

whence

$$w = 35 \text{ lbs./inch, nearly.}$$

126. Oblique, or Unsymmetrical Bending.—It has been remarked and emphasized in § 118 that, in order that the bending may take place entirely in planes parallel to the plane of the applied bending moment, the latter must contain one of the principal axes of inertia of all cross sections of the beam. When this is not the case we may proceed in various ways.

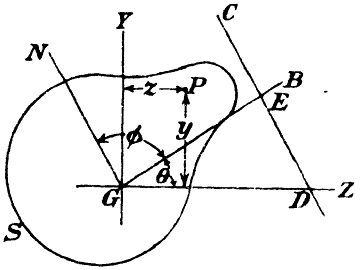

FIG. 162.

In Fig. 162 let GZ and GY be the principal axes of the section whose boundary is S, and let GB be the trace of the plane of the applied bending moment, M. The bending moment, which is a couple, can be resolved into two bending moments (couples) $M \cos \theta$ in the plane GZ and $M \sin \theta$ in the plane GY. Let the principal moments of inertia be I_y and I_z about GY and GZ respectively.

Let the co-ordinates of any point P in the section be (y, z) referred to GY, GZ.

The bending moment $M \cos \theta$ in the plane GZ produces at P a stress $Mz \cos \theta./I_y$, and the bending moment $M \sin \theta$ in the plane GY produces a stress $My \sin \theta./I_z$, since GZ and GY are principal axes. Therefore, by the principle of superposition, the total stress at P is

$$p = \frac{Mz \cos \theta}{I_y} + \frac{My \sin \theta}{I_z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

On the neutral axis $p = 0$. Therefore, the equation of the neutral axis GN is

$$\frac{z \cos \theta}{I_y} + \frac{y \sin \theta}{I_z} = 0$$

or

$$\frac{y}{z} = -\frac{I_z}{I_y} \tan \theta = -\frac{k_z^2}{k_y^2} \tan \theta. \quad . \quad . \quad . \quad . \quad (18)$$

where k_z and k_y are the principal radii of gyration. Referring to the equation of the momental ellipse in § 134, the equation (18) shows that the neutral axis is the diameter of the ellipse which is conjugate to GB, the trace of the plane of the applied bending moment M on the plane YGS.

From here we can proceed by three ways:—

FIRST METHOD.—Find the principal axes of the section and draw the momental ellipse. Next, draw the diameter conjugate to the plane of the applied bending moment; this diameter will be the neutral axis GN and its equation is (18) above.

Let φ be the angle between the neutral axis GN and GB the plane of the applied bending moment.

Let I be the moment of inertia of the cross section about the neutral axis.

Let η be the perpendicular distance of any point P in the section from the neutral axis GN.

The component of M about the neutral axis is $M \sin \varphi$. Hence, applying the formula (5) for bending about the neutral axis, the stress at P is given by

$$p = \frac{(M \sin \varphi) \cdot \eta}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

SECOND METHOD.—Find the principal axes and calculate the stress at any point by the method of superposition, using equation (17) above.

This method will be satisfactory when the position of the point of maximum stress is obvious, otherwise it will usually be best to use the first method, for when the neutral axis is drawn the most highly stressed point can be seen at once.

We might also proceed by drawing the neutral axis to find the most highly stressed point, and then use the method of superposition to calculate the stresses.

THIRD METHOD.—This is due to L. J. Johnson * and is useful in a certain class of problem, particularly when it is desired to find the plane of bending which will produce the greatest stress at any point for a given bending moment.

Equation (17) may be written in the form

$$p = \frac{M}{Z_\theta}$$

where

$$\frac{1}{Z_\theta} = \frac{\cos \theta}{I_y/z} + \frac{\sin \theta}{I_z/y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and Z_θ may be called the modulus of the section for the plane of bending GB.

Now (20) is the polar equation of a straight line referred to GZ (Fig. 162) as initial line, the radius vector being Z_θ and the inclination of the radius vector to the initial line being θ. This line makes intercepts I_y/z on GZ and I_z/y on GY, so it is easily drawn for any given point (y, z) when the principal moments of inertia have been calculated.

Let CD (Fig. 162) be this line for the point P. Then for any plane of bending such as GB, the section modulus Z_θ for that plane for the point P is given by the length of the radius vector GE to a certain scale. It is then easy to see what plane of bending will cause the greatest stress at P: it will be the direction for which GE is least, i.e. when GE is perpendicular to CD.

When the boundary of the section is polygonal, lines such as CD can be drawn for each angular point, and it will then be easy to pick out the plane for which a given bending moment will cause the greatest stress anywhere in the section: it will be given by the shortest perpendicular from G to any of the CD lines.

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* *Trans. Am. Soc. of Civil Engineers*, Vol. LVI (1906), p. 169. See also a paper by C. Batho, *Journal of Franklin Institute*, Aug., 1915.

If the boundary is a curve a similar process can be carried out for a series of points round the boundary, the points being chosen as close together as may be convenient.

It must be noted that, in general, the flexure of a beam of unsymmetrical section involves torsion as well.* In order that a beam may be bent without twisting, the line of action of the load must pass through a certain point, called the "flexural centre." The flexural centre will usually only coincide with the centroid of the section when there is symmetry about an axis perpendicular to the axis of bending. For instance, for a beam of thin triangular section, bending in a plane perpendicular to its axis of symmetry, the flexural centre is at a point about $4a/5$ from the point, where a is the width of the beam.

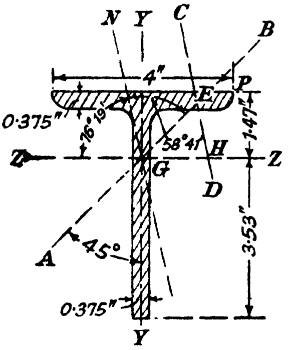

FIG. 163.

Example.—A 4″ × 5″ × 11 lbs. T bar, having the cross-section shown in Fig. 163, is subjected to a bending moment, the plane of which has the trace AB. If the maximum fibre stress is to be limited to 7·5 tons per sq. inch, what maximum bending moment can be applied ?

Find the direction of the neutral axis.

I_z = Moment of inertia about $ZZ = 7\cdot77$ ins.^{4}

I_y = „ „ about $YY = 1\cdot89$ ins.^{4}

(Intercoll. Exam., Cambridge, 1919.)

By equation (18) the neutral axis is given by

$$\frac{y}{z} = -\frac{7\cdot77}{1\cdot89}\tan 45^\circ = -4\cdot11$$

which is a line GN inclined to GZ at an angle 76° 19′. By inspection the point P is easily seen to be the point of the section which is most distant from the neutral axis, its distance being 2·29″.

The angle, φ, between the plane of the applied bending moment and the neutral axis is 58° 41′.

The moment of inertia about the neutral axis is

$$I = 1\cdot89\sin^2 76^\circ\ 19' + 7\cdot77\cos^2 76^\circ\ 19' = 2\cdot22 \text{ ins.}^4$$

Hence, if M is the applied bending moment, the stress at P is

$$p = \frac{M\sin 58^\circ\ 41'}{2\cdot22} \times 2\cdot29$$

$$= \frac{M \times 0\cdot854 \times 2\cdot29}{2\cdot22} = 0\cdot88M$$

If $p = 7\cdot5$ tons/ins.^{2}, $M = 8\cdot52$ tons. ins.

Alternatively we could proceed thus : having discovered that P is the point of maximum stress, we have from (17)

$$p = \frac{M\cos 45^\circ \times 2}{1\cdot89} + \frac{M\sin 45^\circ \times 1\cdot47}{7\cdot77}$$
$$= 0\cdot88M$$

which, of course, gives the same value of M as before.

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* A. A. Griffiths, *Aeronautical Research Committee*, R. and M., 399. See also *Proceedings of the Second International Congress of Applied Mechanics*, 1927, p. 434; *Proc. Royal Soc.*, Series A., Vol. 96, p. 211; *Proc. London Math. Soc.*, Vol. 20, p. 398.

Or, if we follow the third method of § 126, we have for the point P, $y = 1{\cdot}47''$, $z = 2''$. Also $I_y = 1{\cdot}89$ ins.^{4} and $I_z = 7{\cdot}77$ ins.^{4} Hence the line CD (Fig. 163) makes intercepts $\dfrac{7{\cdot}77 \text{ in.}^4}{1{\cdot}47 \text{ ins.}} = 5{\cdot}29$ in.^{3} on the axis GY and $\dfrac{1{\cdot}89 \text{ in.}^4}{2 \text{ ins.}} = 0{\cdot}945$ in.^{3} on the axis GZ.

Drawing this line in Fig. 163 we find that it cuts AB at E, and the length GE represents $1{\cdot}14$ ins.^{3} on the scale that GH represents $0{\cdot}945$ in.^{3}

Hence for a bending moment in the plane AB we have $Z_\theta = 1{\cdot}14$ in.^{3} for the point P. This gives, for the stress at P, $p = 0{\cdot}877M$, with $p = 7{\cdot}5$ tons/in.^{2}, the value of M will be $8{\cdot}55$ tons. ins.

127. Geometrically Similar Beams.—Suppose we have two beams A and B, that all the dimensions of the cross sections of B are n times those of A, and that they are made of the same material.

A moment of inertia of an area by definition is of dimensions (length)^4; therefore the moment of inertia of B will be n^4 times that of A, which we express by the equation

$$I_b = n^4 I_a.$$

The "y" of B is n times the "y" of A.

Therefore, with a given bending moment M, the fibre stresses in the two beams will be

$$p_a = \frac{My_a}{I_a} \text{ and } p_b = \frac{My_b}{I_b} = \frac{My_a}{n^3 I_n} = \frac{p_a}{n^3}.$$

$$\therefore \frac{p_b}{p_a} = \frac{1}{n^3},$$

the suffixes a and b denoting the two beams. On the other hand, if the bending moments which will cause equal fibre stresses p in the two beams are M_a and M_b, we have

$$M_a = \frac{pI_a}{y_a} \text{ and } M_b = \frac{pI_b}{y_b} = \frac{pn^3 I_a}{y_a} = n^3 M_a$$

$$\therefore \frac{M_b}{M_a} = n^3.$$

Suppose the lengths of the beams A and B are l and ml, and that they carry distributed loads w_a and w_b per unit length, being freely supported at the ends. Then the maximum bending moments are

$$M_a = \frac{w_a l^2}{8} \text{ and } M_b = \frac{w_b m^2 l^2}{8}.$$

The fibre stresses are

$$p_a = \frac{w_a l^2 y_a}{8I_a} \text{ and } p_b = \frac{w_b m^2 l^2 y_b}{8I_b} = \frac{w_b m^2 l^2 y_a}{8n^3 I_a}$$

$$\therefore \frac{p_b}{p_a} = \frac{m^2}{n^3} \cdot \frac{w_b}{w_a}.$$

If the stresses are equal we must have

$$\frac{w_b}{w_a} = \frac{n^3}{m^2}.$$

Similarly for concentrated loads W_a and W_b, the ratio required to produce equal stresses is

$$\frac{W_b}{W_a} = \frac{n^3}{m}.$$

STRAIN ENERGY OF BEAMS

128. Strain Energy Due to Normal Stresses.—Neglecting the shearing stresses, the strain energy per unit volume of the bent beam is, from equation (8) of Chapter I,

$$\frac{p^2}{2E},$$

where p is the direct stress, i.e. My/I. Hence the strain energy contained in a slice of beam between two cross sections, at distance δx apart, is

$$\delta U = \delta x \int \frac{M^2 y^2}{2EI^2} dS = \frac{M^2 \delta x}{2EI^2} \int y^2 dS,$$

where dS is an element of area of the cross section and the integral is taken over the whole section. But $\int y^2 dS = I$,

$$\therefore \delta U = \frac{M^2}{2EI} \delta x.$$

Hence the whole strain energy of the beam is

$$\mathrm{U} = \int_0^l \frac{\mathrm{M}^2}{2\mathrm{EI}} \mathrm{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where l is the length of the beam. If the section of the beam is constant this can be written

$$U = \frac{1}{2EI} \int_0^l M^2 dx.$$

Thus, if M be constant, $U = \dfrac{M^2 l}{2EI}$.

In the case of a beam freely supported at its ends carrying distributed load w per unit length, we have (§ 102)

$$M = \frac{w}{2}(lx - x^2)$$

where x is measured from one end.

$$\therefore U = \frac{1}{2EI} \int_0^l \frac{w^2}{4}(l^2x^2 - 2lx^3 + x^4)dx = \frac{w^2 l^5}{240EI}.$$

129. Change of Cross Section in Uniform Bending.—In § 117 we pointed out the change which takes place in the shape of the cross section when a beam is bent, and remarked on its general nature, observing that it is usually so small that it can be disregarded. In a footnote on p. 158 we showed the existence of radial pressures between the longitudinal filaments of a beam bent into a circular arc, and stated that they could be neglected; this is true in the general run of examples met with in engineering practice, but it is not universally true. If we begin our investigation on the hypothesis that the radial pressures can be neglected, we find that the cross section becomes curved, so that the top and bottom edges of a section, which was originally rectangular, are strained into concentric circular arcs with their centre on the opposite side of the beam to the axis of bending.

The top and bottom surfaces of the beam then have *anticlastic* curvature, the general nature of the strain being as shown in Fig. 164; it

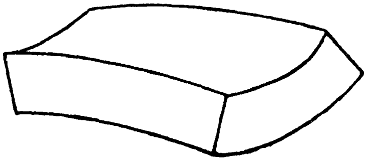

FIG. 164.

can be readily observed by bending a flat piece of indiarubber. If the original section was a rectangle of width $2a$ and depth $2b$, and the beam is bent to a mean radius R, we find that the cross sections are bent to a mean radius $R' = mR$, provided R is large compared with $a^2/2bm$. But with a thin flat beam, where a/b is large, it is quite possible for R to be small compared with $a^2/2bm$, and in this case the lateral curvature does not take place: it is neutralized by the radial forces which we have neglected, so that further analysis is required.*

130. Secondary Stresses in Beams.—By secondary stresses we mean those which, in most practical instances, are of only secondary importance; in general, these stresses are proportional to the squares of the displacements. For instance, we have seen above, in § 118, that each element of a beam is subjected to a radial force arising from the longitudinal stresses. This radial force produces a transverse bending moment, and therefore transverse tensile and compressive stresses in directions perpendicular to the plane of bending; moreover, the general effect of these transverse stresses is to reduce the moment of inertia of the section. When the section is difficult to distort, for example, a solid square or circular section, the effect of the transverse stresses is negligible, but in certain cases, particularly hollow thin sections,† large stresses may be induced. Another instance of secondary stresses is afforded by the type of beam failure considered below in Chapter XXIII: the tendency of a narrow deep beam to twist when it is bent in the plane of its depth. The secondary failure in these cases arises from want of torsional stiffness. Again, when the section is not symmetrical about an axis in the plane of bending, torsion occurs,‡ with the corresponding stresses, unless the load is applied in a unique manner; these torsional stresses are secondary stresses. In general, the equations * involved in the theory of secondary stresses are too complicated to be solved.

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* See Appendix I (2) below, p. 546.

† For instance, see an article by the author: *Phil. Mag.*, Feb., 1923, and Timoshenko's *Strength of Materials*, p. 465.

‡ Cf. p. 170.

APPENDIX: MOMENTS OF INERTIA

131. General Properties of Moments of Inertia.—If (y, z) be the co-ordinates of a point P, within a plane area S, referred to rectangular axes Oy, Oz in the plane of the area, and if dS denote an element of area enclosing the point P, the values of the integrals

$$I_y = \int z^2 \,.\, dS \text{ and } I_z = \int y^2 \,.\, dS,$$

taken over the whole area, are called the moments of inertia of the area S about Oy and Oz respectively.

The value of the integral $\int zy.dS$, taken over the whole area, is called the product of inertia for the axes Oy, Oz. Under certain circumstances the product of inertia for two rectangular axes through the centre of area is zero; these axes are then called the principal axes of the area; the quantities I_z and I_y are then called the principal moments of inertia.

132. Given the Moments of Inertia about the Principal Axes, to Find the Moments of Inertia about any other Line through the Centroid of the Area.—Let I_z and I_y be the moments of inertia about the principal axes GZ, GY (Fig. 165).

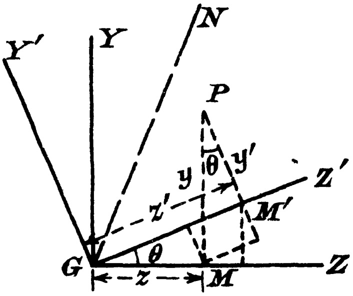

FIG. 165.

Let I_z' be the moment of inertia about any other line through G, such as GZ'.

Draw GY' perpendicular to GZ'.

Let (y, z) be the co-ordinates of any point P in the area S referred to GY, GZ; let $(y'z')$ be the co-ordinates of the same point referred to GY', GZ'. Then we have:

$$\begin{aligned} PM' = y' &= PM \cos\theta - GM \sin\theta \\ &= y\cos\theta - z\sin\theta \\ GM' = z' &= GM\cos\theta + PM\sin\theta \\ &= z\cos\theta + y\sin\theta. \end{aligned}$$

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* The matter is dealt with by A. A. Griffith: *Aeronautical Research Committee*, R. and M., 468. "Second Order Flexural Stresses."

Then, if dS stand for an element of area at P,

$$I_z' = \int y'^2 dS = \int (y \cos\theta - z \sin\theta)^2 dS$$

$$= \cos^2\theta . \int y^2 . dS - 2 \cos\theta \sin\theta \int yz dS + \sin^2\theta \int z^2 . dS .$$

Hence
$$\mathbf{I_z' = I_z \cos^2\theta + I_y \sin^2\theta}, \quad . \quad . \quad . \quad . \quad (22)$$

for $\int yzdS = 0$, GY and GZ being principal axes.

Similarly, the moment of inertia about GY' is

$$\mathbf{I_y' = I_z \sin^2\theta + I_y \cos^2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Adding (24) and (25) gives

$$\mathbf{I_z' + I_y' = I_z + I_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

that is, the sum of the moments of inertia about any two perpendicular axes through the centroid of an area is constant.

133. To Find the Principal Moments of Inertia.—When an area has an axis of symmetry it is evident that this axis is a principal axis. For, suppose GY is an axis of symmetry, then for every point in the area whose co-ordinates are (y, z) there will be a corresponding point whose co-ordinates are $(y, -z)$; therefore $\int yzdS$ taken over that part of the area which lies on the positive side of GY will have the same magnitude, but opposite sign, as $\int yz.dS$ taken over the negative side of GY; therefore the total value of $\int yz.dS$ will be zero.

When there is no axis of symmetry we can proceed as follows: find the centroid of the area by any convenient method, and take any pair of rectangular axes GY', GZ' with the centroid as origin.

Find the moments of inertia I_y' and I_z' about these axes.

Also find the moment of inertia I_N about a line GN bisecting the angle $Z'GY'$. (See Fig. 165.)

Let GY and GZ be the required principal axes.

Then, from (22), if I_y and I_z are the principal moments of inertia,

$$\begin{aligned} I_N &= I_z \cos^2(45^\circ + \theta) + I_y \sin^2(45^\circ + \theta) \\ &= \tfrac{1}{2}I_z\{1 + \cos(90^\circ + 2\theta)\} + \tfrac{1}{2}I_y\{1 - \cos(90^\circ + 2\theta)\} \\ &= \tfrac{1}{2}I_z(1 - \sin 2\theta) + \tfrac{1}{2}I_y(1 + \sin 2\theta) \\ &= \tfrac{1}{2}(I_z + I_y) + \tfrac{1}{2}(I_y - I_z)\sin 2\theta \\ &= \tfrac{1}{2}(I_z' + I_y') + \tfrac{1}{2}(I_y - I_z)\sin 2\theta, \end{aligned}$$

by (24).

$$\therefore (I_y - I_z)\sin 2\theta = 2I_N - (I_z' + I_y') \quad . \quad . \quad . \quad (\text{i})$$

Again, subtracting (22) from (23), we have

$$I_y' - I_z' = I_z(\sin^2\theta - \cos^2\theta) + I_y(\cos^2\theta - \sin^2\theta) = (I_y - I_z)\cos 2\theta$$

or
$$(I_y - I_z)\cos 2\theta = I_y' - I_z' \quad . \quad . \quad . \quad . \quad (\text{ii})$$

Dividing (i) by (ii) we obtain

$$\tan 2\theta = \frac{2I_N - (I_z' + I_y')}{I_y' - I_z'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

whence θ may be calculated. Then, to find I_z and I_y, we have the equations (24) and (ii), which can be written in the form:

$$\left.\begin{aligned} I_z + I_y &= I_z' + I_y' \\ I_z - I_y &= (I_z' - I_y') \sec 2\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

which are easily solved in a numerical case.

134. Ellipse of Inertia, or Momental Ellipse.—In Fig. 166 let GY and GZ be the principal axes of a given area S, and let the principal moments of inertia be written in the form Sk_z^2 and Sk_y^2 so that k_z and k_y are the principal radii of gyration measured at right angles to GZ and GY respectively. Similarly, let the moment of inertia about another line GY' be $I_y' = Sk_y'^2$.

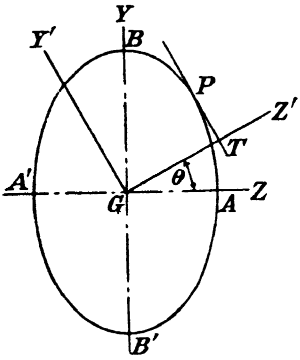


FIG. 166.

Then from (23) it follows that

$$k_y'^2 = k_y^2 \cos^2 \theta + k_z^2 \sin^2 \theta.$$

With G as centre draw the ellipse $ABA'B'$ (semi-axes k_y and k_z). This ellipse is called the ellipse of inertia, or momental ellipse; its equation is

$$\frac{z^2}{k_y^2} + \frac{y^2}{k_z^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Then the length GT of the perpendicular from the centre on to a tangent parallel to GY' is given by the well-known relation

$$GT^2 = k_y^2 \cos^2 \theta + k_z^2 \sin^2 \theta.$$

Hence $GT^2 = k_y'^2$.

Thus, to find the radius of gyration about GY', we draw a tangent to the ellipse (27) and drop a perpendicular GT. The length of this perpendicular gives the required radius of gyration.

135. Given the Moment of Inertia about an Axis through the Centroid of an Area, to Find the Moment of Inertia about any other Parallel Axis.—In Fig. 167 let I_G be the moment of inertia about ZZ which passes through the centroid G. Let $Z'Z'$ be a line parallel to ZZ at a distance h from it, and let I be the moment of inertia about $Z'Z'$.

Let y be the distance of an element of area dS from ZZ, then

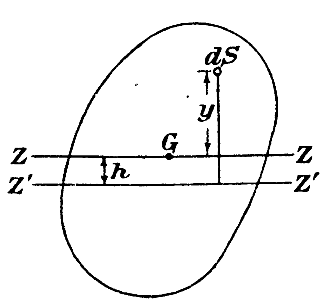


FIG. 167.

$$I = \int (y + h)^2 dS$$

$$= \int y^2.dS + 2h \int y.dS + h^2 \int dS.$$

The first integral is I_G; the second is zero since it represents the moment of the area about a line through the centroid; the third integral is the area. Hence

$$\mathrm{I} = \mathrm{I_G} + \mathrm{Sh}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where S denotes the area under consideration.

136. Graphical Determination of Moment of Inertia of an Irregular Section.—Let $APCQ$ be any area, the moment of inertia of which is required, about an axis through the c.g. parallel to a given direction Oz (Fig. 168).

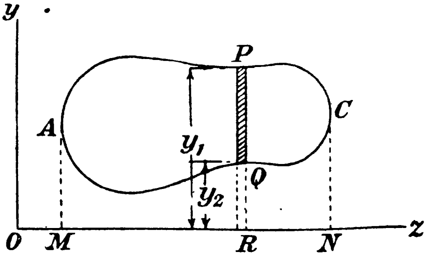


FIG. 168.

The area of the curve is given by

$$S = \int_{OM}^{ON} (y_1 - y_2)dz \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

which can be evaluated by a planimeter or the use of squared paper.

The moment of the area about Oz is given by

$$\mathrm{S\bar{y}} = \tfrac{1}{2} \int_{\mathrm{OM}}^{\mathrm{ON}} (\mathrm{y_1}^2 - \mathrm{y_2}^2)\mathrm{dz} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where $\bar{y}$ denotes the distance of the c.g. from Oz. The integral can be evaluated by plotting curves of $y_1{}^2$ and $y_2{}^2$ and finding their areas.

The moment of inertia about Oz is given by

$$\mathrm{I} = \tfrac{1}{3} \int_{\mathrm{OM}}^{\mathrm{ON}} (\mathrm{y_1}^3 - \mathrm{y_2}^3)\mathrm{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

which can be estimated by plotting curves of $y_1{}^3 - y_2{}^3$.

Finally, since

$$I = I_G + S\bar{y}^2$$

we can find the value of I_G from (29), (30) and (31).

137. Table of Moments of Inertia.—We shall not enter into the calculation of moments of inertia, as the subject is adequately treated in mathematical text-books, and books on mechanics. The following formulæ are given here for convenience of reference; the reader who has not met them before will find no difficulty in verifying them if he is familiar with the ordinary processes of integration.

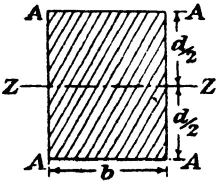
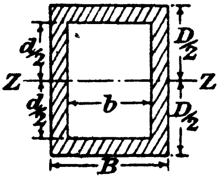
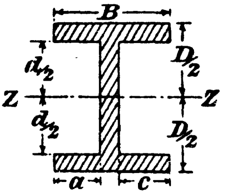
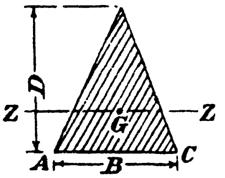
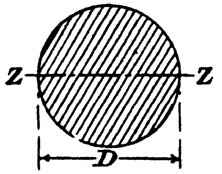
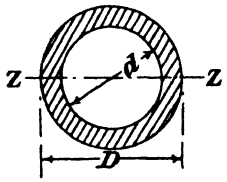
Figure.	Axis.	Moment of Inertia. I	Section Modulus. Z
	ZZ	$\frac{1}{12}bd^3$	$\frac{1}{6}bd^2$
	AA	$\frac{1}{3}bd^3$	
	ZZ	$\frac{1}{12}(BD^3 - bd^3)$	$\frac{1}{6} \cdot \frac{BD^3 - bd^3}{D}$
	ZZ	$\frac{1}{12}(BD^3 - bd^3)$ where $b = a + c$.	$\frac{1}{6} \cdot \frac{BD^3 - bd^3}{D}$
	ZZ	$\frac{1}{36}BD^3$	$\frac{1}{24}BD^2$
	AC	$\frac{1}{12}BD^3$	
	Any diameter.	$\frac{\pi D^4}{64}$	$\frac{\pi}{32}D^3$
		$\frac{\pi}{64}(D^4 - d^4)$	$\frac{\pi}{32} \cdot \frac{D^4 - d^4}{D}$
Circular Tube of outside diam. D and small thickness t.		$\frac{\pi}{8}D^3t$	$\frac{\pi}{4}D^2t$

Fig. 168.

138. Note on I Sections.—Girders and beams of this section are used so frequently in all branches of engineering that they merit special attention. It will be seen from (5) that for a beam of given depth the maximum longitudinal stress varies inversely as the moment of inertia of the section about the neutral axis. Consequently the first desirable quality of a beam section is that the moment of inertia should be as large as possible, whilst the area should be as small as it can be made consistently with this. This end can be attained by concentrating the greatest part of the area towards the top and bottom of the section, and it is by this reasoning that we arrive at the I section as the best for beams under ordinary bending conditions.

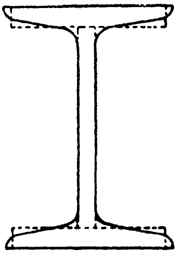


Fig. 169.

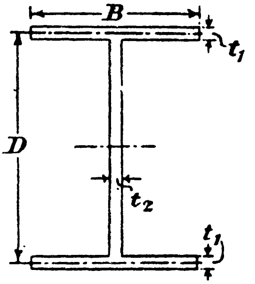


Fig. 170.

With rolled sections the form is usually similar to that shown in Fig. 169, and the moment of inertia can be found accurately by dividing the figure into rectangles, triangles, quadrants of circles, etc., but such a process is very laborious and gives results which are of greater accuracy than is consistent with the data in most cases. A good approximation can be made by drawing by eye an equivalent section composed of rectangles, as shown by the dotted lines. If greater accuracy than this is required the graphical method described above may be used.

In very many instances the thickness of the web and flanges is small compared with the other dimensions, particularly in girders built up of plates, so that we can use an approximate formula for the moment of inertia. This formula we shall proceed to discover.

Let the dimensions of the section be as shown in Fig. 170, D being measured between the centre lines of the two flanges.

The moment of inertia of each flange about its own centre line is

$$\tfrac{1}{12}Bt_1^3.$$

The area of each flange is Bt_1, and the distance of the centroid from the neutral axis of the section is $\frac{D}{2}$. Hence the moment of inertia of each flange about the neutral axis is

$$\tfrac{1}{12}Bt_1^3 + Bt_1 \cdot \frac{D^2}{4}$$

$$= \frac{Bt_1D^2}{4}\left(1 + \tfrac{1}{3}\frac{t_1^2}{D^2}\right)$$

When t_1/D is small we can neglect the second term; in this case the total moment of inertia of both flanges is

$$\frac{Bt_1D^2}{2} \quad \ldots\ldots\ldots \quad \text{(i)}$$

The moment of inertia of the web is

$$\tfrac{1}{12}t_2(D - t_1)^3 \quad \ldots\ldots\ldots \quad \text{(ii)}$$

Adding together (i) and (ii) we have the moment of inertia of the whole section. The practical advantage of this method of calculation is this: if we use the formula $\frac{1}{12}(BD^3 - bd^3)$ of § 137 we depend on the difference of two large numbers, the difference being small compared with either of them, and this makes accurate arithmetic very difficult, but by the above method we avoid this, and, using a slide rule in both cases, probably obtain greater accuracy with very thin sections. The method amounts to this: to the moment of inertia of the web add the joint area of the flanges × (semi-mean distance between flanges)^2.

EXAMPLES XI

1. A beam of I section is 10" deep and has equal flanges 4" broad. The web is 0·3" thick and the flanges 0·5". It may be stressed up to 8 tons/in.^{2}; what bending moment will it carry? (Special Exam., Cambridge, 1919.)

2. Fig. 171 illustrates the front axle of a motor wagon. The axle is of I section: flanges 3" × 1", web 2" × 1". Calculate the tensile stress at the bottom of the axle. (Intercoll. Exam., Cambridge, 1912.)

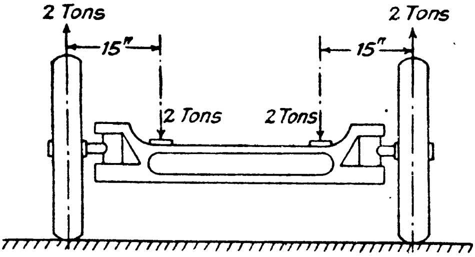


FIG. 171.

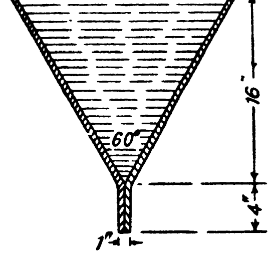


FIG. 172.

3. A water trough 20 ft. long, freely supported at the ends, has the section shown in Fig. 172. It is supported at its extremities and is filled with water. If the metal weighs 480 lbs. per ft.^{3}, and the water 62·5 lbs./ft.^{3}, calculate the greatest longitudinal stress for the middle cross section of the trough. (Intercoll. Exam., Cambridge, 1909.)

4. A built-up steel I-girder is 6 ft. deep over the flanges, each of which consists of four ½" plates, 3 ft. wide, riveted together. The web is ½" thick and is attached to the flanges by four 3½" × 3½" × ½" angle irons. The girder has a clear run of 80 ft. between the supports and carries 2 tons per foot run. Find the maximum longitudinal stress. (Mech. Sc. Trip., 1912.)

5. A beam rests on supports 8 ft. apart and carries a load of 1 ton uniformly distributed. The beam is rectangular in section, 3" deep. How wide should it be if the skin-stress must not exceed 4 tons/in.^{2} ? (R.N.E. College, Keyham, 1920.)

6. An I-section beam has flanges 3" × ½" and web 5" × ½" and rests on supports 9 ft. apart, it carries a concentrated load W at its centre, and a load $\frac{W}{2}$ one quarter of the way along. Calculate the magnitude of W if the maximum stress induced by bending is 4 tons per sq. inch. (Birmingham, 2nd year, 1910.)

7. Calculate the stresses set up by bending due to inertia forces in the locomotive coupling rod shown in Fig. 173, when the engine is running at 65 m.p.h. The diameter of the coupled wheels is 6 ft., and the distance of the coupling pins from the wheel centres is 15". One cubic inch of steel weighs 0·28 lb. Neglect the effect of the enlarged ends, i.e. assume the section shown to continue right up to the centres of the coupling pins and the rod to terminate there. (Birmingham, 3rd year, 1910.)

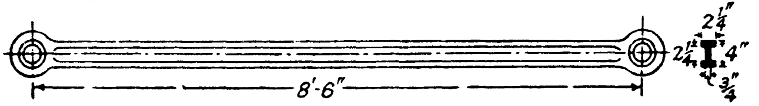


FIG. 173.

8. The construction of a reinforced concrete floor is shown in Fig. 174. Considering a width b of the floor as an independent beam, and assuming that the concrete takes no tension and that stress is proportional to strain in both concrete and steel, show that x, the height of the neutral axis above AB, is given by the equation $2E_sS(y + x) = E_cb(h - x)^2$, where S is the area of the tee section, y is the depth of its centre of gravity below AB, and E_s and E_c are the values of Young's Modulus for steel and concrete respectively.

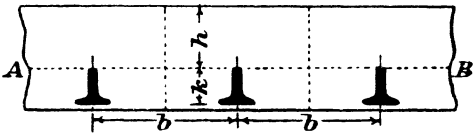


FIG. 174.

If the tees be 4" × 4" × ⅜" for which $S = 2{\cdot}872$ sq. in., and $y = 2{\cdot}894''$, and if $E_s = 15E_c$, find h, the depth of concrete above AB, so that the neutral axis may coincide with AB.

Also find, in this case, the Moment of Resistance of the section when the greatest stress in the concrete is 500 lbs. per sq. inch, the moment of inertia of the tee section about an axis through its centre of gravity, parallel to AB, being $4{\cdot}19$ inches.^{4} (Mech. Sc. Trip. B., 1915.)

9. Fig. 175 gives the dimensions of a concrete beam reinforced with steel rods. E for the concrete and steel is 2×10^6 and 30×10^6 lbs. per sq. inch respectively. The maximum compressive stress in the concrete is not to exceed 600 lbs. per sq. in. and the maximum tensile stress in the steel is not to exceed 16,000 lbs. per sq. in.

Prove that if the steel and concrete are to be efficiently stressed the neutral axis must be situated at a depth of 4·3 ins. below the top of the flange.

Determine also the sectional area of the steel, (1) neglecting tensile stress in the concrete, (2) taking this tensile stress into account. Making assumption (1) calculate the B.M. which will set up the stresses above mentioned. (Mech. Sc. Trip. B., 1911.)

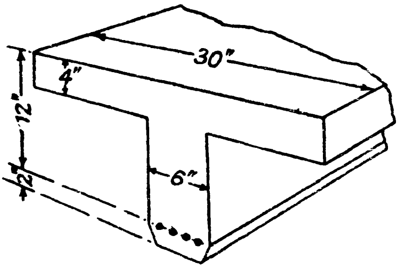


FIG. 175.

10. Fig. 176 shows a section of a reinforced concrete floor made of hollow beams placed side by side and connected together. Calculate the maximum distributed load which can be safely carried on a 10-foot span.

E for concrete = 2×10^6 pounds per sq. in.
E for steel = 3×10^7 „ „
Maximum safe stress, concrete = 500 „ „
„ „ steel = 12,000 „ „

(Mech. Sc. Trip. B., 1914.)

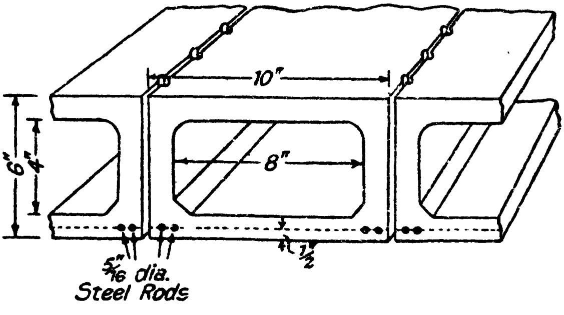


FIG. 176.

11. Draw the principal axes and the momental ellipse (ellipse of gyration) of the section of an angle-iron 3½" × 2½" outside × ¼" thick.

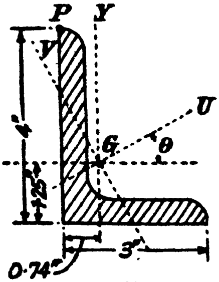


FIG. 177.

12. Fig. 177 shows the section of an unequal angle iron. It is subjected to a bending moment M, the plane of which has GY as trace. Calculate the longitudinal stress at the point P. The principal axes are GU and GV; the moment of inertia about GV is 1·54 in.^{4}, that about GU is 3·19 in.^{4}, and $\tan \theta = 0·55$. (Intercoll. Exam., Cambridge, 1909.)

13. An I beam of 16 ft. span with a web 11″ × ½″ and flanges 6″ × ½″ is inclined so that the web makes an angle of 10° with the vertical. The beam carries a vertical load of 400 lbs./ft. run. Determine the direction of the neutral axis and the greatest stress in the flanges. (Mech. Sc. Trip., 1910.)

14. Fig. 178 represents the cross section of a standard bulb angle bar. Explain how to find the principal axes and the principal moments of inertia of the section.

If the principal axes are inclined at $\tan^{-1} 0{\cdot}205$ with the sides A and B of the section and the principal moments of inertia are 13·52 and 1·42 inches^4, find the moment of inertia about the axes parallel to A and B through the centre of gravity of the section. (Mech. Sc. Trip., 1913.)

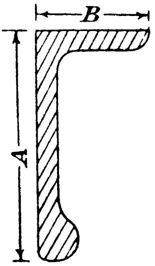


Fig. 178.

15. For the purpose of utilizing narrow strips of timber for planks, the design shown in Fig. 179 was adopted, in which the strips were fitted into a framework of bent steel plate and the whole bolted together by cross bolts. If the longitudinal stress in the timber must not exceed 1 ton/in.^{2} and that in the steel 7·5 tons/in.^{2}, estimate the moment of resistance to bending of the composite plank, about an axis AB, and show that it is about 2¾ per cent. greater than that of a solid plank of the same timber with a cross section 9″ × 1·5″. Take E for steel 20 times E for the timber. (Mech. Sc. Trip., 1921.)

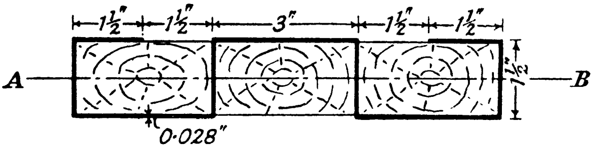


Fig. 179.

16. A reinforced concrete beam of rectangular section is required to span 20 ft. and support a load of 1,200 lbs./ft. If the width of the beam is 40 per cent. of its depth to the reinforcement, determine its size and the reinforcement required. The tensile stress allowed in the steel is 10,000 lbs./in.^{2}, and the compressive stress in the concrete 500 lbs./in.^{2}. $E_s/E_c = 12$. (Mech. Sc. Trip., 1920.)

17. Compare the strain-energy of a cantilever of uniform square cross section, loaded at the end with a weight W, with that of the same rod in simple tension the maximum stress being the same in each case.

18. Find the resilience of a beam of length l, and rectangular section, depth d and breadth b, carrying a load W at a point midway between the ends. Express the average resilience per unit volume in terms of the maximum stress f.

CHAPTER XII

BENDING STRESSES AND DIRECT STRESSES COMBINED

139. Introductory.—Very many instances arise in practice where a member undergoes bending combined with a thrust or pull, and we must investigate the stresses which arise in such cases. When the stiffness of the member is comparatively small so that the applied bending moments cause appreciable deflection, the end load, if a thrust, will increase the bending moment and deflection; if a tension it will decrease them. Thus, on the left of Fig. 180, AB is acted on by a thrust P and lateral loads; if the latter cause the axis of the beam at a section C to deflect an amount δ, the bending moment at C will be increased by $P\delta$; similarly on the right, where P is a tension, the B.M. at C is decreased by $P\delta$. These considerations lead us to a class of problems distinct in itself and treated below in Chapter XXII; for the present we shall suppose that the deflections are negligible. The type of problem to which our present considerations apply should be made clear by the examples on p. 190 and those at the end of the Chapter.

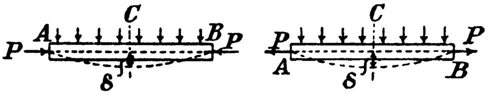


FIG. 180.

140. Stress Due to Combined Bending and Thrust.—Let AB be a small portion of the axis of a member (Fig. 181), which is subjected to an end thrust represented by the resultant P and a bending moment M in the plane $AYBY'$, which is assumed to be a plane of symmetry. Let O be the centroid of the cross section, and let ZOZ' be perpendicular to YOY' in the plane of the cross section.

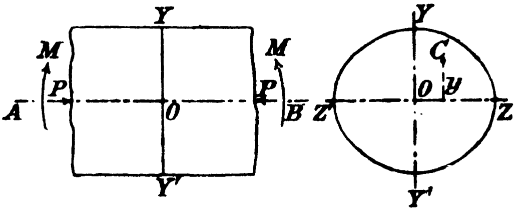


FIG. 181.

Let I = the moment of inertia of the cross section about ZZ', and
S = the area of the cross section.

Then the thrust P produces a compressive stress of intensity

$$\frac{P}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

at all points in the cross section.

At a point C, distant y from ZZ', the bending moment M will produce a stress of intensity

$$\frac{My}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

which will be compressive on one side of ZZ' and tensile on the other side. Combining (i) and (ii) we see that the total compressive stress at C is

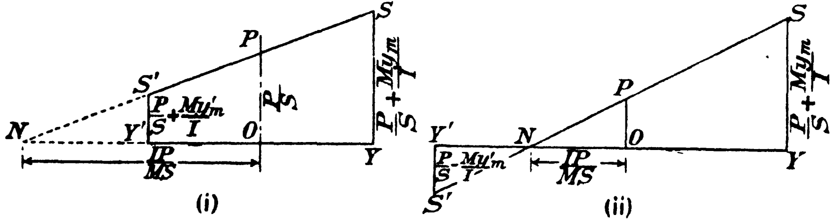
$$p = \frac{P}{S} + \frac{My}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(1)}$$

y being negative when measured towards the convex side of the member.

If y_m denote the maximum value of y on the concave side, and y'_m the maximum numerical value of y on the convex side, the maximum and minimum stresses will be

$$\frac{P}{S} + \frac{My_m}{I} \text{ and } \frac{P}{S} - \frac{My'_m}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(2)}$$

The stress intensity will vary in a linear manner from one side to the other, as shown in Fig. 182, where YS and $Y'S'$ represent the maximum and minimum stresses respectively, OP representing the mean stress. Two cases are shown in Fig. 182; in (i) the stress does not change sign, in (ii) it does. From equation (1) we see that the stress will be zero on a line for which


S
P
S'
N
Y'
O
Y
$\frac{P}{S}+\frac{My'_m}{I}$
$\frac{P}{S}$
$\frac{P}{S}+\frac{My_m}{I}$
$\frac{IP}{MS}$
S
P
Y'
N
O
Y
$\frac{P}{S}-\frac{My'_m}{I}$
$\frac{IP}{MS}$
S'
$\frac{P}{S}+\frac{My_m}{I}$

(i) (ii)

FIG. 182.

$$\frac{P}{S} + \frac{My}{I} = 0$$

or

$$y = -\frac{IP}{MS}$$

If this line falls outside the cross section, as in Fig. 182 (i), the stress will not change sign; if it falls within the section, as in (ii) the stress will change sign.

In cases where the longitudinal load is a tension instead of a thrust it is only necessary to change the sign of P above.

The results of this article are perfectly general and are not limited by the remarks made above in § 139. The effect of the considerations mentioned there is felt in their influence on the value of the bending moment; when the correct value of the bending moment has been found, the above results can be applied.

141. Eccentric End Load.—If the bending moment M above is due to an eccentric end load P as in Fig. 183, we have $M = Ph$, h being the distance of the line of action of the resultant force P from the centroid of the cross section. Equation (1) above then becomes

$$p = \frac{P}{S} + \frac{Phy}{I} = \frac{P}{S}\left(1 + \frac{Shy}{I}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The stress changes sign where

$$\frac{Shy}{I} = -1$$

or

$$y = -\frac{I}{Sh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The condition that the stress does not change sign is that the line given by this equation does not fall within the cross section. Hence, referring to Fig. 183, we must have

$$\frac{I}{Sh} \nless OY'$$

or

$$h \ngtr \frac{I}{S.OY'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

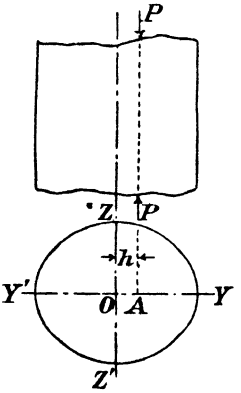


FIG. 183.

If the line of action of the resultant thrust is not at a greater distance than this from the centroid of the cross section, the stress will not change sign. This is of importance in masonry structures where it is not permissible for the material to be under tension.

It must be borne in mind always that h is the distance of the line of action of P from the centroid of the section *in the strained position*. When the bending is so slight that the deflection is negligible, h may be measured in the unstrained state, but when the bending is considerable we must first estimate the deflection which takes place. This problem will be considered later.

142. Circular Section.—When the cross section is a circle of diameter D we have $S = \pi D^2/4$, $I = \pi D^4/64$, and $OY' = D/2$. Hence the maximum value of h for there to be no reversal of stress is, by (5),

$$h = \frac{D}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

that is, the stress will not become tensile provided the eccentricity of the end load is not greater than one-eighth of the diameter of the cross section.

143. Rectangular Section.—Let the cross section be a rectangle of breadth b and length d as shown in Fig. 184, where A represents the line of action of the end thrust. We have then $I = \frac{1}{12}bd^3$, $S = bd$, $OY' = \frac{d}{2}$, so that (5) gives for the limiting value of h:

$$h = \frac{d}{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Similarly, if A be on the other side of O, its distance from O must not exceed $\frac{d}{6}$ if the stress is not to become tensile. Thus the line of action of the thrust must cut YY' somewhere in the middle third of its length if the stress is not to change sign anywhere in the section; this is a most important rule to remember in the design of masonry structures.

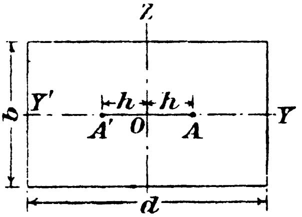


FIG. 184.

144. Unsymmetrical Bending with Eccentric End Load.—We have assumed above that the line of action of the resultant thrust cuts the cross section on a line of symmetry, i.e. on a principal axis. When this is not the case the bending moment, arising from the eccentricity of the end load, may be dealt with by the methods of § 126.

145. Core of Rectangular Section.—We shall find that there is a certain area within which the line of action of the resultant thrust must cut the cross section if the stress is not to become tensile; this area is called the *core*, or *kernel*, of the section. We shall consider a rectangular section. In Fig. 185 let YY' and ZZ' be the principal axes of the rectangular section $ABCD$; let F be the point where the line of action of the thrust P cuts the section, the co-ordinates of F being $(y'z')$ referred to OY and OZ.

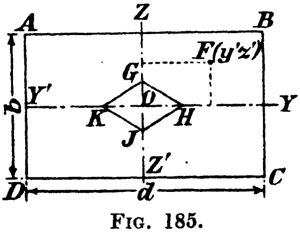


FIG. 185.

The force P at F is equivalent to a force P at O, together with couples Py' about OZ and Pz' about Oy. At any point whose co-ordinates are (y, z) these couples give rise to stresses

$$\frac{Py'y}{\frac{1}{12}bd^3} \text{ and } \frac{Pz'z}{\frac{1}{12}b^3d},$$

whilst the direct stress due to the thrust is $\frac{P}{bd}$. Hence the total compressive stress at (y, z) is

$$p = P\left(\frac{1}{bd} + \frac{12y'y}{bd^3} + \frac{12zz'}{b^3d}\right)^*$$

$$= \frac{P}{bd}\left(1 + \frac{12y'y}{d^2} + \frac{12zz'}{b^2}\right)$$

This will be greatest at B, where $y = \frac{d}{2}$ and $z = \frac{b}{2}$; it will be least at D where $y = -\frac{d}{2}$, $z = -\frac{b}{2}$, y' and z' being taken positive.

We have then, for the maximum and minimum stresses,

$$p_{max.} = \frac{P}{bd}\left(1 + \frac{6y'}{d} + \frac{6z'}{b}\right)$$

$$p_{min.} = \frac{P}{bd}\left(1 - \frac{6y'}{d} - \frac{6z'}{b}\right)$$

If the stress is not to become tensile we must have

$$\frac{6y'}{d} + \frac{6z'}{b} < 1.$$

Therefore $(y'z')$ must be within the space enclosed by the axes OY, OZ and the straight line

$$\frac{6y}{d} + \frac{6z}{b} = 1.$$

This line makes intercepts $\frac{d}{6}$ and $\frac{b}{6}$ on the axes and is shown as GH in Fig. 185: if the stress is not to become tensile at D, the line of action of P must cut the section within the triangle OGH. Similarly, if the stress is not to become tensile at B the thrust must act within the triangle OJK, where $OJ = \frac{b}{6}$ and $OK = \frac{d}{6}$. In general the thrust must act within the rhombus $GHJK$, which is the core of the section.

Example 1.—Find the maximum stress on the section AB of the cramp

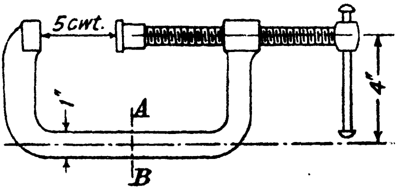


FIG. 186.

shown in Fig. 186 when a pressure of 5 cwt. is exerted by the screw. The section is rectangular 1" × $\frac{3}{8}$". (Special Exam., Cambridge, 1912.)

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* It should be noted that $\frac{1}{bd} + \frac{12y'y'}{bd^3} + \frac{12zz'}{b^3d} = 0$ is the equation of the neutral axis.

The section AB is subjected to a tension of 560 lbs., and a bending moment $560 \times 4 = 2{,}240$ lbs. ins.

The area of the section $= 0{\cdot}375$ in.²

$$\therefore \text{ the direct tensile stress } = \frac{560}{0{\cdot}375} = 1{,}490 \text{ lb./in.}^2$$

$$\text{The moment of inertia } = \frac{0{\cdot}375 \times 1}{12} = 0{\cdot}03125 \text{ ins.}^4$$

Therefore the maximum bending stresses due to the couple of 2,240 lb. ins. are equal to

$$\frac{2{,}240 \times 0{\cdot}5}{0{\cdot}03125} = 35{,}800 \text{ lbs./in.}^2$$

Hence the maximum tensile stress on the section

$$= 35{,}800 + 1{,}490 = 37{,}290 \text{ lbs./in.}^2$$

The maximum compressive stress is

$$35{,}800 - 1{,}490 = 34{,}310 \text{ lbs./in.}^2$$

Example 2.—A masonry pier has a cross section 6′ 0″ × 4′ 0″, and is subjected to a load of 100 tons, the line of the resultant being 2′ 4″ from one of the shorter sides, and 1′ 8″ from one of the longer sides. Find the maximum tensile and compressive stresses produced. (Intercoll. Exam., Cambridge, 1923.)

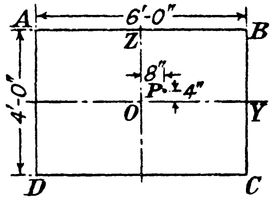

Fig. 187.

In Fig. 187, P represents the line of action of the thrust. The bending moments are

$4 \times 100 = 400$ tons. ins. about OY

$8 \times 100 = 800$ tons. ins. ,, OZ.

The moments of inertia are

$$I_y = \text{M. of I. about } OY = \frac{6 \times 64 \times 144^2}{12} = 32 \times 144^2 \text{ ins.}^4$$

$$I_z = \text{,, ,, } OZ = \frac{4 \times 216 \times 144^2}{12} = 72 \times 144^2 \text{ ins.}^4$$

The area of the section $= 48 \times 72$ in.²

For a point whose distances from OZ and OY are y, z, the stress is (§ 145)

$$p = \frac{100}{48 \times 72} + \frac{400z}{32 \times 144^2} + \frac{800y}{72 \times 144^2} \text{ tons/in.}^2$$

$$= \frac{100}{144 \times 24}\left(1 + \frac{z}{48} + \frac{y}{54}\right).$$

The compressive stress is a maximum at B, where $y = 36''$, $z = 24''$:—

$$p = \frac{100}{144 \times 24}(1 + \tfrac{1}{2} + \tfrac{2}{3})$$

$$= \frac{100 \times 13}{144 \times 24 \times 6} \text{ tons/in.}^2$$

$$= \frac{100 \times 13 \times 2{,}240}{144 \times 24 \times 6} \text{ lbs./in.}^2 = 140 \text{ lbs./in.}^2$$

The stress at D is

$$p = \frac{100}{144 \times 24}(1 - \tfrac{1}{2} - \tfrac{2}{3}) = -\frac{100}{144 \times 24 \times 6} \text{ tons/in.}^2$$

$$= -10{\cdot}8 \text{ lbs./in.}^2$$

Hence the maximum compressive stress is 140 lbs./in.², and the maximum tensile stress 11 lbs./in.² nearly.

BENDING COMBINED WITH END LOAD IN REINFORCED CONCRETE*

146. Bending and Axial Thrust: No Tensile Stresses.—In the first instance we shall assume that the loads are such that no tensile stresses are set up, either in the concrete or the reinforcement. In dealing with homogeneous beams above, the bending moment on the cross section was referred to an axis through the centroid of the section, which axis represents the neutral layer of the beam when it is only subjected to bending. Similarly, in dealing with reinforced concrete beams, the bending moment must be referred to an axis which would be the neutral axis if there were no end load; we shall call this the *bending axis* of the section.

Now, by the principle of superposition, the final value of the compressive stress on the convex side of the beam will be the resultant of the compressive stress due to the axial load, and the tensile stress which would be induced if the bending moment were applied separately. Consequently in seeking the position of the bending axis of the section, we must assume that the concrete can temporarily bear tension until the end thrust is applied. We can then find the position of the bending axis as described below, where we shall only consider a rectangular section.

In Fig. 188 let $ABCD$ be the cross section of the concrete, AB being in the compression face of the beam, which is supposed acted on only by a bending moment M.

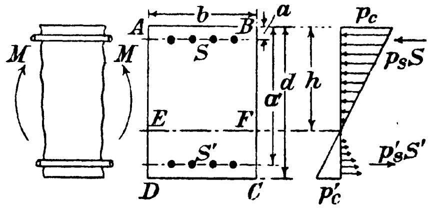

FIG. 188.

Let S = total area of reinforcement near the edge AB, at a distance a from it.

,, S' = ditto, near the face DC, at a distance a' from AB.

,, p_s and p_s' be the compressive and tensile stresses in the steel S and S', due to a bending moment M.

,, p_c and p_c' be the maximum compressive and tensile stresses in the concrete due to the bending moment M.

,, h = the distance of the bending axis from the compression edge AB.

,, E_s and E_c be Young's Modulus for steel and concrete, assumed the same for tension and compression.

Then, neglecting the reduction of the concrete area by the reinforcement, the total normal force on the section is, with a linear distribution of stress,

$$\frac{p_c - p_c'}{2} \cdot bd + p_sS - p_s'S' \quad . \quad . \quad . \quad . \quad \text{(i)}$$

which must be zero, since at present there is no total end thrust.

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* See also *Engineering*, Jan. 18, 1929.

On the assumption that cross sections remain plane we must have (see § 124, p. 166)

$$\frac{p_s}{p_c} = \frac{h-a}{h} \cdot \frac{E_s}{E_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{p_s'}{p} = \frac{a'-h}{h} \cdot \frac{E_s}{E_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{p'_c}{p_c} = \frac{d-h}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Substituting for p_s, p_s' and p_c' from these in terms of p_c in (i) and equating the result to zero we find, by solving the resulting equation,

$$h = \frac{\frac{d}{2}bd + \frac{E_s}{E_c}(aS + a'S')}{bd + \frac{E_s}{E_c}(S + S')} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If desired, for greater accuracy the actual nett area of the concrete can be inserted instead of bd in this expression. This equation gives the distance of the bending axis from the concave face of the beam, and it is to this axis that the bending moment must be referred.

The bending moment equivalent to a compressive stress p_c in the concrete is found by summing up the moments of the stresses about EF, thus:

$$M = \tfrac{1}{3}p_c.bh^2 + \tfrac{1}{3}p_c'b(d-h)^2 + p_sS(h-a) + p_s'S'(a'-h).$$

Substituting for p_c', p_s, p_s' from (ii), (iii) and (iv) this gives

$$M = \frac{p_c}{h}\left[\tfrac{1}{3}bh^3 + \tfrac{1}{3}b(d-h)^3 + \frac{E_s}{E_c}\left\{(h-a)^2S + (a'-h)^2S'\right\}\right] \quad (12)$$

If the bending moment is known this enables us to calculate p_c; p_c', p_s, p_s' can then be found from (8), (9), (10).

Now let p_{c0} and p_{s0} be the compressive stresses set up in the concrete and steel by an axial thrust P. Then we must have, if there be no slip between the steel and the concrete,

$$\frac{p_{c0}}{p_{s0}} = \frac{E_c}{E_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and, neglecting again the reduction of the area of the concrete by the reinforcement,

$$p_{c0} \cdot bd + p_{s0}(S + S') = P \quad . \quad . \quad . \quad . \quad (ii)$$

From (ii) and (13) we find

$$p_{c0} = \frac{P}{bd + \frac{E_s}{E_c}(S + S')} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

From this we can calculate p_{c0}, and then p_{s0} is given by (13).

Under the combined action of P and M, the maximum compressive stress in the concrete will be

$$p_c + p_{c0},$$

whilst the minimum will be

$$p_{c0} - p_c'$$

Similarly, the maximum and minimum compressive stresses in the steel are $p_s + p_{s0}$ and $p_{s0} - p_s'$ respectively.

For the minimum compressive stress in the concrete to be zero we must have

$$p_{c0} = p_c' = \frac{d-h}{h} p_c$$

This requires, from (12) and (14)

$$\frac{M}{P} = \frac{\frac{1}{3}bh^3 + \frac{1}{3}b(d-h)^3 + \frac{E_s}{E_c}\{(h-a)^2 S + (a'-h)^2 S'\}}{(d-h)\{bd + \frac{E_s}{E_c}(S+S')\}} \quad . \quad (15)$$

If M/P be greater than this, part of the concrete will be thrown into tension, and a different method of procedure is required. This we shall now explain.

147. Bending and Axial Thrust: When there are Tensile Stresses.—In Fig. 189, let O be the centroid of the section, and let the axial thrust and bending moment be referred to O. Let h = the distance of the neutral axis from AB, the compression face of the beam.

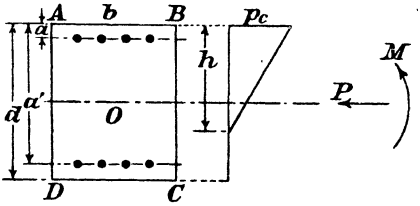

FIG. 189.

Let p_c = the maximum compressive stress in the concrete, and neglect the tensile stress in the concrete due to the combined action of P and M.

,, p_s and p_s' be the compressive and tensile stresses in the steel respectively, due to the combined action of P and M.

For the rest the notation is the same as above.

Then we must have:

$$P = \frac{p_c}{2} bh + p_s S - p_s S' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

$$M = \frac{p_c}{2} . bh\left(\frac{d}{2} - \frac{h}{3}\right) + p_s S\left(\frac{d}{2} - a\right) + p_s' S'\left(a' - \frac{d}{2}\right) \quad . \quad (ii)$$

We also have, as before,

$$\frac{p_s}{p_c} = \frac{E_s}{E_c} \cdot \frac{h-a}{h} \text{ and } \frac{p_s'}{p_c} = \frac{a'-h}{h} \cdot \frac{E_s}{E_c} \quad . \quad . \quad . \quad (16)$$

Substituting for p_s and p_s' in (i) and (ii) gives

$$P = \frac{p_c}{h}\left[\frac{bh^2}{2} + \frac{E_s}{E_c}\left\{(h-a)S - (a'-h)S'\right\}\right] \quad . \quad . \quad . \quad (17)$$

and

$$M = \frac{p_c}{h}\left[\frac{bh^2}{2}\left(\frac{d}{2} - \frac{h}{3}\right) + \frac{E_s}{E_c}\left\{(h-a)\left(\frac{d}{2} - a\right)S + (a'-h)\left(a' - \frac{d}{2}\right)S'\right\}\right] \quad (18)$$

From these two equations h and p_c can be found for a given load when the dimensions are known; p_s and p_s' are then given by (16). Eliminating p_c from (17) and (18) by division, and rearranging gives a cubic equation for h :—

$$\frac{b}{6}h^3 + \frac{b}{2}\left(\frac{M}{P} - \frac{d}{2}\right)h^2 - \frac{E_s}{E_c}\left[\left(\frac{d}{2} - a\right)S - \left(a' - \frac{d}{2}\right)S' - \frac{M}{P}(S+S')\right]h$$
$$+ \frac{E_s}{E_c}\left[aS\left(\frac{d}{2} - a\right) - a'S'\left(a' - \frac{d}{2}\right) - \frac{M}{P}(aS + a'S')\right] = 0 \quad . \quad (19)$$

When this has been solved for h, p_c is given by equation (17).

Frequently the reinforcement is symmetrical and symmetrically arranged, so that $S' = S$, and $a' = d - a$. In this case the equation for h reduces to *

$$\frac{b}{6}h^3 + \frac{b}{2}\left(\frac{M}{P} - \frac{d}{2}\right)h^2 + 2\frac{E_s}{E_c} \cdot \frac{MS}{P}h - \frac{E_s}{E_c}S\left[2\left(\frac{d}{2} - a\right)^2 + \frac{M}{P}d\right] = 0 \quad (20)$$

and the equation for p_c becomes

$$p_c = \frac{Ph}{\frac{bh^2}{2} + \frac{E_s}{E_c}(2h-d)S} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

In practice, where the area of reinforcement has to be estimated, a first approximation may be made as follows: Calculate the stresses as for a homogeneous concrete section, and then let all the tensile stress be taken by steel reinforcing bars. In this way a first idea of the size of the steel rods required may be obtained. Since this neglects the reinforcement on the compression side it cannot be expected to give great accuracy; it will usually be found to give the compression stress in the concrete with accuracy, but to overestimate the stress in the steel considerably.

---

* This can be solved by trial or by the following method:
To solve $x^3 + ax^2 + bx + c = 0$, write $x = z - a/3$, and the equation takes the form $z^3 + pz + q = 0$, the roots of which are

$$z = \sqrt[3]{\sqrt{\frac{q^2}{4} + \frac{p^3}{27}} - \frac{q}{2}} + \sqrt[3]{-\sqrt{\frac{q^2}{4} + \frac{p^3}{27}} - \frac{q}{2}}$$

This is Cardan's solution of a cubic equation.

148. Bending and Axial Tension Combined.—This case can be treated exactly as in § 146 above, the same equations obtaining if the sign of P be changed throughout. The same approximate method referred to above can be used as a first approximation in design.*

Example.—The section shown in Fig. 190 is subjected to a bending moment of 82,000 lbs. ft., and an axial thrust of 54,000 lbs. Find the maximum stresses in the steel and concrete, taking $E_s/E_c = 15$.

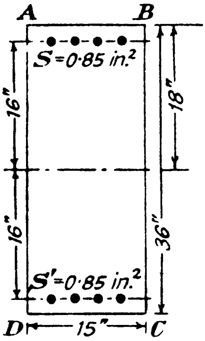

FIG. 190.

Since the reinforcement is symmetrically placed we can use equation (20) to find the depth of the neutral axis from the compression face AB. We have

$$b = 15'', \quad \frac{M}{P} = \frac{82{,}000 \times 12}{54{,}000} = 18{\cdot}222''. \quad d = 36''. \quad S = 2{\cdot}14 \text{ in.}^2$$

$$\frac{M}{P} - \frac{d}{2} = 0{\cdot}222''$$

$$2\frac{E_s}{E_c} \cdot \frac{MS}{P} = 30 \times 18{\cdot}222'' \times 2{\cdot}14 = 1{,}170$$

$$a = 2''\,; \quad \frac{d}{2} - a = 16''\,; \quad 2\left(\frac{d}{2} - a\right)^2 = 512$$

$$\frac{M}{p}d = 656\,; \quad \frac{E_s}{E_c}S = 32{\cdot}1$$

Hence the equation for h is:

$$2{\cdot}5h^3 + 7{\cdot}5 \times 0{\cdot}222h^2 + 1{,}170h - 37{,}500 = 0$$
$$h^3 + 0{\cdot}666h^2 + 468h - 15{,}000 = 0.$$

To solve this put

$$h = x - 0{\cdot}222$$

and the equation becomes

$$x^3 + 467{\cdot}8x - 15{,}100 = 0.$$

The real root of this is

$$\sqrt[3]{\sqrt{\left(\frac{467{\cdot}8}{3}\right)^3 + \left(\frac{15{,}100}{2}\right)^2} + 7{,}550} + \sqrt[3]{-\sqrt{\left(\frac{467{\cdot}8}{3}\right)^3 + \left(\frac{15{,}100}{2}\right)^2} - 7{,}550}$$

$$= \sqrt[3]{15{,}347} + \sqrt[3]{-247}$$
$$= 24{\cdot}852 - 6{\cdot}270$$
$$= 18{\cdot}582$$

---

* A graphical treatment of the above problems will be found in Mörsch's *Der Eisenbetonbau* (see footnote, p. 164).

Hence

$$h = 18{\cdot}582 - 0{\cdot}22 = 18{\cdot}36''.$$

Then, from (21) the compressive stress in the concrete is

$$p_c = \frac{54{,}000 \times 18{\cdot}36}{\dfrac{15 \times 336}{2} + 15 \times 0{\cdot}72 \times 2{\cdot}14}$$

$$= \frac{54{,}000 \times 18{\cdot}36}{2{,}543} = 390 \text{ lbs./in.}^2$$

The compressive stress in the steel is, equation (16), given by

$$p_s = 15 \times \frac{16{\cdot}36}{18{\cdot}36} \times 390 = 5{,}220 \text{ lbs./in.}^2,$$

and the tensile stress in the steel is

$$p_s' = 15 \times \frac{19{\cdot}64}{18{\cdot}36} \times 390 = 6{,}250 \text{ lbs./in.}^2$$

Using the approximate method suggested above in § 146 we proceed as follows :

For the concrete section,

$$I = \frac{15 \times 36^3}{12}$$

$$M = 82{,}000 \times 12 \text{ lbs. ins.}$$

Then the stresses due to bending are

$$\frac{82{,}000 \times 12 \times 18}{\dfrac{15 \times 36^3}{12}} = 304 \text{ lbs./in.}^2$$

The area of the section = 15 × 36 = 540 in.², hence the direct stress due to the thrust is

$$\frac{54{,}000}{540} = 100 \text{ lbs./in.}^2$$

Then the maximum compressive stress in the concrete = 404 lbs./in.², and the maximum tensile stress = 204 lbs./in.², giving a stress distribution as shown in Fig. 191. This makes $h = 23{\cdot}9''$. The total tensile stress = 102 × 15 × 12·1 = 18,510 lbs. The area of the steel section being 2·14 in.², the tensile stress in the steel would be 8,650 lbs./in.² Thus the compressive stress in the concrete is given fairly accurately, but the tensile stress in the steel is very badly wrong.

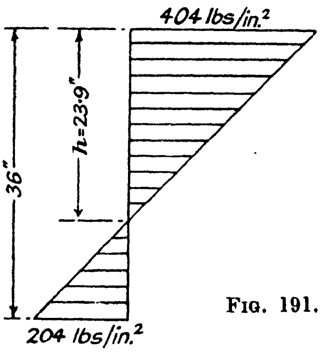

FIG. 191.

EXAMPLES XII

1. Fig. 192 represents a cantilever hydraulic crane. The single rope supports a load of 20 tons and passes over two pulleys and then vertically down the axis of the crane to the hydraulic apparatus. The section AB of the crane is a hollow rectangle. The outside dimensions are 15″ and 30″ and the material is 1″ thick all round, and the longer dimension is in the direction AB. Calculate the maximum tensile and compressive stresses set up in the section, and locate the position of the neutral axis. (Mech. Sc. Trip., 1907.)

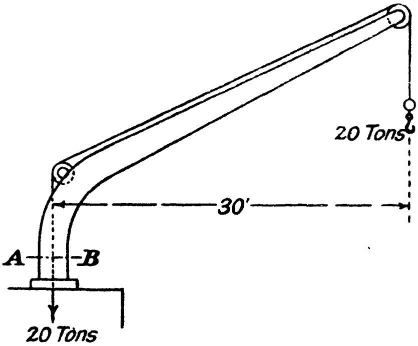

FIG. 192.

2. Fig. 193 shows the horizontal cross section of the cast-iron standard of a vertical drilling machine. The line of thrust of the drill passes through P. Find the greatest value the thrust may have without the tensile stress exceeding 1 ton/in.² What will be the stress along the face AB? (Intercoll. Exam., Cambridge, 1920.)

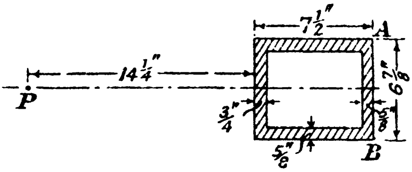

FIG. 193.

3. A vertical masonry chimney has an internal diameter d_1 and an external diameter d_0. The base of the chimney is given a horizontal acceleration a ft./sec.², and the whole chimney moves horizontally with this acceleration. Show that at a section at depth h below the top of the chimney, the resultant normal force acts at a distance $ah/2g$ from the centre of the section. If the chimney behaves as an elastic solid, show that at a depth $g(d^2_0+d^2_1)/4ad_0$ below the top, tensile stress will be developed in the material. (Mech. Sc. Trip., 1910.)

4. A masonry column has a rectangular section $ABCD$; $AB = CD = 6$ ft. and $BC = AD = 4$ ft. It carries a load of 100 tons concentrated at a point P, 2 ft. from AB and 2½ ft. from BC. Determine the greatest and least compressive stress, and the position of the neutral axis, for a horizontal section some distance below $ABCD$. (Intercoll. Exam., Cambridge, 1911.)

5. A horizontal cross section of a short reinforced concrete column is shown in Fig. 194, where P is the point of application of the load. Find the maximum value this load may have if the steel is to be stressed to 16,000 lbs./in.^{2} and the concrete to not more than 500 lbs./in.^{2} What area of steel will be required? Take E for steel $= 12$ times E for concrete. (Mech. Sc. Trip., 1921.)

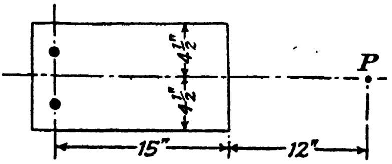

Fig. 194.

6. A link of a valve gear has to be curved in one plane, for the sake of clearance, as shown in Fig. 195. Estimate the maximum tensile and compressive stress in the link if the thrust is 540 lbs. (Mech. Sc. Trip., 1920.)

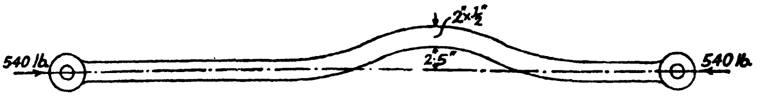

Fig. 195.

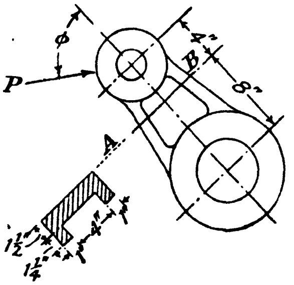

Fig. 196.

7. Fig. 196 shows a cast-iron crank with a section on the line AB. Show how to determine the greatest compressive and tensile stresses at AB, normal to the section, due to the thrust P of the connecting rod at the angle φ shown.

Prove that in order that the stresses at the section shall not exceed 3 tons per sq. in. either in tension or compression the thrust P must not exceed 10·2 tons. (Mech. Sc. Trip., 1919.)

8. A stanchion is built up of two channels $12'' \times 4'' \times \frac{5}{8}''$ placed back to back, 8" apart, and riveted to two plates $16'' \times \frac{1}{2}''$. The stanchion carries a bracket bolted up against one of the 16" plates, and the effective load is 15 tons at a distance of 3′ 6″ from the centre line of the stanchion. Find the limits of stress in a section of the stanchion. (Mech. Sc. Trip., 1915.)

9. Fig. 197 illustrates a cast-iron bracket carrying a bearing. The load on the bearing is 1,000 lbs. The form of the section AB is given. Calculate the greatest tensile stress across the section AB and the distance of the neutral axis of the section from the centre of gravity of the section. (Intercoll. Exam., Cambridge, 1914.)

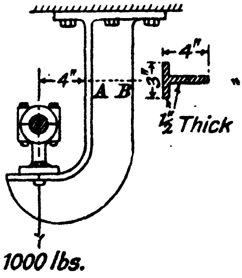

Fig. 197.

CHAPTER XIII

SHEARING STRESSES IN BEAMS

149. Introductory.—In § 117 we have referred to the existence of shearing stresses and have given a general physical indication of their nature in bent beams. We then proceeded to investigate the longitudinal stresses in a beam when shear is absent, and stated that the result could be applied with sufficient accuracy to the cases when shear is present. That is to say, that the effect of shear *on the longitudinal stresses* is not of any practical importance, but it does not imply that the shearing stresses can be neglected altogether: they must be considered for their own importance, not for their effect on longitudinal stresses. In the elementary treatment which we shall give here, we consider only the shear stresses in transverse planes parallel to the shearing force due to the load, and the complementary shear stresses in longitudinal planes parallel to the axis of the beam. The presence of other shearing stresses is disregarded. On these assumptions we examine the distribution of shearing stress over the cross section.

150. Elementary Treatment of the Distribution of Shearing Stress.—In what follows, we neglect the variation of the intensity of shear stress over the width of the section. This is not strictly accurate, but no great error is involved, and in the majority of practical cases the width of the beam, in the part which carries most of the shear, is small compared with the depth.

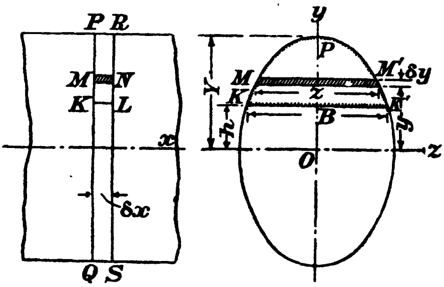

FIG. 198.

Let PQ and RS be two normal cross sections of the beam, separated by a distance δx (Fig. 198); let the bending moment on the section PQ

be M, and that on the section RS be $M + \delta M$, so that, in the limit when δx is made indefinitely small, the shearing force on either section is given by (§ 96)

$$F = -\frac{dM}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{i})$$

Now the longitudinal stress across the section PQ, at a distance y from the neutral axis is

$$p = \frac{My}{I}$$

where I has the same meaning as previously. Similarly, if the cross section of the beam is constant, the longitudinal stress across RS at the same height is

$$p + \delta p = \frac{(M + \delta M)y}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{ii})$$

Now consider a small slice MNM' bounded at the ends M and N by the sections PQ and RS. Let the thickness of the slice be δy and the width z, so that the areas of its ends are $z.\delta y$. Then the end M is acted on by a thrust

$$p(z.\delta y) = \frac{My}{I}z.\delta y,$$

and the end N is acted on by a thrust

$$(p + \delta p)\,(z.\delta y) = \frac{M + \delta M}{I}yz.\delta y.$$

Hence there is a resultant thrust acting on the slice, in the direction NM, given by

$$\frac{\delta M}{I}yz.\delta y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{iii})$$

Next consider the volume of the beam above some level KLK' (Fig. 198) included between the sections PQ and RS, and let the height of KL above the neutral axis be h; also let B be the width KK'. Then this piece of beam will be acted on by an unbalanced thrust in the direction LK equal to

$$\frac{\delta M}{I}\int_h^Y yz.dy$$

where Y is the extreme height OP. This force is balanced by the shear on the surface KLK', the area of which is $B.\delta x$.

Let q be the mean intensity of the shearing stress on KLK', then the total tangential force is $B\delta x.q$. Hence, for equilibrium, we must have

$$B\delta x.q = -\frac{\delta M}{I}\int_h^Y yz.dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{iv})$$

q being positive when it acts to the left on $PRLK$.

Therefore, in the limit, when δx is made indefinitely small, we have

$$q = -\frac{1}{IB}\cdot\frac{dM}{dx}\int_h^Y zy.dy$$

$$= \frac{F}{IB}\int_h^Y zy.dy \quad . \quad . \quad \text{[by (i)]} \quad . \quad . \quad . \quad . \quad (1)$$

Now the integral represents the moment, about the neutral axis, of the area of the cross section included between the level h and the top of the beam.

Let S = the area of the part KPK' of the cross section,

$\bar{y}$ = the distance of the centroid of this area from the neutral axis, then we have, from the above expression for q,

$$\mathbf{q} = \frac{\mathbf{FS\bar{y}}}{\mathbf{IB}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is a very important formula.

When the depth of the beam is not uniform a different treatment is required, since it is not then necessary that the shear stress should vanish at the upper and lower surfaces. (See an article by Filon in *Engineering*, Dec. 12, 1924.)

151. Special Cases ; Beams of Constant Section.—

(i) Rectangular Cross Section (Fig. 199).—In this case z is constant and equals B, which is the same at all depths ; hence from (1) of § 150,

$$q = \frac{F}{I}\int_h^{D/2} ydy = \frac{12F}{BD^3}\left(\frac{D^2}{8} - \frac{h^2}{2}\right) \quad . \quad . \quad . \quad \text{(see p. 180)}$$

$$= \frac{6F}{BD^3}\left(\frac{D^2}{4} - h^2\right)$$

The maximum occurs when $h = 0$, and is given by

$$\mathbf{q}_{\max} = \frac{3}{2}\cdot\frac{\mathbf{F}}{\mathbf{BD}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since BD is the area of the cross section, we see that the maximum shear stress is 50 per cent. greater than the mean over the whole section. It is zero at the top and bottom, and the curve of q plotted against h is a parabola as shown in Fig. 199.

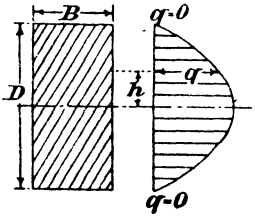

FIG. 199.

(ii) Circular Cross Section.—Referring to Fig. 200 we have now

$$z = 2R\cos\theta \qquad B = 2R\cos\varphi$$

$$y = R\sin\theta \qquad h = R\sin\varphi$$

$$\therefore\ dy = R\cos\theta.d\theta.$$

Therefore, by (1), § 150,

$$q = \frac{F}{2R.I\cos\varphi}\int_{\phi}^{\pi/2} 2R^3\cos^2\theta\sin\theta.d\theta$$

$$= \frac{2F}{\pi R^5\cos\varphi}.2R^3.\left[\frac{-\cos^3\theta}{3}\right]_{\phi}^{\frac{\pi}{2}}$$

$$= \frac{4F}{\pi R^2\cos\varphi}.\frac{\cos^3\varphi}{3}$$

$$= \frac{4F}{3\pi R^2}\cos^2\varphi.$$

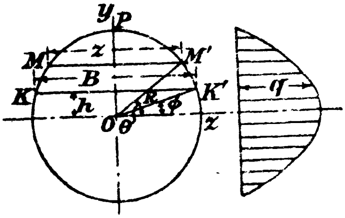

FIG. 200.

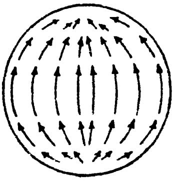

FIG. 201.

From the figure $\sin\varphi = \frac{h}{R}$, hence

$$q = \frac{4F}{3\pi R^2}\left(1 - \frac{h^2}{R^2}\right)$$

This is a maximum when $h = 0$, and the maximum value is given by

$$q_{max} = \frac{4F}{3\pi R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The area of the section is πR^2, so that the mean intensity of shear stress is $F/\pi R^2$. Hence the maximum is $\frac{4}{3}$ times the mean; the shear stress is zero at the top and bottom, and varies according to a parabolic curve.

The value of q found above is only to be regarded as giving the average shear stress on a horizontal strip at a distance h from the neutral axis. The actual distribution of shear stress on a vertical section is somewhat as indicated in Fig. 201.

(iii) Thin Circular Tube (see Fig. 202).—

Let R = the mean radius of the tube,

t = the radial thickness.

Then $I = \pi R^3 t$ approximately (p. 180), and

$S\bar{y}$ = the moment about Oz of the area KPK'

$$= 2\int_{\phi}^{\pi/2} Rd\theta.t.R\sin\theta$$

$$= 2R^2t\cos\varphi.$$

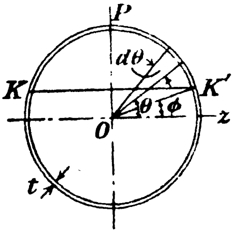

FIG. 202.

And b, the thickness of the section, measured parallel to Oz, at the level KK', is approximately :

$$b = 2t \sec \varphi.$$

Hence, by equation (2)

$$\mathbf{q} = \frac{F \times 2R^2 t \cos \varphi}{\pi R^3 t \times 2t \sec \varphi} = \frac{\mathbf{F}}{\pi \mathbf{Rt}} \cos^2 \boldsymbol{\varphi} \quad . \quad . \quad . \quad . \quad (5)$$

The maximum value is $\frac{F}{\pi Rt}$, when φ is zero, i.e. on a horizontal diameter. The mean intensity over the whole section is $\frac{F}{2\pi Rt}$, so that the maximum is twice the mean.

(iv) **Rectangular I Section.**—In the I section shown in Fig. 203, the distribution of shearing stress in the flanges must be similar to that indicated by the arrows at the top of the figure, and is not open to calculation. The formulæ found above will tell us the *average* shearing stress at any height above the neutral axis, and that is all ; we shall see in a moment that this is all that matters in most practical cases.

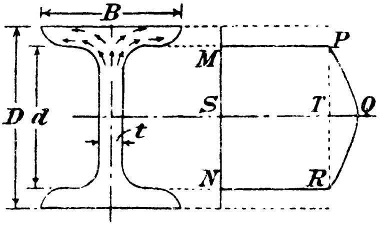

FIG. 203.

For a height y above the neutral axis, when $0 < y < \frac{d}{2}$, we have approximately

$$S\bar{y} = t\left(\frac{d}{2} - y\right)\tfrac{1}{2}\left(\frac{d}{2} + y\right) + \frac{B(D-d)}{2} \times \frac{D+d}{4}$$

$$= \frac{t}{2}\left(\frac{d^2}{4} - y^2\right) + \frac{B}{8}(D^2 - d^2).$$

Hence, in the web,

$$q = \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right) + \frac{FB}{8It}(D^2 - d^2) \quad . \quad . \quad . \quad (\mathrm{i})$$

The maximum occurs when $y = 0$, so that

$$q_{max} = \frac{F}{8I}\left\{d^2 + \frac{B}{t}(D^2 - d^2)\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{ii})$$

The variation of q between $y = -\frac{d}{2}$ and $y = \frac{d}{2}$, as given by (i), is shown by the parabola PQR, q being plotted horizontally. When $y = \pm \frac{d}{2}$, we have

$$q = \frac{FB}{8It}(D^2 - d^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{iii})$$

This is represented by MP or NR in Fig. 203, whilst q_{max} is represented by SQ.

By subtraction we have

$$TQ = SQ - MP = \frac{Fd^2}{8I}.$$

Hence the parabolic area PQR is equal to

$$\tfrac{2}{3} \cdot \frac{Fd^2}{8I} \cdot d = \frac{Fd^3}{12I}.$$

The rectangular area $MPRN$ is equal to

$$\frac{FBd}{8It}(D^2 - d^2),$$

from (iii). Hence the total area $MPQRN$ is

$$\frac{F}{It}\left[B(D^2 - d^2)\frac{d}{8} + \frac{d^3t}{12}\right]$$

Therefore the total shear taken by the web is

$$\frac{F}{I}\left[B(D^2 - d^2)\frac{d}{8} + \frac{d^3t}{12}\right]. \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Now, approximately, I is given by

$$I = 2\frac{B(D-d)}{2}\left(\frac{D+d}{4}\right)^2 + \frac{d^3t}{12}$$

$$= B(D^2 - d^2)\frac{D+d}{16} + \frac{d^3t}{12} \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

If the flanges are thin, D and d are not very different, so that $\frac{D+d}{16}$ is not very different from $\frac{d}{8}$. Hence, comparing (iv) and (v), we see that, when the flanges are thin, the shear taken by the web is very nearly equal to F, the whole shearing force on the section. Thus the distribution of shear stress in the flanges is not of much practical importance, and it has been proved by experiment that the variation through the thickness of the web is very slight.

It will also be found that, in the case of rolled steel sections, or built-up plate girders, the variation of q between the top and bottom of the web is very slight, so that we can, in these cases, assume, with sufficient accuracy, that the whole shearing force is uniformly distributed over the web. This gives

$$\mathbf{q} = \frac{\mathbf{F}}{\mathbf{td}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This underestimates the shearing stress on the neutral axis, where it is usually not important, and overestimates it at the top and bottom of the web, where its importance is greatest (cf. § 154, and Example 2, p. 208).

152. Shear in Built-up Plate Girders.*—When a girder is built of plates and angles riveted together as in Fig. 204, the rivets have to carry the longitudinal shear between the flanges and the web. In such cases we can safely disregard the variation of shear stress in the web and take it as F/td, where F is the total shearing force on the section. Then this, by the principle of complementary shear stresses, will also be the shear, per unit area, of the top or bottom of the web, between the web and the flange. Hence if x is the pitch of the rivets, the force to be resisted by the rivets is

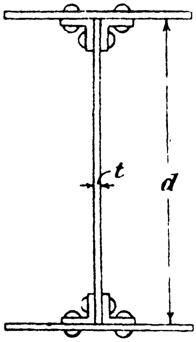

FIG. 204.

$$\frac{F}{td} \times tx = \frac{Fx}{d}.$$

From this the size of the rivets may be calculated.

153. General Remarks on Shearing Stresses in Beams.—It must be remembered that the value of the shear stress we have calculated above is the mean value over the width of the section, and not necessarily the maximum value. In the rolled or built-up I, T, and box sections so largely used, this will not differ greatly from the maximum, but in solid circular sections and the like the shear stress in the middle may be appreciably greater than at the sides. Some interesting experiments by A. A. Griffiths † on the distribution of shear stress in beams show that in I sections such as are used for wooden aeroplane spars, where the thickness of the web is much larger compared with the depth than in the case of rolled or built-up steel sections, the above method of calculation gives a very good approximation to the true value of the shear stress in the web, and that the variation across the thickness is very small. The experiments confirm what is indicated by the above; that most of the shearing force on an I girder is taken by the web.

In our approximate calculations we have neglected altogether the possibility of the existence of the shear stresses q_{xz} and have considered only the q_{xy} stresses. Griffith's experiments show that in I and T sections the q_{xz} shear stresses are a maximum in the radius of the corners where the flanges join the web, and that they are not of importance with the radii ordinarily met with in practice. No gain in strength seems to be obtained by making the radii more than about one-sixteenth the thickness of the web.

It is perhaps advisable to point out here a fundamental error which is made in the above treatment of shear stresses, as it is one which may easily be made in other investigations. We have considered only the equations of equilibrium, without any regard to the strains. We have stated that the simple theory of pure bending gives a good approximation to the longitudinal stresses in more general cases, and we have given an approximate treatment of the shearing stress distribution. But we have not paused to see whether the resulting strains are consistent.* In point of fact they are not, but the error is not of practical importance, in this instance, for the majority of sections met with in engineering practice.

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* An article on Shearing Stresses in Ships, by U. Suychira, will be found in *Engineering*, Vol. XCIV (1912), p. 894; see correction Jan. 20, 1922.

† *Aeronautical Research Committee*, R. and M., 399.

154. Principal Stresses in Beams.—We have shown separately how to find the longitudinal stress at any point in a beam due to bending moment, and the mean horizontal and vertical shearing stresses, within the limits between which the theory of pure bending is applicable. But it does not follow that these are the greatest direct or shear stresses. Within the limits of our present theory we can employ the formulæ of §§ 64 and 66 to find the principal stresses and the maximum shear stress.

We can draw, on a side elevation view of the beam, lines showing the directions of the principal stresses. Such lines are called the lines of principal stress; they are such that the tangent at any point gives the direction of principal stress. As an example, the lines of principal stress have been drawn in Fig. 205 for a simply supported beam of uniform rectangular cross section, carrying a uniformly distributed load. The stresses are a maximum where the tangents to the curves are parallel to the axis of the beam, and diminish to zero when the curves cut the faces of the beam at right angles. On the neutral axis, where the stress is one of shear, the principal stress curves cut the axis at 45°.

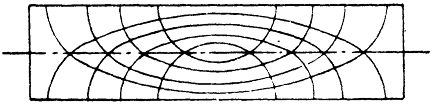

FIG. 205.

Example 1.—The web of a girder of I section (Fig. 206) is 18″ deep × $\frac{3}{8}$″ thick; the flanges are each 9″ × $\frac{1}{2}$″. The girder, at some particular section, has to withstand a total shearing force of 20 tons. Calculate the shearing stress at the top, and middle, of the web, and the total amount of the shear taken by the web. (Intercoll. Exam., Cambridge, 1908.)

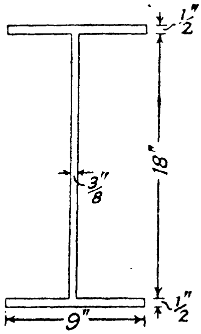

FIG. 206.

The moment of inertia of the web $= \dfrac{0{\cdot}375 \times 18^3}{12} = 182$ in.^{4}

„ „ „ „ each flange about neutral axis $= 4{\cdot}5 \times 9{\cdot}25^2 = 385$ in.^{4}

I = Total moment of inertia = 952 in.^{4}

---

* See Love's *Theory of Elasticity*, 3rd Ed., p. 351.

At a height y, in the web, above the neutral axis, the shear stress is given by (i), p. 204 :—

$$q = \frac{20}{2 \times 932}(81 - y^2) + \frac{20 \times 9}{8 \times 932 \times 0{\cdot}375}(19^2 - 18^2)$$

$$= \frac{810 - 10y^2}{952} + \frac{2{,}220}{952} = \frac{3{,}030 - 10y^2}{952} \text{ tons/in.}^2$$

At the top of the web $y = 9$,

$$q = \frac{2{,}220}{952} = 2{\cdot}36 \text{ tons/in.}^2$$

At the middle $y = 0$,

$$q = \frac{3{,}030}{952} = 3{\cdot}18 \text{ tons/in.}^2$$

The total shearing force on the web, between the heights

$$y \text{ and } y + dy = q \times dy \times \frac{3}{8};$$

hence the total shear on the web is

$$\int_{-9}^{9} \frac{3}{8} q\,dy = \frac{3}{8 \times 952} \int_{-9}^{9} (3{,}030 - 10y^2)dy$$

$$= \frac{3}{8 \times 952}\left[3{,}030y - \frac{10y^3}{3}\right]_{-9}^{9}$$

$$= \frac{3 \times 49{,}680}{8 \times 952} = 19{\cdot}6 \text{ tons}$$

that is 98 per cent. of the whole shearing force on the section.

If we assume the whole shearing force to be uniformly distributed over the web, we find that the mean shearing stress is 2·74 tons/in.²

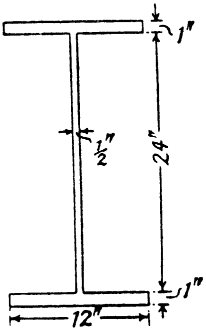

FIG. 207.

Example 2.—The flanges of an I girder (Fig. 207) are 12″ × 1″ and the web is 24″ × ½″. At a particular section the B.M. is 150 tons. ft. and the shearing force is 50 tons. Consider a point in the section at the top of the web and calculate for this point : (i) the longitudinal stress, (ii) the shear stress, (iii) the principal stresses. (Intercoll. Exam., Cambridge, 1913.)

First calculate the moment of inertia about the neutral axis :

is Moment of inertia of web $= \frac{0{\cdot}5 \times 24^3}{12} = 576$ in.⁴

casi

equ ,, ,, ,, flanges $= 12 \times 12{\cdot}5^2 = 1{,}870$ in.⁴ (each).

state ∴ Total moment of inertia $= 4{,}316$ in.⁴

to th ext, for a point at the top of the web,

$$\bar{Sy} \text{ (p. 202)} = 12 \times 12{\cdot}5 = 150 \text{ in.}^3$$

$$t = 0{\cdot}5''$$

* An
Engineer
† *Aero*

Then for this point we have, with M = 1,800 tons. ins.

$$p = \frac{1{,}800 \times 12}{4{,}316} = 5 \text{ tons/in.}^2, \text{ compression.}$$

$$q = \frac{50 \times 150}{4{,}316 \times 0{\cdot}5} = 3{\cdot}48 \text{ tons/in.}^2$$

Then, § 64, the principal stresses are:

$$\begin{aligned} & -2{\cdot}5 \pm \tfrac{1}{2}\sqrt{25 + 48{\cdot}4} \\ &= -2{\cdot}5 \pm 4{\cdot}28 \\ &= -6{\cdot}78 \text{ and } +1{\cdot}78 \text{ tons/in.}^2, \end{aligned}$$

the former being compressive, and the latter tensile.

It should be noticed that the greater principal stress is about 13 per cent. greater than the longitudinal stress. At the top of the flange the longitudinal stress = 5·42 tons/in.², so that the greatest principal stress at the top of the web is 12·5 per cent. greater than the maximum longitudinal stress.

It will be instructive to examine the strength of this girder according to the various theories of complex stresses. Let us suppose that the elastic limit of steel in simple tension is 15 tons/in.², and let n denote the multiple of the above bending moment and shearing force which will cause the beam to reach the elastic limit.

(i) If the maximum direct stress be taken as the criterion,

$$n = \frac{15}{6{\cdot}78} = 2{\cdot}21$$

(ii) If the maximum shear be taken: the greatest shear stress under the above loads = 4·28 tons/in.², and in simple tension the maximum shear = 7·5 tons/in.² at the elastic limit. Hence

$$n = \frac{7{\cdot}5}{4{\cdot}28} = 1{\cdot}75.$$

(iii) Maximum strain theory: the simple stress which will produce the same maximum strain (§ 72) is, with $m = 10/3$,

$$\begin{aligned} \bar{p} &= 0{\cdot}35 \times 5 + 0{\cdot}65 \times 8{\cdot}56 \\ &= 7{\cdot}31 \text{ tons/in.}^2 \end{aligned}$$

giving

$$n = \frac{15}{7{\cdot}31} = 2{\cdot}05.$$

(iv) Haigh's Strain-Energy Theory, with $m = \frac{10}{3}$. The elastic limit will be reached when (see § 81, iv.)

$$p_1^2 + p_2^2 - \tfrac{3}{5}p_1p_2 = 225$$

With the above loads $p_1 = 6{\cdot}78$, $p_2 = -1{\cdot}78$, and

$$p_1^2 + p_2^2 - \tfrac{3}{5}p_1p_2 = 56{\cdot}5,$$

hence

$$n^2 = 225/56{\cdot}5 = 3{\cdot}98, \text{ or } n = 2 \text{ nearly.}$$

The variations of the maximum shear stress (q_m), the greatest principal stress (p_1), the equivalent simple stress for equal strain ($\bar{p}$), and the function $\sqrt{p_1^2 + p_2^2 - \frac{3}{5}p_1p_2}$ are plotted in Fig. 208, which shows clearly that the top point of the web is the weakest point according to theory (i), (iii) or (iv) above, but that according to (ii) the strength is practically constant throughout the depth of the web, the centre being very slightly weaker than the top. In all cases the top of the web is weaker than the flange.

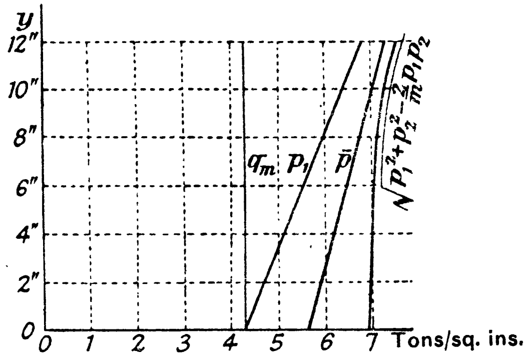

FIG. 208.

155. Superimposed Beams.

—If we make a girder by placing one beam on the top of another, there will be a tendency for the two beams to slide over each other along the plane of contact AB (Fig. 209), and this sliding will take place unless it is prevented in some way. Thus the moment of resistance (see p. 160) will simply be the sum of the moments of resistance of the two beams separately: each beam will act independently of the other, so that if they are similar beams the strength of the compound girder will be double the strength of either beam.

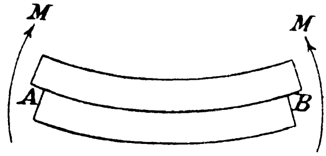

FIG. 209.

If, however, this sliding is prevented, i.e. if the tendency to shear is provided for, the compound beam will behave as a solid member, and the moment of resistance will be found by estimating the moment of inertia of the double section about its neutral axis considered as a single area. Thus, in the case of two equal beams of rectangular section, the strength of the compound beam will be $2^2 = 4$ times as strong as one beam by itself, if the shear between them is prevented.

In the case of flanged iron or steel beams, the flanges may be riveted together. In the case of timber beams the shear may be provided against by the use of shear keys and bolts, as indicated in Fig. 210, where K denotes square sectioned keys fitted between the two beams, and B denotes bolts passing vertically through the beams between the shear keys. The bolts are necessary in order to keep the two halves of the girder together; on account of the shearing stress along the plane AC,

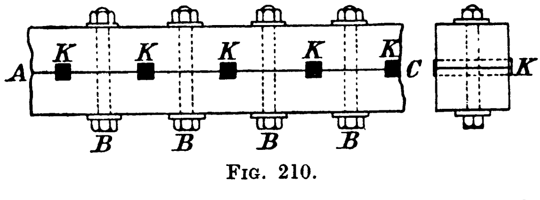

FIG. 210.

the beams tend to slide over each other and so to rotate the keys *K*, as in Fig. 211, where the large arrows *S* indicate the tendency of the two beams to slide over each other, and the small arrows indicate the pressure of the beams on the keys. The keys will begin to rotate in a clockwise direction into the position shown by the dotted square, trying to lever the two beams apart in the direction of the arrows *A*; this separation is prevented by the bolts *B* in Fig. 210.

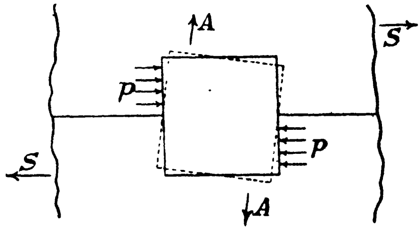

FIG. 211.

156. Shear in Reinforced Concrete Beams.*—As an illustration of the method of dealing with shear stresses in reinforced concrete beams, we shall consider a beam of rectangular section with reinforcement on the tension side only. The method to be followed will be the same for any section, but the general formulæ for any other case are undesirably complicated. As in § 123, we shall neglect the tensile stress in the concrete, below the neutral axis, and we adopt the same notation as in § 125.

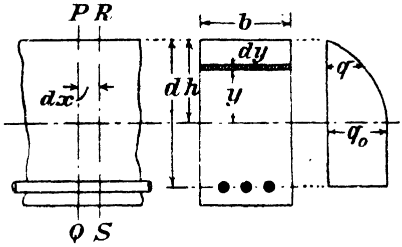

FIG. 212.

Let *PQ* and *RS* be two normal sections of the beam (Fig. 212) separated by a distance δx, the bending moments on the two sections being M and $M + \delta M$ respectively.

Let p_c denote the maximum compressive stress in the concrete; then the stress at height y from the neutral axis, on the section *PQ*, is

$$\frac{y}{h}p_c = \frac{y}{h}\cdot\frac{2M}{bh\left(d-\frac{h}{3}\right)} = \frac{2My}{bh^2\left(d-\frac{h}{3}\right)}$$

by (15) of § 125. Hence the total thrust on a strip of width δy and length (across the beam) b will be

$$\frac{2My\delta y}{h^2\left(d-\frac{h}{3}\right)}$$

---

* For a full treatment of this subject reference must be made to works on reinforced concrete (see footnote, p. 164.) See also *Engineering*, Oct. 28, 1927.

Similarly the thrust on a corresponding strip of the section RS will be

$$\frac{2(M+\delta M)}{h^2\left(d-\frac{h}{3}\right)}y\delta y.$$

Then there will be an unbalanced thrust on a horizontal slice of the beam (as in § 150) equal to

$$\frac{2\delta M}{h^2\left(d-\frac{h}{3}\right)}y\delta y.$$

Hence the total unbalanced thrust between the level y and the top of the beam will be

$$\frac{2\delta M}{h^2\left(d-\frac{h}{3}\right)}\int_y^h y\,dy=\frac{\delta M(h^2-y^2)}{h^2\left(d-\frac{h}{3}\right)}$$

This must be balanced by the shear between the two parts of the beam above and below the level y. If q denote the shear stress at height y, we must therefore have

$$qb\delta x=\frac{\delta M(h^2-y^2)}{h^2\left(d-\frac{h}{3}\right)}$$

$$\therefore q=\frac{\delta M}{\delta x}\cdot\frac{h^2-y^2}{bh^2\left(d-\frac{h}{3}\right)}$$

In the limit, when δx is indefinitely small, $\delta M/\delta x=F$, the shearing force on either section, so that

$$\mathbf{q}=\frac{\mathbf{F}}{\mathbf{b}\left(\mathbf{d}-\frac{\mathbf{h}}{3}\right)}\left(1-\frac{\mathbf{y}^2}{\mathbf{h}^2}\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When $y=h$, i.e. at the compression face of the beam, $q=0$.

Let q_0 denote the value of q on the neutral axis ($y=0$), then

$$\mathbf{q}_0=\frac{\mathbf{F}}{\mathbf{b}\left(\mathbf{d}-\frac{\mathbf{h}}{3}\right)}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since we are assuming that there is no longitudinal stress in the concrete when y is negative, i.e. below the neutral axis, the shearing stress will retain the value q_0 down to the level of the reinforcement. Hence we obtain the diagram of shear stress shown in Fig. 212.

At the level of the reinforcement the longitudinal shear is taken up by the reinforcing bars.

At PQ the tension is

$$T = \frac{M}{d - \frac{h}{3}}$$

by (16) of § 125, and at RS it is

$$T + \delta T = \frac{M + \delta M}{d - \frac{h}{3}}$$

The shear to be taken up is $q_0.b.\delta x$. Hence we must have

$$q_0 b.\delta x = \delta T = \frac{\delta M}{d - \frac{h}{3}}$$

Thus

$$q_0 b = \frac{\delta M}{\delta x} \cdot \frac{1}{d - \frac{h}{3}} = \frac{F}{d - \frac{h}{3}},$$

in the limit. This represents the total adhesion which must exist between the concrete and steel, per unit length of the latter. Hence the adhesive stress, per unit surface of reinforcement, is

$$\frac{q_0 b}{\text{total perimeter of reinforcement}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The surface required for adhesion must be estimated by considering the section where the shearing force is greatest.

Adhesion must not be relied on entirely to take the shear, but the ends of the rods should be bent into the form of a hook as shown in Fig. 213, the diameter of the circular part of the hook being at least five times the diameter of the rod. Provision for shear is also frequently provided by diagonal or vertical reinforcement, but the subject is too lengthy to treat here.*

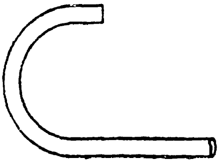

FIG. 213.

157. Shear in Oblique Bending.—When the applied bending moment is not in a plane containing a principal axis of the cross sections the above treatment may be applied, the correct position of the neutral axis being found in the manner explained in § 126.

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* For a very complete treatment of shear in reinforced concrete beams see the work referred to in the footnote on p. 164, where a most interesting account of experimental investigations will be found.

EXAMPLES XIII

1. A plate web girder consists of 4 plates, in each flange, of 12″ width. The web is 24″ deep and is connected to the flanges by 4″ × 4″ × ½″ angles, riveted with ¾″ rivets. Assuming the maximum bending moment to be 370 tons. ft., and the shearing force to be 38 tons, obtain suitable dimensions for (i) the thickness of the flange plates, (ii) the pitch of the rivets. Take the tensile stress as 7 tons/in.^{2}, and the shear stress in the rivets as 5 tons/in.^{2} (R.N.E.C., Keyham, 1920.)

2. In a small gantry for unloading goods from a railway waggon, it is proposed to carry the lifting tackle on a rolled steel joist, 9″ × 4″, wt. 21 lbs. per ft., supported at the ends, and of effective length 14 ft. The equivalent dead load on the joist due to the load to be raised is 3 tons, and this may act at any point of the middle 12 ft. By considering the fibre stress and the shear, examine whether the joist is suitable. The flanges are 4″ × 0·46″ and the web is 0·3″ thick. The allowable fibre stress is 7·5 tons/in.^{2}, and the allowable shear stress 5 tons/in.^{2} (Intercoll. Exam., Cambridge, 1919.)

3. A girder of I section has flanges 9″ × 1″, and web 24″ × ½″. At a certain section the sagging B.M. is 200 tons. ft., and the shearing force 40 tons. Determine (i) the tensile stress at the bottom of the girder, (ii) the principal stresses at the level where the web joins the bottom flange. (Intercoll. Exam., Cambridge, 1914.)

4. A steel beam of I section 30 ft. long is supported horizontally at its extremities and carries a load of 20 tons concentrated at a point 10 ft. from one end. The web is 20″ × ⅝″ and each flange is 8″ × 1″. Calculate for a section 15 ft. from the end : (*a*) the maximum longitudinal stress, (*b*) the maximum shearing stress in the web, (*c*) the principal stresses at the top and bottom of the web. (Intercoll. Exam., Cambridge, 1909.)

5. A girder of I section has a web 24″ × ½″ and flanges 12″ × 1″. The girder is subjected to a bending moment of 100 tons. ft. and a shearing force of 100 tons at a particular section. Calculate how much of the shearing force is carried by the web, and how much of the bending moment by the flanges. (Mech. Sc. Trip., 1910.)

6. Estimate the greatest shearing stress in the web of the girder of question 4, p. 183.

7. A girder of I section has flanges 6″ × 1″ and a web 12″ × ½″, giving a sectional area of 18 in.^{2} and a principal moment of inertia of 580 in.^{4} At a certain section the B.M. is 50 tons. ft., and the shearing force 15 tons. Find the magnitude of the principal stresses at a point in this section at the top of the web. (Mech. Sc. Trip., 1915.)

8. The shear at a given section of a built-up I girder is 100 tons and the depth of the web is 5′ 6″. The web is joined to the flanges by angle irons with 1″ rivets. Determine the thickness of the web plate and the pitch of the rivets, allowing a shear stress of 3 tons/in.^{2} in the web and rivets, and a bearing pressure on the rivets of 8 tons/in.^{2}

9. A reinforced concrete beam 12″ wide has for reinforcement two rods ½″ diameter, with their centres 3½″ from the compression face. It carries a load of 150 lbs. per foot run. Taking $E_s/E_c = 15$, find the adhesive stress between the steel and the concrete. The length of the beam is 12 ft.

10. A reinforced concrete T beam has the flange 98″ × 4″, and the web 11″ wide by 20″ deep. There are 5 round steel rods 1⅛″ diameter 2″ from the bottom of the web. The shearing force is 23,000 lbs. Calculate the adhesive stress between the steel and the concrete, with $E_s/E_c = 15$.

11. A steel pipe 24″ diameter × ⅜″ thick has its ends closed and is full

of water. The length is 60 ft. and the ends are freely supported. Draw a curve showing the distribution of shear stress over the middle cross section.

12. A compound girder is built of one 18″ × 7″, wt. 75 lbs. per foot, rolled steel joist with *two* 10″ × $\frac{5}{8}$″ steel plates riveted to each flange. If the ends are simply supported and the effective span is 30 ft., what is the maximum uniformly distributed load which can be supported by the girder? (See Fig. 214.) If the plates are riveted to the flanges with $\frac{7}{8}$″ rivets, and in a cross section there are two rivets in each flange, what should be the pitch?

Allowable fibre stress for plates = 7·5 tons/in.^{2}
,, shear ,, ,, rivets = 4 ,,
,, bearing pressure ,, ,, = 7 ,,

(Mech. Sc. Trip., 1920.)

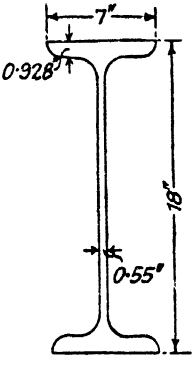

FIG. 214.

13. The section of an open rectangular cast-iron channel for carrying water is 12″ wide and 8″ deep, measured externally, the metal being 1″ thick. The weight of the cast iron and the water it holds is 125 lbs. per ft. run. The channel is 20′ long and is supported at its ends.

Calculate the intensity of the shear stress in the vertical sides at a point 7″ from the top in a section 5′ from the end. (Intercoll. Exam., Cambridge 1911.)

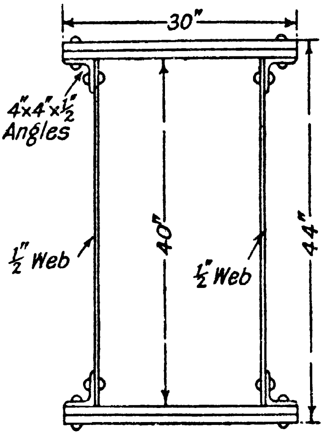

FIG. 215.

14. The section of a girder is shown in Fig. 215. The girder carries a number of concentrated loads and for the length of girder between two of these loads the shearing force is 80 tons. Show that, for this region, the pitch of the rivets connecting angle and web, which will limit the horizontal force to be resisted by each rivet to 3 tons, is 3·4″ approximately. (Intercoll. Exam., Cambridge, 1912.)

15. If the cross section of a beam varies, but the depth is constant, show that the formula (2) on p. 202 becomes

$$q = \frac{S\bar{y}}{B}\left(\frac{F}{I} - \frac{M}{I} \cdot \frac{dI}{dx}\right).$$

16. A cast-iron channel 20 ft. long and of semi-circular section, has a mean radius of 4″ and thickness $\frac{1}{16}$″. It is supported at the ends and filled with water. Find the maximum longitudinal stress and the maximum shear stress produced. Density of cast iron 460 lbs./ft.^{3} (Mech. Sc. Trip., 1921.)

CHAPTER XIV

THE DEFLECTION OF BEAMS

158. Introductory.—In our study of beams we have seen how to estimate the total action between two contiguous portions of a beam supported freely at two points, or built in at one end and free at the other, this action being conveniently resolved into a bending moment and a shearing force ; we have also seen how to calculate, as accurately as we require in engineering, the stresses due to these bending moments and shearing forces. This may be summarized by saying that we have seen how to calculate the strength of such beams ; but there is another aspect of the problem of flexure which remains to be treated, namely, the calculation of the stiffness of beams. In most practical cases it is necessary that a beam should be not only strong enough for its purpose, but also that it should have the requisite stiffness, that is, that it should not deflect from its original position by more than a certain amount. Again, there are certain classes of beams, such as those carried by more than two supports and beams with their ends held in such a way that they must keep their original direction, for which we cannot calculate the bending moments and shearing forces without studying the deformation of the axis of the beam.

In this chapter, then, we shall consider the following problem : given the external forces applied to a beam, the cross section of the beam, and the geometrical conditions attaching to the ends of the beam, to discover the curve assumed by the axis of the beam under the given circumstances.

159. General Equations.—It was shown in § 118 that a beam of uniform cross section, acted on by end-couples only, bends into a circular arc of radius R, given by

$$\frac{1}{R}=\frac{M}{EI}$$

where I is the moment of inertia of the cross section about its neutral axis, and M is the value of the bending moment due to the end couples. At the same time we derived an expression for the longitudinal stress across a normal section. These two relations are accurate in the particular case for which they were found, and we explained that the formula for the stress could be applied with sufficient accuracy to practically all the cases of straight beams met with in engineering. In cases when there is shear as well as bending, i.e. when the bending moment is not constant, the axis does not bend into a circular arc, but to some other curve determined by the loading. To deal with these cases we assume that the above formula for R is still applicable, but that R is now the radius of curvature of the strained axis at the point where the bending moment is M.

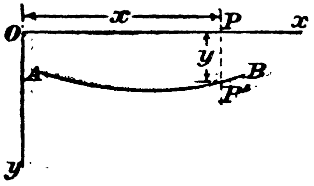

FIG. 216.

As before we shall suppose, for convenience, that in the unstrained state, the axis of the beam is horizontal and straight; we shall also assume that the lateral load acts vertically downwards. In Fig. 216 let OP be a portion of the unstrained axis of the beam, O being any convenient point of reference which is taken as origin of co-ordinates, and P being at a distance x from O. Let AB be a portion of the axis of the beam in its strained position, P' being the displaced position of P.

Draw Oy vertically downwards, and let y be the vertical displacement of P. Then we shall suppose that both y and $\frac{dy}{dx}$ are so small that their squares can be neglected, and we neglect the horizontal displacement of P.

Let M = the bending moment at P or P'.

R = the radius of curvature of AB at P'.

I = the moment of inertia of the cross section, for the present supposed constant.

E = Young's Modulus.

Then the fundamental equation from which we must deduce the curve assumed by the axis of the beam is

$$\frac{EI}{R} = M.$$

Now the radius of curvature R is given by

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\pm\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

Since we assume that $\left(\frac{dy}{dx}\right)^2$ is negligible, we can write

$$\frac{1}{R} = \pm\frac{d^2y}{dx^2}$$

and our fundamental equation becomes

$$\pm EI\frac{d^2y}{dx^2} = M.$$

Since $\frac{d^2y}{dx^2} = 0$ is the condition for a point of inflexion on a curve we see that the curvature of the beam is reversed wherever $M = 0$ (cf. § 107).

We must now make a convention concerning signs. In Chapter IX, p. 115, we agreed to consider the bending moment positive when it tends to make the axis of the beam concave upwards. With the axes drawn as in Fig. 216, y will increase with x, i.e. $\frac{dy}{dx}$ will be positive, when the tangent slopes downwards to the right. Then, with the axis concave upwards, $\frac{dy}{dx}$ will be decreasing as x increases, i.e. $\frac{d^2y}{dx^2}$ will be negative. Thus we must write our fundamental equation in the form

$$-\mathrm{EI}\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = \mathrm{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where M is the "sagging" bending moment.

This equation can be written in other forms which are sometimes convenient: since the shearing force, F, is equal to $-dM/dx$ (§ 96), we have, on differentiating (1)

$$\mathrm{F} = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) = \mathrm{EI}\frac{\mathrm{d}^3\mathrm{y}}{\mathrm{dx}^3}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if the beam be of uniform section so that E and I are constant, and F be considered positive when the right-hand part of the beam tends to move upwards relative to the left.

Again, since $\frac{dF}{dx} = w$ (equation (1), § 96), by differentiating (2) we have

$$\mathrm{w} = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = \mathrm{EI}\frac{\mathrm{d}^4\mathrm{y}}{\mathrm{dx}^4}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

if E and I are constant and w is the distributed load per unit length, acting downwards.

If we know w, F or M as functions of x, we can obtain the equation to the curve taken up by the axis of the beam by direct integration of one of these equations. Each integration will involve one arbitrary constant, and these constants must be chosen so that the conditions which obtain at the ends * of the beam are satisfied. To find the deflection curve of the beam, then, we proceed as follows:

(i) Choosing a convenient point as origin, write down the general expression for M, F or w in terms of x, and substitute in the suitable equation, (1), (2) or (3) above. It will usually be most convenient to take the origin at a fixed point in the beam, e.g. at one of the supports.

(ii) Integrate the equation so formed. In the case of equation (1) this will introduce two arbitrary constants, equation (2) three, and equation (3) four. On this account it will always shorten the subsequent work if we select equation (1) when we can.

(iii) Find the values of these constants from the prescribed conditions of the problem in hand. We shall now illustrate the method by applying it to certain particular cases.

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* Or some other points whose displacements are known.

160. Reinforced Concrete Beams.—In the case of reinforced concrete, EI is replaced by (E_c × moment of inertia of compression area of concrete) + (E_s × moment of inertia of the steel), both moments of inertia being measured about the neutral axis of the beam. Similar modifications must be made in the case of other composite beams.

161. Cantilever with Concentrated Load.—In Fig. 217 let OAB be a cantilever with the end O fixed in a horizontal position, carrying a weight W at A distant a from O. Let l be the total length of the beam.

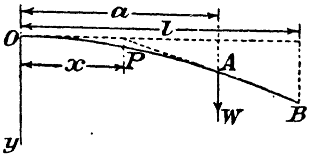

FIG. 217.

For any point P between O and A we have

$$M = -W(a-x).$$

Hence the deflection equation is

$$EI\frac{d^2y}{dx^2} = W(a-x)$$

$$\therefore\ EI\frac{dy}{dx} = W\left(ax - \frac{x^2}{2}\right) + A \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and

$$EIy = W\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) + Ax + B \quad . \quad . \quad . \quad \text{(ii)}$$

where A and B are constants to be determined. The conditions at the end O ($x = 0$) are $y = 0$, and $\frac{dy}{dx} = 0$. The first gives $B = 0$, and the second requires $A = 0$ from (i). Hence

$$\frac{dy}{dx} = \frac{W}{EI}\left(ax - \frac{x^2}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$\text{and}\quad y = \frac{W}{2EI}\left(ax^2 - \frac{x^3}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Let φ denote the slope of the tangent at A, and y_A the deflection at A. Substituting $x = a$ in (iii) and (iv) gives

$$\varphi = \frac{Wa^2}{2EI}$$

$$y = \frac{Wa^3}{3EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since there is no load on the part AB, this will remain straight, and in the direction of the tangent at A. Hence, if y_B denote the deflection at B,

$$\begin{aligned} y_B &= y_A + (l-a)\varphi \\ &= \frac{Wa^3}{3EI} + \frac{Wa^2(l-a)}{2EI} \end{aligned}$$

or

$$y_B = \frac{Wa^2}{6EI}(3l-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

162. Cantilever with Uniformly Distributed Load.—Let OA be the cantilever (Fig. 218) with the end O fixed in position and direction, and let w be the distributed load per unit length. Then at any point P,

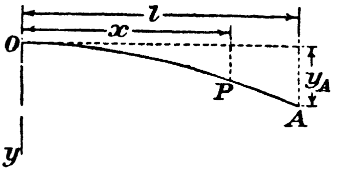

FIG. 218.

$$M = -\frac{w}{2}(l-x)^2$$

Hence

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(l-x)^2$$

$$\therefore\ EI\frac{dy}{dx} = -\frac{w}{6}(l-x)^3 + A \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and

$$EIy = \frac{w}{24}(l-x)^4 + Ax + B \quad . \quad . \quad . \quad \text{(ii)}$$

The constants A and B are given by the conditions $y = 0$ when $x = 0$, and $\frac{dy}{dx} = 0$ when $x = 0$.

From (ii) the first condition requires

$$\frac{wl^4}{24} + B = 0. \qquad \therefore\ B = -\frac{wl^4}{24}.$$

From (i) the second condition requires

$$-\frac{wl^3}{6} + A = 0. \qquad \therefore\ A = \frac{wl^3}{6}.$$

Hence, from (ii) we have

$$EIy = \frac{w}{24}(l-x)^4 + \frac{wl^3x}{6} - \frac{wl^4}{24}$$

or

$$y = \frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4) \quad . \quad . \quad . \quad \text{(iii)}$$

At the end A, where $x = l$, the deflection y_A is

$$y_A = \frac{wl^4}{24EI}(6 - 4 + 1) = \frac{wl^4}{8EI} \quad . \quad . \quad . \quad . \quad (6.)$$

163. Supported Cantilever with Distributed Load.—Suppose the cantilever of the last article has a support at a distance a from the fixed end O. We cannot, by the methods of pure statics, find the pressure of the support and are not able to draw the bending moment diagram for the beam, but the theory of deflections enables us to surmount this difficulty.

Let R be the pressure between the support and the beam.

If the cantilever were unsupported the *downward* deflection of a point on the axis, distant a from the fixed end, would be

$$\frac{w}{24EI}(6l^2a^2 - 4la^3 + a^4)$$

by (iii) of § 162.

If the cantilever were acted on only by the upward force R the *upward* deflection of the same point, by (4) of § 161, would be

$$\frac{Ra^3}{3EI}.$$

Therefore, under the action of both R and w, the downward movement of the supported point would be

$$\frac{wa^2}{24EI}(6l^2 - 4la + a^2) - \frac{Ra^3}{3EI}.$$

To determine R we must know the properties of the support:

(i) If the support is unyielding and fixed at the level of the built-in end of the cantilever, we must have

$$\frac{wa^2}{24EI}(6l^2 - 4la + a^2) - \frac{Ra^3}{3EI} = 0,$$

which gives

$$R = \frac{wl}{8}\left(6\frac{l}{a} - 4 + \frac{a}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If $a = l$, i.e. if the support be at the outer end of the cantilever, $R = 3wl/8$.

(ii) If the support sinks a distance δ, we have

$$\frac{wa^2}{24EI}(6l^2 - 4la + a^2) - \frac{Ra^3}{3EI} = \delta$$

or

$$R = \frac{wl}{8}\left(6\frac{l}{a} - 4 + \frac{a}{l}\right) - \frac{3EI\delta}{a^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When R has been found, the bending moment, shearing force and deflection curves can be found by superposing those due to R upon those due to w, paying due attention to sign.

Example 1.—A steel rod 2″ diameter protrudes 5 ft. horizontally from a wall. (i) Calculate the deflection due to a load of 2 cwt. hung on the end of the rod. The weight of the rod may be neglected. (ii) If a vertical steel wire 10 ft. long, 0·1″ diameter, supports the end of the cantilever, being taut but unstressed before the load is applied, calculate the end deflection on application of the load. Take $E = 30 \times 10^6$ lbs./in.^{2} (R.N.E.C., Keyham, 1923.)

(i) The length of the cantilever is $l = 60''$.
The load on the end is $W = 224$ lbs.
The moment of inertia of the cross section is

$$I = \frac{\pi}{64} \times 2^4 = 0{\cdot}7854 \text{ in.}^4$$

Hence, from (4) § 161, the deflection at the end is ($a = l = 60''$)

$$\frac{224 \text{ lbs.} \times 60^3 \text{ in.}^3}{3 \times 30 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}7854 \text{ in.}^4} = 0{\cdot}685''.$$

(ii) Let T = the tension in the wire (lbs.). The area of the cross section $= 0{\cdot}007854$ in.^{2} Then the elongation is

$$\frac{120'' \times T \text{ lbs.}}{30 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}007854 \text{ in.}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

The load on the end of the cantilever $= 224 - T$ lbs. This will produce a deflection, using equation (4), § 161.

$$\frac{(224 - T)\text{ lbs. } 60^3\text{ ins.}^3}{3 \times 30 \times 10^6\text{ lbs./in.}^2 \times 0{\cdot}7854\text{ ins.}^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

We must therefore have, equating (i) and (ii),

$$120T = \frac{(224 - T)216{,}000}{3 \times 100}$$

$$252{,}000T = 224 \times 216{,}000$$

$$T = 192.$$

Then the deflection is, from (i)

$$\frac{120 \times 192}{30 \times 10^6 \times 0{\cdot}007854}\text{ ins.}$$

$$= 0{\cdot}0978\text{ ins.}$$

Example 2.—A platform carrying a uniformly distributed load rests on two cantilevers projecting l feet from a wall, as shown in Fig. 219. The distance between them is $\frac{1}{2}l$ feet. In what ratio might the load on the platform be increased if the ends were supported by a cross girder of the same section as the cantilevers, resting on a rigid column in the centre, as shown? It may be assumed that when there is no load on the platform the cantilevers just touch the cross girder without pressure. (Mech. Sc. Trip., 1914.)

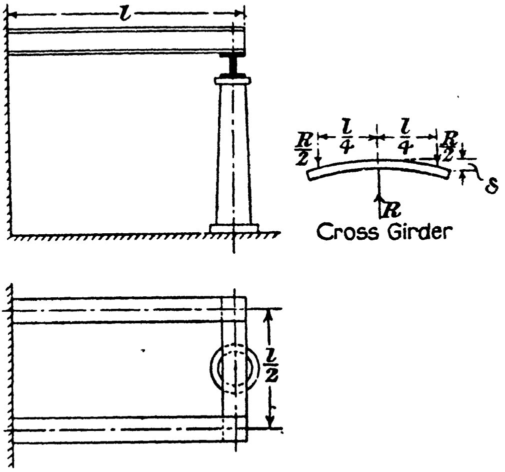

FIG. 219.

Let w_1 = the safe load per ft. on each cantilever when unsupported.

Then the maximum bending moment $= \dfrac{-w_1 l^2}{2}$.

Let w_2 = the safe load when supported,

δ = the deflection of the end of each cantilever,

$\frac{1}{2}R$ = the pressure between each cantilever and the cross girder.

Then from (8) of § 163, putting $a = l$, the pressure is

$$\frac{R}{2} = \frac{3}{8}w_2l - \frac{3EI\delta}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

We see from the figure above and (4) of § 161, that

$$\delta = \frac{(R/2)(l/4)^3}{3EI} = \frac{Rl^3}{384EI},$$

I having the same value for the cantilevers and the cross girder. Substituting this value of δ in (i) gives

$$\frac{R}{2} = \frac{3w_2l}{8} - \frac{R}{128}$$

or

$$R = \frac{48}{65}w_2l.$$

The upward pressure on the end of each cantilever is $\frac{R}{2} = \frac{24}{65}w_2l$, giving a bending moment at the wall equal to $\frac{24}{65}w_2l^2$. The bending moment of opposite sign due to the distributed load is $\frac{1}{2}w_2l^2$. Hence it is clear that the maximum bending moment due to both acting together must occur at the wall and is equal to $\left(\frac{1}{2} - \frac{24}{65}\right)w_2l^2 = \frac{17}{130}w_2l^2$. If this is to be equal to $\frac{1}{2}w_1l^2$, we must have $w_2 = \frac{65}{17}w_1$; in other words, the load on the platform can be increased in the ratio 65/17, or nearly 4/1.*

164. Beam with Uniform Bending Moment.

—Let a beam OA be acted on at its ends by terminal couples M, as shown in Fig. 220, and let l be the length of the beam. Then the bending moment at any point P will be M. We have then

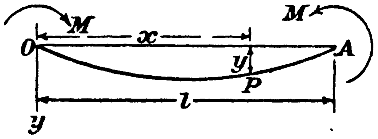

FIG. 220.

$$EI\frac{d^2y}{dx^2} = -M$$

$$\therefore EI\frac{dy}{dx} = -Mx + A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

and

$$EIy = -\frac{Mx^2}{2} + Ax + B. \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

where A and B are constants to be determined. If the conditions are that the ends O and A do not move, we must have

$$y = 0 \text{ when } x = 0 \text{ or } l.$$

The first condition requires that $B = 0$, and the second that

$$-\frac{Ml^2}{2} + Al = 0$$

$$\therefore A = \frac{Ml}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

---

* The B.M. at the centre of the cross girder is $6w_2l^2/65$, which is less than that at the wall.

Hence we have from (ii)

$$y = \frac{Mx}{2EI}(l - x).$$

At the centre of the beam, where $x = \frac{l}{2}$, we have

$$\mathbf{y_0} = \frac{\mathbf{M}l^2}{\mathbf{8EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

y_0 being the central deflection.

From (i) and (iii) we see that the slope at the ends is given by $Ml/2EI$.

These results might easily be obtained by pure geometry, since we know that the curve is a circular arc of radius EI/M ; we leave this as an exercise for the reader.

165. Beam Simply Supported at the Ends and carrying a Uniformly Distributed Load.—

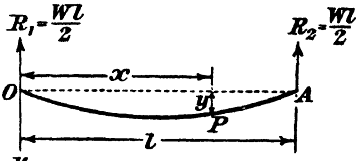

FIG. 221.

In Fig. 221 let OA be the beam, of length l, and let the distributed load be w per unit length, w being constant. Then the reactions at the ends are each equal to $wl/2$, and, taking the origin at O, the bending moment at P is given by (§ 102)

$$M = \frac{w}{2}(lx - x^2).$$

Hence from equation (1), § 159,

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(lx - x^2)$$

$$\therefore\ EI\frac{dy}{dx} = -\frac{w}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + A$$

and

$$EIy = -\frac{w}{2}\left(\frac{lx^3}{6} - \frac{x^4}{12}\right) + Ax + B \quad . \quad . \qquad \text{(i)}$$

where, as before, A and B are constants to be determined. If the supports do not move, y must vanish at each end, i.e. when $x = 0$ or l. Hence

$$B = 0$$

and

$$-\frac{w}{2}\left(\frac{l^4}{6} - \frac{l^4}{12}\right) + Al = 0$$

$$\therefore\ A = \frac{wl^3}{24}$$

Then (i) becomes

$$EIy = -\frac{w}{2}\left(\frac{lx^3}{6} - \frac{x^4}{12}\right) + \frac{wl^3x}{24}$$

or

$$y = \frac{w}{24EI}(l^3x - 2lx^3 + x^4) \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

The central deflection, y_0, is given by

$$y_0 = \frac{5wl^4}{384EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

166. Freely Supported Beam with Concentrated Load.—In Fig. 222 let the beam *OB* be freely supported at the ends *O* and *B*, and let it carry a load *W* at *A*, the dimensions being as shown. The reactions are

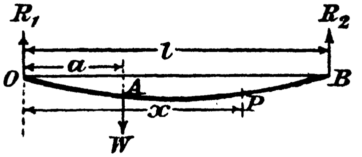

FIG. 222.

$$R_1 = \frac{l-a}{l}W \text{ and } R_2 = \frac{a}{l}W$$

Taking moments on the left of *P*, the sagging bending moment at *P*, when x is greater than a, is

$$M = R_1x - W(x-a),$$

Therefore we have, $x > a$.

$$EI\frac{d^2y}{dx^2} = -R_1x + W(x-a) \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Now, when $x < a$, the second term on the right-hand side of this equation will be absent, so that apparently we shall have to deal with two differential equations, one for *OA* and one for *AB*. The integration of these equations will introduce four constants, which will be determined from the conditions $y = 0$ at each end, and the conditions that y and $\frac{dy}{dx}$ should be continuous at *A*. We can, however, avoid these complications by the following artifice.*

Integrate equation (i) thus:

$$EI\frac{dy}{dx} = -\frac{R_1x^2}{2} + \frac{W}{2}\{x-a\}^2 + A$$

$$EIy = -\frac{R_1x^3}{6} + \frac{W}{6}\{x-a\}^3 + Ax + B \quad . \quad . \quad \text{(ii)}$$

If we agree to omit the { } terms when $x < a$, the equations will have the correct form for the part *OA*; also, since the second term on the right vanishes when $x = a$, it is clear that, if we use these equations for the whole beam, y and $\frac{dy}{dx}$ will be continuous at the point *A*, whatever the values of the constants *A* and *B*.

---

* Due to R. Macaulay of King's College, Cambridge: *Messenger of Mathematics*, Jan., 1919, xlviii.

We find A and B from the conditions $y = 0$ when $x = 0$ and $x = l$. The condition $y = 0$ when $x = 0$ gives $B = 0$, since the $\{ \}$ term is omitted now.

The condition $y = 0$ when $x = l$ gives

$$0 = -\frac{R_1 l^3}{6} + \frac{W}{6}(l - a)^3 + Al.$$

Hence

$$\begin{aligned} A &= \frac{R_1 l^2}{6} - \frac{W}{6l}(l - a)^3 \\ &= \left[\frac{(l - a)l}{6} - \frac{(l - a)^3}{6l}\right] W \\ &= \frac{Wa}{6l}(l - a)(2l - a) \end{aligned}$$

Substituting in (ii) we get

$$y = \frac{W}{6EI} \frac{x(l - a)}{l}(2al - a^2 - x^2) + \frac{W}{6EI}\{x - a\}^3 \quad . \quad \text{(iii)}$$

This is the equation of the deflection curve of the beam, it being understood that the term $\{ \}$ is rejected when the quantity inside is negative. Putting $x = a$, we find for the deflection under the load :

$$\frac{Wa^2(l - a)^2}{3EIl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

When W is at the centre of the beam, i.e. when $a = \frac{l}{2}$, this gives for the deflection

$$\frac{Wl^3}{48EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It should be noted that (11) is not the maximum deflection of the beam, except when a has the value $\frac{l}{2}$.

167. Rules for applying Macaulay's Method.—The above method will be found extremely useful in dealing with deflection problems when the bending moment is discontinuous, and further illustrations of its use will be found below. In applying the method, it will be found convenient to follow these rules :

(*a*) Take the origin at the left-hand end of the beam.

(*b*) Write down the bending moment for a point in the last portion of the beam to the right (e.g. the part *CB* in Fig. 223, the part *DB* in Fig. 225, and so on), taking moments to the *left*.

(*c*) Integrate such expressions as $(x - a)$, which drop out for the part of the beam where $x < a$, in the form $\frac{1}{2}\{x - a\}^2$, using the $\{ \}$ to denote that these terms are rejected when the part inside the brackets becomes negative.

(*d*) Uniformly distributed loads must always be made to extend to the right-hand extremity of the beam, introducing negative loads if necessary (see Case ii, p. 228).

(*e*) In beams loaded symmetrically, use the condition $\frac{dy}{dx} = 0$ when $x = \frac{l}{2}$, from symmetry, to find the constant A above; the constant B is always zero.

168. Freely Supported Beam with Distributed Load over a Portion of the Span.—

CASE I (Fig. 223).—Suppose the load is w per unit length over the portion CB of the span. Then, choosing a point P between C and B, the bending moment at P is

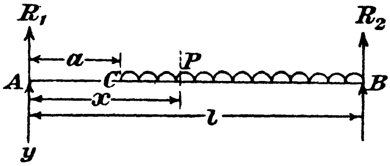

FIG. 223.

$$M = R_1x - \frac{w\{x-a\}^2}{2}$$

where
$$R_1 = \frac{w\{l-a\}^2}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

The general equation for the deflection is therefore

$$EI\frac{d^2y}{dx^2} = -R_1x + \frac{w}{2}\{x-a\}^2$$

where the last term is rejected if $x < a$.

The integral of this equation is

$$EIy = -\frac{R_1x^3}{6} + \frac{w}{24}\{x-a\}^4 + Ax + B \quad . \quad . \quad (\text{ii})$$

the term { } being rejected when $x < a$.

To find A and B we have the conditions $y = 0$ when $x = 0$ and when $x = l$. These conditions give

$$0 = B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iii})$$

$$0 = -\frac{R_1l^3}{6} + \frac{w}{24}\{l-a\}^4 + Al + B \quad . \quad . \quad . \quad (\text{iv})$$

From (i), (iii), (iv) we have $B = 0$, and

$$A = \frac{w}{24l}(l-a)^2(l^2 + 2al - a^2).$$

Hence the equation of the deflexion curve is, from (ii),

$$EIy = -\frac{w(l-a)^2x^3}{12l} + \frac{w}{24}\{x-a\}^4 + \frac{wx}{24l}(l-a)^2(l^2+2al-a^2) \quad (13)$$

the term { } being rejected when $x < a$.

CASE II (Fig. 224), when the load does not reach to either support. This can be treated in the same way as the last by supposing the load w to extend from C to B and superimposing a load $-w$ from D to B,

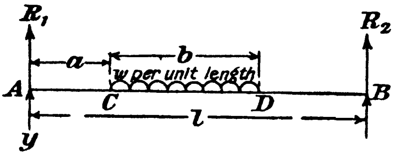

FIG. 224.

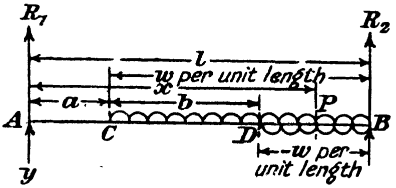

FIG. 225.

as in Fig. 225. Then, taking a point P between D and B, the bending moment at P is

$$M = R_1 x - \frac{w\{x - a\}^2}{2} + \frac{w\{x - \overline{a + b}\}^2}{2},$$

the second term being rejected when $x < a$, and the third when $x < (a + b)$. The deflection is then given by

$$EI\frac{d^2y}{dx^2} = -R_1 x + \frac{w}{2}\{x - a\}^2 - \frac{w}{2}\{x - \overline{a + b}\}^2,$$

from which we have as before

$$EIy = -\frac{R_1 x^3}{6} + \frac{w}{24}\{x - a\}^4 - \frac{w}{24}\{x - \overline{a + b}\}^4 + Ax + B,$$

the terms { } being rejected when the quantity inside the brackets becomes negative. The constants A and B are found as before from the conditions $y = 0$ when $x = 0$ or l. As the rest of the work proceeds exactly as in Case (i) we shall not give it here. Other similar cases can be dealt with in the same way.

169. Beam Supported at Each End, with a Couple Applied at an Intermediate Point.*—In Fig. 226 let the beam AB, supported at

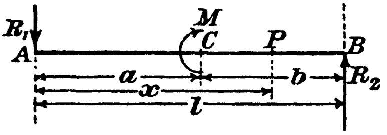

FIG. 226.

A and B, be acted on by a couple M applied at the point C, then the reactions R_1 and R_2 are given by M/l.

The bending moment at P is $-R_1 x + \{M\}$, $(x > a)$. Hence we have

$$EI\frac{d^2y}{dx^2} = R_1 x - \{M\}$$

---

* This extension of the Macaulay method is due to H. A. Webb.

The integral of this can be written in the form

$$EI\frac{dy}{dx} = \frac{R_1x^2}{2} - M\{x - a\} + A \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\therefore\ EIy = \frac{R_1x^3}{6} - \frac{M}{2}\{x - a\}^2 + Ax + B \quad . \quad . \quad . \quad \text{(ii)}$$

As usual, B is zero, since $y = 0$ when $x = 0$.

The condition $y = 0$ when $x = l$ gives

$$A = -\frac{R_1l^2}{6} + \frac{M}{2l}(l - a)^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Substituting this value of A in (ii) we get the equation of the deflection curve. We shall content ourselves with finding the deflection at C, where $x = a$.

Let y_c = the deflection at C, then

$$EIy_c = \frac{R_1a^3}{6} - \frac{R_1al^2}{6} + \frac{Ma}{2l}(l - a)^2$$

$$= \frac{Ma}{6l}(a^2 - l^2) + \frac{Ma}{2l}(l - a)^2$$

Hence

$$y_c = \frac{Ma}{3EIl}(l - a)(l - 2a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The slopes at the ends are obtained from (i), and are given by

$$\left.\begin{aligned} \left(\frac{dy}{dx}\right)_{x=0} &= \frac{M}{6EIl}(2l^2 - 6al + 3a^2) \\ \left(\frac{dy}{dx}\right)_{x=l} &= \frac{M}{6EIl}(3a^2 - l^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Example 1.—A beam rests on two supports 20 ft. apart and carries a uniformly distributed load of 3 tons per foot run. The moment of inertia of the cross section is 2,700 ins.^{4}, and E = 13,500 tons/in.^{2} Find the maximum deflection.

Since E and I are in inch units we will reduce the length and the load to the same units.

$$l = 240 \text{ ins.}$$

$$w = \tfrac{3}{12} = 0{\cdot}25 \text{ ton per inch.}$$

Hence, from (10) the deflection at the centre is

$$\frac{5 \times 0{\cdot}25 \text{ tons/ins.} \times 240^4 \text{ ins.}^4}{384 \times 13{,}500 \text{ tons/in.}^2 \times 2{,}700 \text{ in.}^4}$$

$$= \frac{5 \times 0{\cdot}25 \times 33 \times 10^8 \text{ ins.}}{384 \times 1{\cdot}35 \times 0{\cdot}27 \times 10^8} = 0{\cdot}295''.$$

Example 2.—If the beam in Example 1 be propped up to the level of the supports, the prop being 8 ft. from one end, what will be the pressure on the prop ?

Let y be the deflection of the unsupported beam at a point 8 ft. (= 96″) from one end, and let R be the pressure on the prop when it is in position.

Then, from (ii) of § 165, putting $x = 96$,

$$y = \frac{0{\cdot}25}{24 \times 13{,}500 \times 2{,}700}(96 \times 240^3 - 480 \times 96^3 + 96^4)$$

$$= \frac{0{\cdot}25 \times 144 \times 144}{24 \times 13{,}500 \times 2{,}700}(64{,}000 - 20{,}480 + 4{,}096)$$

$$= \frac{216 \times 47{,}616}{13{,}500 \times 2{,}700} = 0{\cdot}282''.$$

Now a force R tons acting upwards on the beam at 96″ from one end would produce an upward deflection (§ 166, (11))

$$\frac{R \times 96^2 \times 144^2}{3 \times 13{,}500 \times 2{,}700 \times 240} = 0{\cdot}00728R \text{ ins.}$$

Since the effect of R is to neutralize the deflection due to the distributed load we must have

$$0{\cdot}00728R = 0{\cdot}282$$
$$R = 38{\cdot}7 \text{ tons.}$$

Example 3.—A beam, 16 ft. long, has an I section 1 ft. deep, and is simply supported at the ends. The moment of inertia of the cross section about the neutral axis is 400 ins.^{4} A concentrated load of 5 tons is hung 12 ft. from a support. What is the maximum deflection, and the deflection at the middle point of the beam? (R.N.E.C., Keyham, 1919.) (See Fig. 227.)

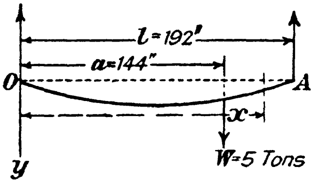

FIG. 227.

Taking $E = 13{,}500$ tons/in.^{2}, the value of EI is 13,500 tons/in.^{2} × 400 ins.^{4} = $5{\cdot}4 \times 10^6$ tons. in.^{2}

We proceed exactly as in § 166, and use equation (iii) of that article, putting $a = 144''$, $l = 192''$, $W = 5$ tons. This gives

$$y = \frac{5x \times 48}{6 \times 5{\cdot}4 \times 10^6 \times 192}(34{,}560 - x^2) + \frac{5}{6 \times 5{\cdot}4}\{x - 144\}^3 \text{ inches.}$$

or
$$6{\cdot}48 \times 10^6 y = 8{,}640x - \frac{x^3}{4} + \{x - 144\}^3 \quad . \quad . \quad \text{(i)}$$

When $x = 96''$ this gives

$$6{\cdot}48 \times 10^6 y = 8{,}640 \times 96 - \frac{96^3}{4}$$
$$= 608{,}000$$
$$y = 0{\cdot}094''.$$

To find when y is a maximum we must solve the equation $dy/dx = 0$. But a difficulty arises in so far as we do not know which form of equation (i) we should use, as we do not know whether the value of x which makes dy/dx vanish is less or greater than 144″. To decide this we investigate the value of dy/dx when $x = 144''$: we have from (i)

$$6{\cdot}48 \times 10^6\frac{dy}{dx} = 8{,}640 - \frac{3x^2}{4} + 3\{x - 144\}^2 \quad . \quad . \quad \text{(ii)}$$

When $x = 144''$ this gives

$$6{\cdot}48 \times 10^6\frac{dy}{dx} = 8{,}640 - 15{,}550 = -6{,}910.$$

Thus $\frac{dy}{dx}$ is negative, that is y is decreasing, and it will go on decreasing to the end $x = 192.''$ Therefore the maximum value of y will occur when x is less than $144''$; to find this value of x we take the equation

$$\frac{3x^2}{4} - 8{,}640 = 0$$

which gives $x = 107{\cdot}3''$.

Substituting in (i) we find

$$y_{max} = 0{\cdot}0985''.$$

170. Beam with Terminal Couples and Distributed Load.—Let the ends of the beam be supported in such a way that no constraint is offered to their angular deflection, and let external couples M_1 and M_2 be applied to the ends as in Fig. 228. For the rest, the notation is as before.

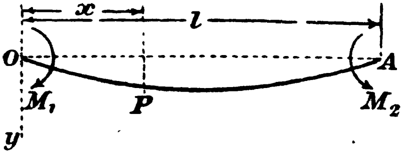

FIG. 228.

At any point P, the bending moment is, § 107, equation (18),

$$M = \frac{l-x}{l}M_1 + \frac{x}{l}M_2 + \frac{w}{2}(lx - x^2) \quad . \quad . \quad . \quad \text{(i)}$$

Hence the equation of the deflection curve is given by

$$EI\frac{d^2y}{dx^2} = -\frac{l-x}{l}M_1 - \frac{x}{l}M_2 - \frac{w}{2}(lx - x^2)$$

Integrating this twice we have

$$EI\frac{dy}{dx} = \frac{(l-x)^2}{2l}M_1 - \frac{x^2}{2l}M_2 - \frac{w}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + A \quad . \quad . \quad . \quad \text{(ii)}$$

$$EIy = -\frac{(l-x)^3}{6l}M_1 - \frac{x^3}{6l}M_2 - \frac{w}{2}\left(\frac{lx^3}{6} - \frac{x^4}{12}\right) + Ax + B \quad . \quad . \quad . \quad \text{(iii)}$$

If the ends of the beam remain at the same horizontal level we must have

$$y = 0 \text{ when } x = 0$$
$$y = 0 \quad ,, \quad x = l.$$

Therefore

$$-\frac{l^2}{6}M_1 + B = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

and

$$-\frac{l^2}{6}M_2 - \frac{wl^4}{24} + Al + B = 0 \quad . \quad . \quad . \quad . \quad \text{(v)}$$

From these we obtain

$$A = \frac{wl^3}{24} - \frac{M_1 - M_2}{6}l$$

$$B = \frac{l^2}{6}M_1$$

Substituting for A and B in (iii) we find

$$y = \frac{x(l-x)(x-2l)}{6EIl}M_1 + \frac{x(l^2-x^2)}{6EIl}M_2 + \frac{wx}{24}(l^3 - 2lx^2 + x^3). \quad (16)$$

The slopes at the ends are given by:

$$EI\left(\frac{dy}{dx}\right)_{x=0} = \frac{lM_1}{2} + A$$

$$EI\left(\frac{dy}{dx}\right)_{x=l} = -\frac{lM_2}{2} - \frac{wl^3}{12} + A$$

or

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{lM_1}{3EI} + \frac{lM_2}{6EI} + \frac{wl^3}{24EI} \quad . \quad . \quad . \quad . \quad (17)$$

$$\left(\frac{dy}{dx}\right)_{x=l} = -\frac{lM_1}{6EI} - \frac{lM_2}{3EI} - \frac{wl^3}{24EI} \quad . \quad . \quad . \quad . \quad (18)$$

These results will be useful later.

171. Relative Movement of Supports.—Suppose that, in the case just considered, the end A sinks a distance δ relative to the end O. Then, when $x = l$, we must have $y = \delta$ instead of $y = 0$. The equations for finding A and B are then

$$-\frac{l^2}{6}M_1 + B = 0,$$

and

$$-\frac{l^2}{6}M_2 - \frac{wl^4}{24} + Al + B = EI\delta,$$

instead of (iv) and (v) above. These give

$$A = -\frac{l}{6}(M_1 - M_2) + \frac{wl^3}{24} + \frac{EI\delta}{l}$$

$$B = \frac{l^2}{6}M_1.$$

Then from (ii) we have, when $x = 0$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{lM_1}{3EI} + \frac{lM_2}{6EI} + \frac{wl^3}{24EI} + \frac{\delta}{l} \quad . \quad . \quad . \quad . \quad (19)$$

and, when $x = l$,

$$\left(\frac{dy}{dx}\right)_{x=l} = -\frac{lM_1}{6EI} - \frac{lM_2}{3EI} - \frac{wl^3}{24EI} + \frac{\delta}{l} \quad . \quad . \quad . \quad . \quad (20)$$

The deflection curve is obtained by adding a term $x\delta/l$ to (16) above.

172. Beams with Non-Uniformly Distributed Load : Graphical Treatment.*—When a beam carries a load which is not uniformly distributed the methods of the previous articles can still be employed if M and $\int M.dx$ are both integrable functions of x, for we have in all cases

$$-EI\frac{d^2y}{dx^2} = M,$$

---

* For alternative graphical method see Appendix I (3).

which can be written in the form

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{M}{EI}$$

If I is uniform along the beam the first integral of this is

$$\frac{dy}{dx} = A - \frac{1}{EI}\int M dx \qquad \text{. . . . . (21)}$$

where A is a constant. The second integral is

$$y = Ax + B - \frac{1}{EI}\iint M.dx.dx \qquad \text{. . . . . . (22)}$$

If M and $\int M dx$ are integrable functions of x the process of finding y can be continued analytically, the constants A and B being found from the terminal conditions. Failing this the integrations must be performed graphically. This is most readily done by plotting the bending-moment curve, and from that deducing a curve of areas representing $\int Mdx$. From this curve a third is deduced representing $\iint Mdx.dx$. We shall explain the process in detail for two cases, and illustrate them by examples.

173. Simply Supported Beam.—We have shown how to deduce the bending moment diagram for any loading in Chapter IX, so we shall assume here that it is known. Let the B.M. diagram for a beam OC be the curve OMC in Fig. 229.

Then, starting at O, draw a curve OND such that at any point the ordinate PN represents the area OMP of the B.M. diagram. In Fig. 229 we have drawn the curves on separate bases for the sake of clearness, but this is not necessary. Then for this curve

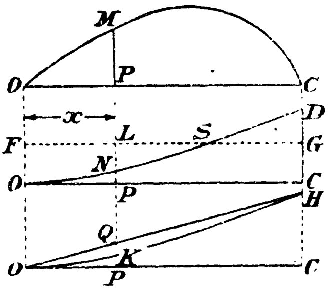

FIG. 229.

$$NP = \int_0^x Mdx.$$

Next draw the curve OKH such that the ordinate PK represents the area ONP of the previous curve, i.e.

$$PK = \int_0^x PN.dx = \int_0^x\int_0^x M.dx.$$

Join OH.

Now at O we have $x = 0$, $y = 0$, $\int_0^x\int_0^x Mdx = 0$. Hence (22) shows that $B = 0$.

At C we have $x = l$, $y = 0$, and $\int_0^x \int_0^x M dx dx$ is represented by CH. Therefore from (22)

$$0 = Al - \frac{1}{EI} . CH.$$

$$\therefore A = \frac{CH}{EIl}$$

Then at P we shall have

$$PQ = \frac{x}{l} CH = EI.A.x.$$

$$\begin{aligned} \therefore QK &= PQ - PK \\ &= EI.Ax - \int_0^x \int_0^x M.dx.dx \\ &= EI\left\{Ax - \frac{1}{EI} \int_0^x \int_0^x Mdx.dx\right\} \\ &= EIy. \end{aligned}$$

That is, the ordinates of the curve OKH, measured from the line OH, represent EI times the deflections of the corresponding points of the beam.

Again, if we draw a line FG at a height above OB equal to $\frac{CH}{l} = EI.A$, we have

$$\begin{aligned} LN &= LP - NP \\ &= EI.A - \int_0^x Mdx \\ &= EI\left\{A - \frac{1}{EI} \int_0^x M.dx\right\} \\ &= EI\frac{dy}{dx}. \end{aligned}$$

Therefore the ordinates of the curve OND, measured from the line FG, represent EI times the slopes at corresponding points on the axis of the beam, positive slopes being when the curve is below the line FG.

SCALES.—Let the scales of the B.M. diagram be

$1'' = s$ inches.
$1'' = m$ lbs. ins.

Then 1 in.^{2} of the area of the B.M. diagram will represent ms lbs. in.^{2} Let the curve OD be drawn to a scale of

$1'' = n$ sq. ins. of B.M. curve.
$= mns$ lbs. in.^{2}

Then the scale of the slopes, measured between FG and OD will be

$$1'' = \frac{mns}{EI} \text{ radians.}$$

Let the curve *OH* be drawn to the scale

$$1'' = p \text{ sq. ins. of the slope curve.}$$
$$= pmns^2 \text{ lbs. in.}^3$$

Then the scale of deflections, measured between the line *OH* and the curve *OH* will be

$$1'' = \frac{pmns^2}{EI} \text{ ins.}$$

174. Cantilever with Irregular Load.—In Fig. 230 *BM* is the bending moment diagram for the cantilever *OB* fixed at the end *O*, the bending moment being negative for a downward load. The curve *OS* represents $\int M dx$ and its ordinates are drawn downwards since *M* is negative. Therefore, if we reckon these ordinates as positive they will represent $-\int M.dx$, i.e. $+EI\frac{dy}{dx}$. The curve *OD* is then drawn from *OS*, representing $-\iint Mdx.dx$, i.e. EIy. The constants *A* and *B* both vanish, since the curves make y and $\frac{dy}{dx}$ vanish with x, so that *OB* is the base line in all cases. The scales are determined as in the last article.

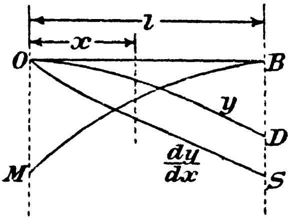

FIG. 230.

Example.—A beam of length 14 ft., freely supported at the ends, carries a total load of 9 tons. The load increases uniformly from one end to double the intensity at the other end. The moment of inertia of the section is 300 ins.^{4}, and $E = 13{,}500$ tons/in.^{2} Calculate the maximum deflection.

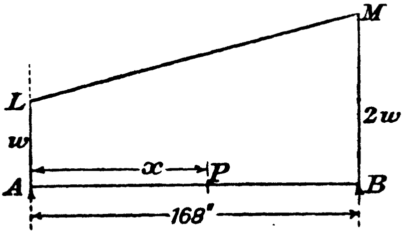

FIG. 231.

In Fig. 231 let *AB* be the beam, and *LM* the load distribution curve. Let w tons/in. be the load per ft. at *A* and $2w$ the load per ft. at *B*. The total load on the beam is then $1{\cdot}5w \times 14 \times 12$ tons. Hence

$$w = \frac{9}{1{\cdot}5 \times 14 \times 12} = 0{\cdot}0357 \text{ tons/inch.}$$

At any point *P* distant x from *A* the loading is

$$0{\cdot}0357 + \frac{x}{168} \times 0{\cdot}0357 = 0{\cdot}0357 + 0{\cdot}00212x \text{ tons/inch.}$$

Hence, § 96, the shearing force at P is

$$F = \int (0{\cdot}0357x + 0{\cdot}00212x)dx + A$$

$$F = 0{\cdot}0357x + 0{\cdot}00106x^2 + A \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where A is a constant of integration, and the bending moment is

$$M = -\int Fdx + B$$

or from (i):

$$M = -(0{\cdot}01785x^2 + 0{\cdot}000353x^3 + Ax + B) \quad . \quad . \quad . \quad \text{(ii)}$$

where B is a constant. Since the beam is freely supported, M must vanish when $x = 0$ and $x = l$. This requires $B = 0$, and

$$A = -0{\cdot}01785l - 0{\cdot}000353l^2.$$

Then (ii) becomes

$$M = 0{\cdot}01785(lx - x^2) + 0{\cdot}000353(l^2x - x^3) \text{ tons. ins.}$$

The deflection curve is given by

$$-EI\frac{d^2y}{dx^2} = M.$$

$$EI = 13{,}500 \times 300 = 4{\cdot}05 \times 10^6 \text{ tons. in.}^2$$

Hence

$$4{\cdot}05 \times 10^6\frac{d^2y}{dx^2} = -0{\cdot}01785(lx - x^2) - 0{\cdot}000353(l^2x - x^3).$$

Integrating this we obtain:

$$4{\cdot}05 \times 10^6\frac{dy}{dx} = -0{\cdot}01785\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) - 0{\cdot}000353\left(\frac{l^2x^2}{2} - \frac{x^4}{4}\right) + C, \quad \text{(iii)}$$

and

$$4{\cdot}05 \times 10^6y = -0{\cdot}01785\left(\frac{lx^3}{6} - \frac{x^4}{12}\right) - 0{\cdot}000353\left(\frac{l^2x^3}{6} - \frac{x^5}{20}\right) + Cx + D, \quad \text{(iv)}$$

The constants C and D are given by the conditions $y = 0$ when $x = 0$ and $x = l$. This requires $D = 0$ and

$$C = 0{\cdot}01785\left(\frac{l^3}{6} - \frac{l^3}{12}\right) + 0{\cdot}000353\left(\frac{l^4}{6} - \frac{l^4}{20}\right)$$
$$= 124{,}000.$$

Substituting this value of C, and $l = 168''$, in (iii) and (iv) gives:

$$4{\cdot}05 \times 10^6\frac{dy}{dx} = 124{,}000 - 344x^2 + 123{\cdot}5x^3 + 2{\cdot}16x^4 \quad . \quad \text{(v)}$$

and

$$4{\cdot}05 \times 10^6y = 124{,}000x - 114{\cdot}6x^3 + 30{\cdot}9x^4 + 0{\cdot}432x^5 \quad . \quad \text{(vi)}$$

The deflection is a maximum when $\frac{dy}{dx} = 0$, i.e. when

$$2{\cdot}16x^4 + 123{\cdot}5x^3 - 344x^2 + 124{,}000 = 0.$$

The real positive root of this is found by trial to be 85·2. Substituting this value of x in (vi) we find

$$y = 0{\cdot}138''.$$

175. Beams of Varying Section.—When the section of the beam changes from point to point, the moment of inertia I must be brought under the sign of integration in equations (21) and (22) of § 172, which then read :

$$\left.\begin{aligned} \frac{dy}{dx} &= A - \frac{1}{E}\int \frac{M}{I}dx \\ y &= Ax + B - \frac{1}{E}\iint \frac{M}{I}dx.\,dx \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The general method of procedure follows exactly the same lines as before. If M/I and $\int \frac{M}{I}dx$ are integrable functions of x, the work can be done analytically, otherwise graphical methods must be employed. In the latter case a curve of M/I must be taken as the starting point instead of a curve of M.

Example 1.—A steel strip 4′ 6″ long and 2″ wide has a varying thickness. The thickness at the centre is $\frac{3}{8}''$ and it decreases to zero at the ends in such a manner that the thickness at a distance x ins. from an end is $\frac{1}{8}\sqrt[3]{x}$. The strip is supported at its ends and carries a central load W. If the centre line in the unstrained state is straight, show that the load W will bend it into a circular arc.

Calculate the value of W to give a central deflection of 1 inch, taking $E = 30 \times 10^6$ lbs./in.^{2} (Intercoll. Exam., Cambridge, 1913.)

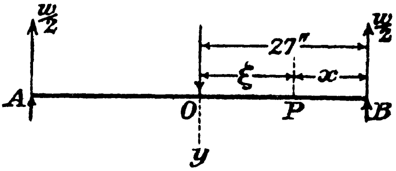

FIG. 232.

In Fig. 232 let AB represent the beam. Since the section of the beam is symmetrical about its middle point O, it will be convenient to take the origin there.

Let ξ denote the distance of a point P from O, so that $x = 27 - \xi$.

The depth of the section at $P = \frac{1}{8}x^{\frac{1}{3}} = \frac{1}{8}(27 - \xi)^{\frac{1}{3}}$. Hence

$$I = \frac{1}{12} \times 2 \times \frac{27 - \xi}{8^3} = \frac{27 - \xi}{3{,}072} \text{ ins.}^4$$

The bending moment at P is

$$M = \frac{W}{2}(27 - \xi), \text{ lbs. ins.}$$

$$\therefore \frac{M}{I} = \frac{W \times 3{,}072}{2} = 1{,}536W$$

$$\frac{d^2y}{d\xi^2} = -\,1{,}536\frac{W}{E}$$

which shows that the radius of curvature is constant, i.e. that the beam bends into a circular arc.

Integrating this equation, we have

$$\frac{dy}{d\xi} = A - 1{,}536\frac{W\xi}{E}$$

$$y = A\xi + B - 768\frac{W\xi^2}{E}.$$

We must have $y = 0$ when $\xi = 27$, and on account of symmetry the tangent to the curved central line will be horizontal when $\xi = 0$, i.e. $\frac{dy}{d\xi} = 0$ when $\xi = 0$. Hence $A = 0$, and

$$B = \frac{768 \times 27^2 W}{E}$$

The deflection at the centre, where $\xi = 0$, is B. If this is to be 1″, we have

$$768 \times 27^2 \frac{W}{E} = 1$$

$$\therefore W = \frac{E}{768 \times 729} = \frac{30 \times 10^6}{768 \times 729} = 53{\cdot}6 \text{ lbs.}$$

Example 2.—In Fig. 233 the curve M represents the bending moment due to air pressure on the blade of an aeroplane propeller at various distances from the axis, the diameter of the propeller being 10 ft. The moments of inertia of the sections of the blade are given in the table below. Regarding the blade as rigid for the first 9″ from the axis, taking $E = 1{\cdot}6 \times 10^6$ lbs./in.^{2}, and neglecting the twist of the blade, it is required to deduce a deflection curve.

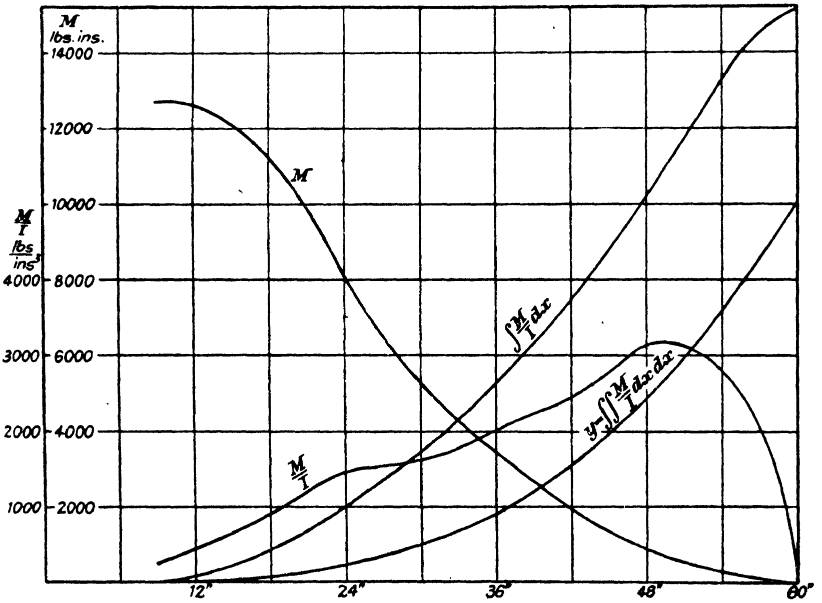

FIG. 233.

Dist. from Axis (ins.)	9	12	15	18	24	30	36	42	48	54
I. (ins.^{4})	50·8	27·4	17·4	12·25	5·65	3·23	1·69	0·783	0·278	0·074
Dist. from Axis	57	60								
I.	0·0298	0								

The blade is treated as a cantilever so that, strictly, the bending moment ought to be considered negative and drawn downwards, according to our conventions, but for convenience we disregard sign and draw all the curves upwards on a horizontal base.

The first step is the calculation of M/I at the various sections, and plotting the results as a curve as shown.

We next plot the areas of this curve, beginning at $x = 9''$, i.e. draw the curve of $\int \frac{M}{I}dx$.

From this the third curve, $y = \iint \frac{M}{I}dx\,dx$ is obtained by the same process.

In the original drawing of Fig. 233, the scales were

1″ represents 6″
1″ ,, 2,000 lbs. ins., bending moment
1″ ,, 1,000 lbs./in.^{3}, M/I.

Hence 1 sq. in. of area of M/I curve represented 6,000 lbs./in.^{2} The curve $\int \frac{M}{I}dx$ was drawn to a scale of 1″ = 2 in.^{2} of M/I curve = 12,000 lbs./in.^{2}

Therefore 1 sq. in. of area of the $\int \frac{M}{I}dx$ curve represented

$$6 \times 12{,}000 = 72{,}000 \text{ lbs./ins.}$$

The curve $\iint \frac{M}{I}dx\,dx$ was drawn to a scale

$$1'' = 5 \text{ in.}^2 \text{ of } \int \frac{M}{I}dx \text{ curve}$$

$$= 360{,}000 \text{ lbs./ins.}$$

The deflection is

$$y = \frac{1}{E}\iint \frac{M}{I}dx\,dx$$

∴ the scale of the $\iint \frac{M}{I}dx\,dx$ curve was

$$1'' = \frac{360{,}000}{1{\cdot}6 \times 10^6} = 0{\cdot}225'' \text{ deflection.}$$

The maximum deflection at the end of the blade

$$= 5{\cdot}05 \times 0{\cdot}225$$
$$= 1{\cdot}14'' \text{ nearly.}$$

176. Non-Uniformly Distributed Load and Terminal Couples: Expressions for the Slopes.—If w vary from point to point along the beam, let M_0 denote the bending moment at any point due to w. Then, instead of (i), § 170, we shall have

$$M = M_0 + \frac{l-x}{l}M_1 + \frac{x}{l}M_2,$$

and the differential equation for the deflection curve becomes

$$EI\frac{d^2y}{dx^2} = -M_0 - \frac{l-x}{l}M_1 - \frac{x}{l}M_2 \quad . \quad . \quad . \quad \text{(i)}$$

Integrating this between the limits 0 and l we have

$$EI\left[\left(\frac{dy}{dx}\right)_{x=l} - \left(\frac{dy}{dx}\right)_{x=0}\right] = -\frac{lM_1}{2} - \frac{lM_2}{2} - \int_0^l M_0 dx \ . \quad \text{(ii)}$$

Again, multiplying (i) by x we have

$$EI\left(x\frac{d^2y}{dx^2}\right)=-\frac{lx-x^2}{l}M_1-\frac{x^2}{l}M_2-xM_0.$$

i.e.
$$EI\frac{d}{dx}\left(x\frac{dy}{dx}-y\right)=-\frac{lx-x^2}{l}M_1-\frac{x^2}{l}M_2-xM_0$$

Integrating this between the limits 0 and l we have

$$EI\left[x\frac{dy}{xd}-y\right]_0^l=-\frac{l^2M_1}{6}-\frac{l^2M_2}{3}-\int_0^l M_0xdx.$$

Since $y=0$ when $x=0$ and when $x=l$, the value of the left-hand side is $lEI\left(\frac{dy}{dx}\right)_{x=l}$ Therefore

$$\left[\frac{dy}{dx}\right]_{x=l}=-\frac{lM_1}{6EI}-\frac{lM_2}{3EI}-\frac{1}{lEI}\int_0^l M_0x.dx \quad . \quad . \text{(iii)}$$

Substituting this in (ii) we have

$$\left[\frac{dy}{dx}\right]_{x=0}=\frac{lM_1}{3EI}+\frac{lM_2}{6EI}+\frac{1}{EI}\int_0^l M_0dx-\frac{1}{lEI}\int_0^l M_0xdx \quad \text{(iv)}$$

The integral $\int_0^l M_0dx$ is given by the area of the bending moment curve due to the lateral loads only, and $\int_0^l M_0xdx$ is the moment of this area about the end $x=0$. Equations (iii) and (iv) then give the slopes at the ends of the beam.

If A = the area of the B.M. diagram due to the lateral loads,* and
$\bar{x}$ = the distance of its centroid from the end $x=0$,
we can write equations (iii) and (iv) in the form

$$\left[\frac{dy}{dx}\right]_{x=0}=\frac{lM_1}{3EI}+\frac{lM_2}{6EI}+\frac{A}{EI}-\frac{A\bar{x}}{lEI}$$

$$\left[\frac{dy}{dx}\right]_{x=l}=-\frac{lM_1}{6EI}-\frac{lM_2}{3EI}-\frac{A\bar{x}}{lEI} \quad . \quad . \quad . \quad . \quad (24)$$

Putting $\bar{x}'=l-\bar{x}$ = the distance of the centroid of the B.M. diagram from the end $x=l$, the first of these becomes

$$\left[\frac{dy}{dx}\right]_{x=0}=\frac{lM_1}{3EI}+\frac{lM_2}{6EI}+\frac{A\bar{x}'}{lEI} \quad . \quad . \quad . \quad . \quad (25)$$

177. Non-Uniformly Distributed Load and Terminal Couples, with Varying Cross Section.

—When the section of the beam is not constant, I must be brought over to the right-hand side of (i) of § 176, which then becomes

$$E\frac{d^2y}{dx^2}=-\frac{M_0}{I}-\frac{M_1}{I}+\frac{M_1-M_2}{l}\cdot\frac{x}{I}.$$

---

* The phrase "lateral loads" does not include the terminal couples.

Proceeding in the same way as before, the equations for finding the slopes at the ends of the beam will be found to be

$$E\left[\left(\frac{dy}{dx}\right)_{x=l}-\left(\frac{dy}{dx}\right)_{x=0}\right]=-\int_0^l\frac{M_0}{I}dx-M_1\int_0^l\frac{dx}{I}+\frac{M_1-M_2}{l}\int_0^l\frac{x}{I}dx \quad . \quad (26)$$

$$E\left(\frac{dy}{dx}\right)_{x=l}=-\frac{1}{l}\int_0^l\frac{M_0x}{I}dx-\frac{M_1}{l}\int_0^l\frac{x}{I}dx+\frac{M_1-M_2}{l^2}\int_0^l\frac{x^2}{I}dx \quad . \quad . \quad (27)$$

It will be necessary to plot five curves : $\frac{M_0}{I}$, $\frac{1}{I}$, $\frac{x}{I}$, $\frac{x^2}{I}$, $\frac{M_0x}{I}$, and to find their areas.

178. Beam Acted on by Terminal Couples and Carrying a Concentrated Load.—As an example of the use of the formulæ established in § 176 we shall consider the case shown in Fig. 234 (upper part).

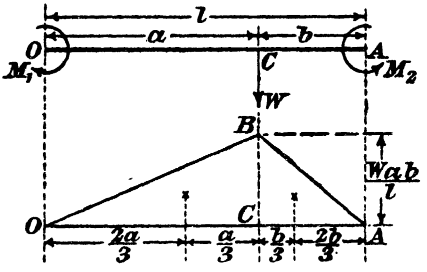

FIG. 234.

The B.M. diagram for the lateral load W is the triangle OBA shown in the lower part of the figure. In the notation of § 176 we have

$$\begin{aligned}A\bar{x} &= \text{moment of area } OBA \text{ about } O\\ &= \text{sum of moments of areas } OBC \text{ and } ABC \text{ about } O\\ &= \frac{2a}{3}\times\frac{Wa^2b}{2l}+\left(a+\frac{b}{3}\right)\frac{Wab^2}{2l}\\ &= \frac{Wab}{6l}(2a^2+3ab+b^2)\\ &= \frac{Wa(l^2-a^2)}{6}\end{aligned}$$

Hence, from (24), p. 240,

$$\left(\frac{dy}{dx}\right)_{x=l}=-\frac{lM_1}{6EI}-\frac{lM_2}{3EI}-\frac{Wa(l^2-a^2)}{6EIl} \quad . \quad . \quad . \quad . \quad (28)$$

Similarly,

$$\left(\frac{dy}{dx}\right)_{x=0}=\frac{lM_1}{3EI}+\frac{lM_2}{6EI}+\frac{Wa(a^2-3al+2l^2)}{6EIl} \quad . \quad . \quad . \quad (29)$$

If the end A sinks a distance δ relative to the end O, we can show, as in § 171, that we must add $\frac{\delta}{l}$ to the above expressions for the values of $\frac{dy}{dx}$ at the ends of the beam.

DEFLECTIONS BY HARMONIC ANALYSIS

179. Introductory.—The following method * of calculating deflections enables us to deal mathematically with any load distribution whatsoever, and will be made use of later when we consider the effects of pulsating loads on girders. Regarding the load distribution as a function of x, measured along the axis of the beam, it can always be expressed as a series of sines or cosines by the method of Fourier's Analysis, so that if we develop formulæ for the case of a load which is distributed along the beam according to a law of sines we can deduce formulæ for any other distribution of load.

180. Freely Supported Beam with Sinusoidal Distribution of Load.—Consider the case of a beam of length l freely supported at each end and subjected to a load

$$w = w_0 \sin \frac{n\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

per unit length, x being measured from one end, and n being any integer.

The equation for the deflection is

$$EI\frac{d^4y}{dx^4} = w = w_0 \sin \frac{n\pi x}{l}.$$

By successive integrations we get

$$EI\frac{d^3y}{dx^3} = -\frac{w_0 l}{n\pi}\cos\frac{n\pi x}{l} + A$$

$$EI\frac{d^2y}{dx^2} = -\frac{w_0 l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} + Ax + B$$

$$EI\frac{dy}{dx} = \frac{w_0 l^3}{n^3\pi^3}\cos\frac{n\pi x}{l} + \frac{A}{2}x^2 + Bx + C$$

$$EIy = \frac{w_0 l^4}{n^4\pi^4}\sin\frac{n\pi x}{l} + \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D$$

where A, B, C, D are constants. The conditions to be satisfied are: $\frac{d^2y}{dx^2} = 0$ and $y = 0$ when $x = 0$, so that $B = D = 0$; $\frac{d^2y}{dx^2}$ must also vanish when $x = l$, therefore $A = 0$; finally, since $y = 0$ when $x = l$ we get $C = 0$. Hence the deflection is given by

$$y = \frac{w_0 l^4}{n^4\pi^4 EI}\sin\frac{n\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

---

* The first investigators to adopt this method appear to have been Prof. S. P. Timoshenko (*Phil. Mag.*, May, 1922), and Prof. C. E. Inglis (*Proc. Inst. C.E.*, 1924).

The bending moment is

$$M = -EI\frac{d^2y}{dx^2} = \frac{w_0 l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Thus a sinusoidal distribution of load leads to sinusoidal distributions of bending moment and deflection, when the beam is freely supported at each end.

181. Freely Supported Beam with Uniformly Distributed Load.—We shall illustrate the application of the Fourier series by analysing the deflection of a beam freely supported at each end, and carrying a uniformly distributed load w per unit length.

We can express the constant quantity w as a Fourier series by the equation *

$$w = \frac{4w}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \ldots\ldots\right) . \quad . \quad (33)$$

Then from the results of the last article we can write down at once

$$M = \frac{4wl^2}{\pi^3}\left(\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} \ldots\ldots\right) \quad . \quad (34)$$

and

$$y = \frac{4wl^4}{\pi^5 EI}\left(\sin\frac{\pi x}{l} + \frac{1}{3^5}\sin\frac{3\pi x}{l} + \frac{1}{5^5}\sin\frac{5\pi x}{l} \ldots\ldots\right) . \quad . \quad (35)$$

182. Freely Supported Beam with Concentrated Load.—As another illustration let us take the case of a freely supported beam bearing a concentrated load W at a point distant a from the end from which we measure x. In this instance it is more convenient to begin with the bending moment. We have

$$M = \frac{W(l-a)}{l}x \text{ from } x = 0 \text{ to } x = a$$

$$M = \frac{Wa}{l}(l - x) \quad ,, \quad x = a \text{ to } x = l.$$

---

* For convenience of reference we give the formulæ for expressing a function $f(x)$ as a series of sines. If $f(x)$ has the values $f_1(x)$ from $x = 0$ to $x = a$, $f_2(x)$ from $x = a$ to $x = b$, $f_3(x)$ from $x = b$ to $x = c$, and so on, we can write

$$f(x) = A_1\sin\frac{\pi x}{l} + A_2\sin\frac{2\pi x}{l} + \ldots + A_n\sin\frac{n\pi x}{l} + \ldots$$

where 0 to l is the whole range of x which we consider, and the coefficients are given by

$$A_n = \frac{2}{l}\left\{\int_0^a f_1(x)\sin\frac{n\pi x}{l}dx + \int_a^b f_2(x)\sin\frac{n\pi x}{l}dx + \ldots\right\}$$

The formula expressed thus enables us to deal with discontinuous load distributions, such as triangular, etc. In the above case we have

$$A_n = \frac{2}{l}\int_0^l w\sin\frac{n\pi x}{l}dx = \frac{2w}{\pi n}(1 - \cos n\pi);$$

hence $A_1 = \frac{4w}{\pi}$, $A_2 = A_4 = \ldots 0$; $A_3 = \frac{4w}{3\pi}$, etc.

Using the formulæ given in the footnote on the last page we find that we can write

$$M = \frac{2Wl}{\pi^2}\left(\sin\frac{\pi a}{l}\sin\frac{\pi x}{l} + \frac{1}{2^2}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + \ldots\ldots\right) \quad . \quad (36)$$

Then, from § 180, we see at once that

$$y = \frac{2Wl^3}{\pi^4 EI}\left(\sin\frac{\pi a}{l}\sin\frac{\pi x}{l} + \frac{1}{2^4}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + \ldots\ldots\right) \quad . \quad (37)$$

and that W can be expressed as

$$W = \frac{2W}{l}\left(\sin\frac{\pi a}{l}\sin\frac{\pi x}{l} + \sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + \ldots\ldots\right) . \quad . \quad (38)$$

In making use of the results of §§ 180–182 in any particular case we can attain to any desired degree of accuracy by retaining sufficient terms in the series.

DEFLECTION OF BEAMS DUE TO SHEAR

183. Introductory.—In calculating the deflections of beams we have neglected the distortion which arises from the shear stresses, and this can generally be done with safety. In certain cases, however, it is desirable to calculate the amount of extra deflection which results from the shear stresses. The exact calculation is not possible by simple processes, but we can make a very fair approximation by the following method, which is based on the foregoing theory of bending which is itself not exact. On the assumption that the intensity of shear stress is uniform over a narrow horizontal strip of cross section, we have shown how to estimate its value in § 150. By equating the corresponding strain energy to the work done by the load in producing shear, we obtain an approximate value for the deflection due to shear. For a more accurate calculation we must employ the exact methods of St. Venant,* if the shape of the cross section permits.

We shall now work out approximate formulæ for the deflections due to shear.

184. Approximate Formula.—Let AB and CD (Fig. 235) be two normal cross sections of the beam separated by a distance dx. Let $d\eta$ be the relative displacement of the two sections. If the shearing force on each section is equal to F, then the work done on the section is $\frac{1}{2}F.d\eta$. (Cf. § 53, p. 61).

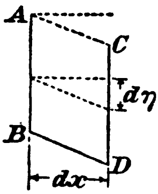

FIG. 235.

Again, if q is the shear stress at height y from the neutral axis, the strain energy of the element $ABDC$ is

$$dx\int\frac{q^2}{2C}\,dS$$

where dS is an element of area of the cross section, and the integral is taken over the whole section.

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* See Todhunter and Pearson's *History of Elasticity*, Vol. II, § 91 and § 96.

In the notation of § 150, p. 202, $q = \frac{F}{IB}\int_h^Y zy.dy$.

Therefore, the strain energy for the slice dx is

$$dx \cdot \frac{F^2}{2C}\int\left\{\frac{1}{I^2B^2}\left[\int_h^Y zy \cdot dy\right]^2 \cdot B\right\}dh,$$

the second integration extending over the whole depth, which may be written

$$k \cdot \frac{F^2}{2C} \cdot dx$$

where

$$k = \frac{1}{I^2}\int\left\{\frac{1}{B}\left(\int_h^Y zy\,dy\right)^2\right\}dh. \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The total strain energy due to shear, for a beam of length l, is therefore

$$U = \frac{1}{2C}\int_0^l k \cdot F^2 \cdot dx.$$

If the cross section be uniform the factor k can be taken outside the integral.

Now, by a generalization of the theorem stated in § 41, p. 40, if η denote the deflection due to a load w, η is given by

$$\eta = \frac{\partial U}{\partial w},$$

η being measured in the same direction as w. If we adhere to our previous conventions with regard to sign, we must write

$$\eta = \frac{\partial U}{\partial w} = -\frac{1}{C}\int_0^l kF \cdot \frac{\partial F}{\partial w} \cdot dx, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

since F is considered positive in an upward direction (e.g. Fig. 118).

Equations (39) and (40) enable us to obtain approximately the deflection of a beam due to shear strains.

The following values of k will be found useful:

(i) Rectangle, breadth B, depth D: $k = \frac{6}{5BD}$.

(ii) Ellipse, semi-axes a and b: $k = \frac{10}{9\pi ab}$.

(iii) I, box, and channel sections: it will usually be sufficiently accurate to assume that all the shearing stress is taken by the web or webs (see p. 204).

185. Special Cases.—(i) Cantilever with concentrated load W at the end, uniform section.

$$\eta = -\frac{1}{C}\int_0^l kF \cdot \frac{\partial F}{\partial W} \cdot dx.$$

F is constant and equal to $-W$, therefore

$$\eta = \frac{k}{C}\int_0^l W \cdot dx = \frac{kWx}{C}; \text{ at free end, } \eta = \frac{kWl}{C},$$

x being measured from the fixed end.

(ii) Cantilever of uniform section, with uniform load w per unit length,

$$F = -w(l - x), \text{ giving } \eta = \frac{kwx}{C}\left(l - \frac{x}{2}\right).$$

At the free end $(x = l)$, $\eta = \dfrac{kwl^2}{2C}$.

(iii) Uniform beam, freely supported at both ends, uniformly distributed load w per unit length,

$$F = w\left(x - \frac{l}{2}\right), \quad \eta = \frac{kwx}{2C}(l - x).$$

At the centre, $\eta = \dfrac{kwl^2}{8C}$.

(iv) Ditto, but with isolated load W at a distance a from the end $x = 0$.

$$\eta = -\frac{Wak}{C}\left(l - \frac{x}{l}\right), \ (a < x < l).$$

EXAMPLES XIV

1. A straight girder of uniform section and length l rests on supports at the ends, and is propped up by a third support in the middle. The weight of the girder and its load is w per unit length. If the central support does not yield, prove that it takes a load equal to $\frac{5}{8}wl$.

2. A beam of I section 16″ × 6″ carries a wall over a span of 20 ft., the load being 2·5 tons per foot run. The greatest moment of inertia of the section is 726 ins.^{4} Find the maximum stress and deflection.

Assuming that these are found to be excessive, find what relief of dip and stress will be given by a cross beam of the same section carried on supports 16″ below the supports of the main beam and having a 10 ft. span, the middle point of the cross beam coming under the middle point of the main beam. (Intercoll. Exam., Cambridge, 1905.)

3. A horizontal steel girder of uniform section, 45 ft. long, is supported at its extremities and carries loads of 12 and 8 tons concentrated at points 10 ft. and 15 ft. from the two ends respectively. I for the section of the girder is 4,000 in.^{4} and $E = 30 \times 10^6$ lbs./in.^{2} Calculate the deflections of the girder at points under the two loads. (Mech. Sc. Trip., 1911.)

4. A cantilever is to be made of several round rods of steel placed side by side. It is to project 2 ft. and is to support a load of 1,000 lbs. at the end. The deflection at the end is to be 0·1″, and the maximum stress in the steel 4 tons/in.^{2} What size and how many rods should be used? (Mech. Sc. Trip., 1913.)

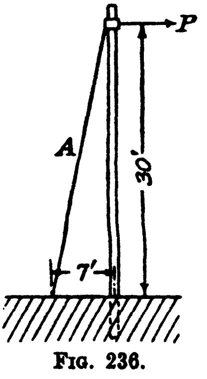

FIG. 236.

5. Fig. 236 represents a wooden mast, with a uniform diameter of 12″, which is built into a concrete block, and is subjected to a horizontal pull at a point 30 ft. from the ground. The wire guy A is to be adjusted so that it becomes taut and begins to take part of the load when the mast is loaded to a maximum stress of 1,000 lbs./in.^{2} Show that when the mast is unloaded there must be about 0·8″ of slack in the guy. Take E for timber $= 2 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip., 1914.)

6. The sides of a doorway are formed of two 6″ × 3″ I beams placed with their flanges facing each other, and rigidly bolted down at ground level. A third beam is to be placed horizontally between the sides, at a height of 10 ft., to form the top. It is found that the top beam is ¼″ longer than the width of the opening, and the sides are accordingly thrust apart by a jack, acting in a horizontal line 6 ft. from the ground. If the greater moment of inertia of the beams is 20·25 ins.^{4}, what force must be exerted by the jack, and what is the greatest normal tensile stress in the section of the beams 6″ from the ground? Take $E = 30 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip., 1915.)

7. A beam 30 ft. long is carried on three supports in the same horizontal line, one at each end, and the third 10 ft. from one end. Each span carries a load of 1 ton at the middle point: find the reaction at each support. (Mech. Sc. Trip., 1916.)

8. A beam 6 ft. long has the Z shaped section shown in Fig. 237. The beam is encastered at one end, AB being horizontal, and its other end is acted on by a vertical load of 2,000 lbs. passing through the c.g. of the end section. Calculate the horizontal and vertical displacements of the outer extremity of the axis of the beam.

Show that by the addition of a horizontal force applied at right angles to the beam, through the c.g. of its outer end, the displacement of the point can be made entirely vertical. Calculate the magnitude of the force and the vertical displacement produced by the combined forces. (Mech. Sc. Trip., 1907.)

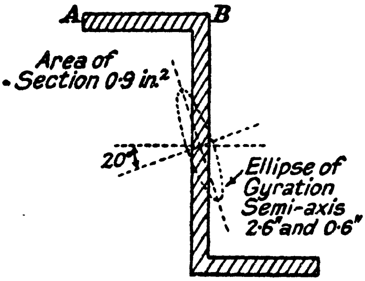

FIG. 237.

9. A bridge across a river has a span $2l$, and is constructed with beams resting on the banks and supported at the middle on a pontoon. When the bridge is unloaded the three supports are all at the same level, and the pontoon is such that the vertical displacement is equal to the load on it multiplied by a constant λ. Show that the load on the pontoon, due to a concentrated load W, placed one-quarter of the way along the bridge, is given by

$$\frac{11W}{16\left(1 + \frac{6EI\lambda}{l^3}\right)}$$

where I is the moment of inertia of the section of the beams. (Intercoll. Exam., Cambridge, 1922.)

10. Two equal steel beams are encastered at one end (see Fig. 238) and connected by a steel rod as shown. Show that the pull in the tie rod is

$$P = \frac{5Wl^3}{32\left(\frac{6aI}{\pi d^2} + l^3\right)}$$

where d is the diameter of the rod, and I is the moment of inertia of the section of each beam about its neutral axis. (Intercoll. Exam., Cambridge, 1923.)

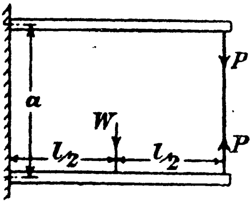

FIG. 238.

11. The floor of a room is carried on 12″ × 5″ rolled steel joists, 40 ft. long, which rest on the walls at their ends and are supported by stanchions

at the middle. The supports were originally all at the same level and the maximum fibre stress 7·5 tons/in.², but it is found that one of the stanchions has sunk 3″ more than the walls. Assuming the load uniformly distributed, show that, due to the settlement, the maximum fibre stress in the joist will be increased by about 36 per cent., if $E = 30 \times 10^6$ lbs./in.² (Mech. Sc. Trip., 1920.)

12. A horizontal cantilever of uniform cross section is deflected ¼″ by a load concentrated at the free end. Calculate the deflection produced by the same load distributed in a uniformly varying manner such that (*a*) the load is zero at the fixed end, (*b*) the load is zero at the free end. (Mech. Sc. Trip., 1921.)

13. The beam in question 21, p. 141, has $I = 320$ ins.⁴, and $E = 13{,}000$ tons/in.² Draw the deflection diagram. (R.N.E.C., Keyham, 1921.)

14. Obtain the equation for the deflection curve of a cantilever of uniform strength carrying a load distributed in such a way that it decreases uniformly from a maximum at the fixed end to zero at the free end. The section of the beam is to be rectangular, the depth being varied in such a way that the longitudinal stress is kept constant along the top and bottom of the beam, while the width is uniform.

15. A beam is loaded and supported in the manner shown in Fig. 239. I for the cross section $= 5{,}000$ ins.⁴, and $E = 30 \times 10^6$ lbs./in.² By graphical means draw the deflection curve for the girder to the following scales: 1″ = 4 ft. horizontally ; deflections 5 times full size. (Mech. Sc. Trip., 1911.)

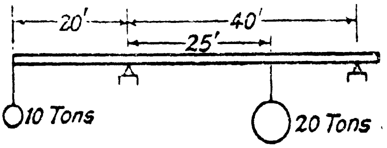

FIG. 239.

16. If a total load W be distributed over a cantilever of uniform section in such a way that the intensity of loading varies uniformly from zero at the free end to a maximum at the fixed end, show that the deflection curve is given by

$$y = \frac{W}{60EIl^2}(4l^5 - 5l^4x + x^5)$$

where x is measured from the free end.

17. A beam carries a load, w per unit length, which extends over a distance a from one end. Measuring x from that end show that the deflection can be expressed in the form

$$y = \frac{2wl^4}{\pi^5 EI}\sum_{k=1}^{k=\infty}\frac{1}{k^5}\left(1 - \cos\frac{k\pi a}{l}\right)\sin\frac{k\pi x}{l},$$

where l is the length of the beam, which is freely supported at each end.

18. A girder, freely supported at both ends, carries a distributed load which varies uniformly in intensity from zero at one end to w_0 per unit length at the other end. Measuring x from the end where the load intensity is zero, show that the deflection is given by

$$y = \frac{2w_0l^4}{\pi^5 EI}\sum_{k=1}^{k=\infty}\frac{(-1)^{k+1}}{k^5}\sin\frac{k\pi x}{l}.$$

CHAPTER XV

BUILT IN, OR ENCASTRÉ, BEAMS

186. Introductory.—In all our investigations of the stresses and deflections of beams having two supports, we have, with one exception,[*] supposed that the supports exercise no constraint on the flexure of the beam, i.e. the axis of the beam has been assumed free to take up any inclination to the line of supports. This has been necessary because, without knowing how to deal with the deformation of the axis of the beam, we were not in a position to find the bending moments on a beam when the supports constrain the direction of the axis. We shall now investigate this problem. When the ends of a beam are fixed in direction so that the axis of the beam has to retain its original direction at the points of support, the beam is said to be built in, or encastré.

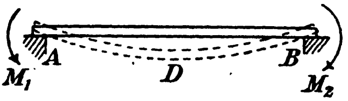

FIG. 240.

Consider a straight beam resting on two supports A and B (Fig. 240) and carrying vertical loads. If there is no constraint on the axis of the beam, it will become curved in the manner shown dotted, the extremities of the beam rising off the supports.

In order to make the ends of the beam lie flat on the horizontal supports, we shall have to apply couples as shown by M_1 and M_2. If the beam is firmly built into two walls, or bolted down to two piers, or in any other manner held in such a way that the axis cannot tip up at the ends in the manner indicated, the couples such as M_1 and M_2 are supplied by the resistance of the supports to deformation. These couples are termed fixing moments, and the main problem of the encastré beam is the determination of these couples; when we have found these we can draw the bending moment diagram in the manner shown in § 107, and calculate the stresses in the usual way. Since the couples M_1 and M_2 must be such as to produce curvature in the opposite direction to that caused by the loads, they will usually be negative, according to our conventions, the positive directions of M_1 and M_2 being opposite to those shown in Fig. 240.

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[*] The propped cantilever of § 163.

187. Encastré Beam with Uniformly Distributed Load.—Let $w =$ the load per unit length. On account of symmetry it is evident that the fixing moments M_1 and M_2 must be equal. In § 169 we have found expressions for the slope at each end of a beam carrying a uniformly distributed load and acted on by terminal couples; in our present case these slopes must be zero. We must have, therefore, from (17), p. 232,

$$\frac{lM_1}{3EI} + \frac{lM_2}{6EI} + \frac{wl^3}{24EI} = 0$$

and $M_1 = M_2$. Hence

$$\mathbf{M}_1 = \mathbf{M}_2 = -\frac{\mathbf{w}\boldsymbol{l}^2}{\mathbf{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, substituting these values in (16) of § 170 we find for the equation of the deflection curve

$$y = \frac{wx^2(l - x)^2}{24EI}$$

At the centre, where $x = \frac{l}{2}$, we find from this that the maximum deflection is

$$\frac{wl^4}{384EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e., one-fifth of the deflection of a freely supported beam carrying the same load.

The bending moment diagram is drawn as in § 107, and is shown in Fig. 241, where the curve ARB is the bending moment diagram for the distributed load acting alone, HK is the bending moment diagram for the fixing moments, and HMK is the resultant bending moment diagram. It will be seen that the greatest bending moment occurs at the ends, and is two-thirds of the greatest bending moment ($wl^2/8$) when the beam is freely supported. The bending moment at the middle is

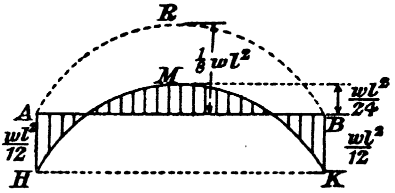

FIG. 241.

$$\tfrac{1}{8}wl^2 - \tfrac{1}{12}wl^2 = \tfrac{1}{24}wl^2,$$

which is one-half that at the ends.

188. Encastré Beam with Single Concentrated Load.—Let a be the distance of the load W from one end of the beam; then, proceeding as above, we have from (28) and (29), p. 241,

$$\frac{lM_1}{3} + \frac{lM_2}{6} + \frac{aW(a^2 - 3al + 2l^2)}{6l} = 0$$

$$\frac{lM_1}{6} + \frac{lM_2}{3} + \frac{aW(l^2 - a^2)}{6l} = 0$$

From these we have

$$\left.\begin{aligned} \mathrm{M}_1 &= -\frac{(l-\mathrm{a})^2\mathrm{a}}{l^2}\mathrm{W} \\ \mathrm{M}_2 &= -\frac{\mathrm{a}^2(l-\mathrm{a})}{l^2}\mathrm{W} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

M_1 being the fixing moment at the end from which a is measured.

The equation for the deflection curve can be found by combining the M_1 and M_2 terms of (16), § 170, with the W terms of (iii), § 166, and using the values of M_1 and M_2 given by (3). At the point where W is fixed, i.e. where $x = a$, we find

$$y = \frac{Wa^3(l-a)^3}{3EIl^3}$$

The maximum deflection can be shown to be

$$\frac{2}{3}\,\frac{Wa^3(l-a)^2}{(l+2a)^2EI}$$

When W is at the middle of the span, the fixing moments are (putting $a = l/2$) $M_1 = M_2 = -\frac{1}{8}Wl$, and the deflection at the centre is $Wl^3/192EI$, which is one-quarter of the deflection the beam would have if freely supported at the ends.

The bending moment diagram can be drawn in the same way as before. It is shown in Fig. 242 for the case when the load is at the middle of the beam : ARB is the bending moment diagram for the load W only, HK is that for the fixing moments, and HMK is the resultant bending moment.

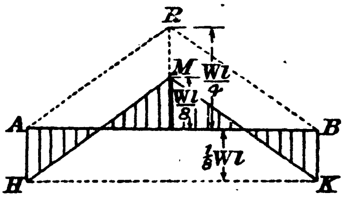

FIG. 242.

189. Encastré Beam with Irregular Loading.—In this case, referring to § 175, the slopes at the ends are given by the equations (iii) and (iv), p. 240, hence the fixing moments are given by :

$$\left.\begin{aligned} M_1 + 2M_2 &= -\frac{6}{l^2}\int_0^l M_0x.dx \\ 2M_1 + M_2 &= \frac{6}{l^2}\int_0^l M_0x.dx - \frac{6}{l}\int_0^l M_0dx \end{aligned}\right\} \quad . \quad . \quad . \quad (i)$$

the section of the beam being uniform.

In the notation of § 176 these equations can be written

$$M_1 + 2M_2 = -\frac{6A\bar{x}}{l^2}$$

$$2M_1 + M_2 = \frac{6A\bar{x}}{l^2} - \frac{6A}{l}$$

which give

$$\left.\begin{aligned} M_1 &= \frac{A}{l}\left(\frac{6\bar{x}}{l} - 4\right) \quad . \quad . \quad . \\ M_2 &= \frac{A}{l}\left(2 - \frac{6\bar{x}}{l}\right) \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Adding equations (i) together we have

$$M_1 + M_2 = -\frac{2}{l}\int_0^l M_0 dx. \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

When the loading is symmetrical we must have $M_1 = M_2$; hence, from (ii), in this case:

$$M_1 = M_2 = -\frac{1}{l}\int_0^l M_0 dx.$$

That is

$-M_1 = -M_2 =$ the mean height of the B.M. diagram for the transverse loads only.

In any case (ii) shows us that the mean value of the fixing moments must be equal and opposite to the mean bending moment due to the transverse loads.

In the above equations M_1 is the fixing moment at the end from which x is measured, M_2 that at the other end.

190. Varying Section.—When the section of the beam is not constant, I must be brought under the signs of integration, as explained in § 177. The right-hand sides of equations (26) and (27) of § 177 must be zero, and thus we have two equations for finding M_1 and M_2:

$$\left.\begin{aligned} M_1\int_0^l \frac{dx}{I} - \frac{M_1 - M_2}{l}\int_0^l \frac{x}{I}dx &= -\int_0^l \frac{M_0}{I}dx \quad . \quad . \quad . \\ M_1\int_0^l \frac{x}{I}dx - \frac{M_1 - M_2}{l}\int_0^l \frac{x^2}{I}dx &= -\int_0^l \frac{M_0 x}{I}dx \quad . \quad . \end{aligned}\right\} \quad . \quad . \qquad (5)$$

the notation being that of § 177.

191. Disadvantages of Built-in Beams.—The results we have obtained above show that a beam which has its ends firmly fixed in direction is both stronger and stiffer than the same beam with its ends freely supported. On this account it might be supposed that beams would always have their ends encastré whenever possible; in practice it is not often done. There are several objections to built-in beams: in the first place a small subsidence of one of the supports will tend to set up large stresses, and, in erection, the supports must be aligned with the utmost accuracy; changes of temperature also tend to set up large stresses. Again, in the case of live loads passing over bridges, the frequent fluctuations of bending moment, and vibrations, would quickly tend to make the degree of fixing at the ends extremely uncertain.

Most of these objections can be obviated by employing the double cantilever construction. Since the bending moments at the ends of an encastré beam are of opposite sign to those in the central part of the beam, there must be points of inflexion, i.e. points where the bending moment is zero. At these points a hinged joint might be made in the beam, the axis of the hinge being parallel to the bending axis, since there is no bending moment to resist. If this is done at each point of inflexion, the beam will appear as a central girder freely supported by two end cantilevers; the bending moment curve and deflection curve will be exactly the same as if the beam were solid and encastré. With this construction the beam is able to adjust itself to changes of temperature or subsidence of the supports.

192. Effect of Sinking of Supports.—Referring to Fig. 228, p. 231, suppose the end A sinks a distance δ, relative to the end O. The fixing moments at the ends of the beam are found, as before, by equating to zero the expressions for the slopes at the ends. Thus, from (19) and (20), p. 232, we find for the encastré beam,

$$\left.\begin{aligned} M_1 &= -\frac{wl^2}{12} - \frac{6EI\delta}{l^2} \\ M_2 &= -\frac{wl^2}{12} + \frac{6EI\delta}{l^2} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

Similarly, when there is a concentrated load W at a distance a from the end O (Fig. 234) we find

$$\left.\begin{aligned} M_1 &= -\frac{(l-a)^2aW}{l^2} - \frac{6EI\delta}{l^2} \\ M_2 &= -\frac{a^2(l-a)W}{l^2} + \frac{6EI\delta}{l^2} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (7)$$

when the ends are fixed in direction.

Example.—A horizontal beam is encastré at both ends, and is 40 ft. long. It carries a uniformly distributed load of 20 lbs./ft. over the first half of the beam, and a concentrated load of 400 lbs. at the centre of the second half. Sketch curves of shearing force and bending moment, and determine the points of contrary flexure. Determine also the deflection under the concentrated load, if $I = 3{\cdot}79$ in.^{4} and $E = 30 \times 10^6$ lbs./in.^{2}. (Mech. Sc. Trip., 1912.)

The arrangement is shown in Fig. 243, and it should be noticed that we have drawn the beam with the distributed load over the right-hand end of the beam, i.e. the half remote from the origin of x; this makes possible the direct application of the rules given in § 167, without the artifice used in § 168, Case ii.

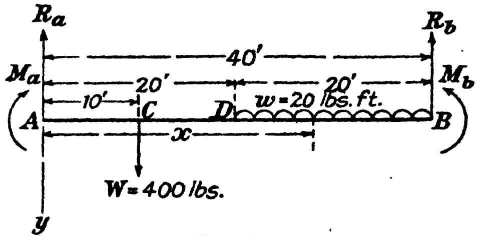

FIG. 243.

The reactions are found in the usual way and are

$$R_a = 400 - \frac{M_a - M_b}{40} \text{ lbs.}$$

$$R_b = 400 + \frac{M_a - M_b}{40} \text{ lbs.,}$$

where M_a and M_b denote the fixing moments at A and B, measured in lbs. ft.

The bending moment at P is given by

$$\begin{aligned} M &= M_a + xR_a - W\{x - 10\} - \frac{w}{2}\{x - 20\}^2 \\ &= M_a\left(\frac{40 - x}{40}\right) + \frac{x}{40}M_b + 400x - 400\{x - 10\} - 10\{x - 20\}^2 \quad \text{(i)} \end{aligned}$$

Hence the deflection curve is given by

$$EI\frac{d^2y}{dx^2} = -M_a\left(\frac{40 - x}{40}\right) - \frac{x}{40}M_b - 400x + 400\{x - 10\} + 10\{x - 20\}^2$$

By integration we get

$$EI\frac{dy}{dx} = \frac{M_a(40 - x)^2}{80} - \frac{x^2}{80}M_b - 200x^2 + 200\{x - 10\}^2 + \frac{10}{3}\{x - 20\}^3 + A$$

$$EIy = -\frac{M_a(40 - x)^3}{240} - \frac{x^3}{240}M_b - \frac{200x^3}{3} + \frac{200}{3}\{x - 10\}^3 + \frac{10}{12}\{x - 20\}^4 + Ax + B$$

We must have $y = 0$ when $x = 0$; therefore

$$B = \frac{800}{3}M_a$$

We also require $y = 0$, when $x = 40$; therefore

$$40A + B = \frac{64{,}000}{240}M_b + \frac{200 \times 64{,}000}{3} - \frac{200 \times 27{,}000}{3} - \frac{10 \times 160{,}000}{12}$$

$$\therefore\ A = -\frac{20M_a}{3} + \frac{20M_b}{3} + \frac{175{,}000}{3}$$

Since the beam is encastered at both ends, we must have $\frac{dy}{dx} = 0$ when $x = 0$, and when $x = 40$. Hence

$$\left.\begin{aligned} 20M_a + A &= 0 \\ -\frac{1{,}600}{80}M_b - 200 \times 1{,}600 + 200 \times 900 + \frac{10}{3} \times 8{,}000 + A &= 0 \end{aligned}\right\}$$

These two conditions give

$$\begin{cases} 40M_a + 20M_b = -175{,}000 \\ 20M_a + 40M_b = -165{,}000, \end{cases}$$

whence we find

$$\begin{cases} M_a = -3{,}082 \text{ lbs. ft.} \\ M_b = -2{,}583 \text{ lbs. ft.} \end{cases}$$

The equation of the bending moment curve is then got from (i); and we have the following values of M:

$$x = 0,\ M = -3{,}082 \text{ lbs. ft.}$$
$$x = 10,\ M = +1{,}045 \text{ lbs. ft.}$$
$$x = 20,\ M = +1{,}167 \text{ lbs. ft.}$$
$$x = 30,\ M = +300 \text{ lbs. ft.}$$
$$x = 40,\ M = -2{,}583 \text{ lbs. ft.}$$

Hence one point of inflexion occurs between $x = 0$ and $x = 10$; the other between $x = 30$ and $x = 40$. To find the first, we have, from (i),

$$\frac{40 - x}{40}M_a + \frac{x}{40}M_b + 400x = 0$$

which gives $x = 7{\cdot}48$ ft.

To find the second point of inflexion, the whole of the expression (i) for M must be used, since $x > 20$. This leads to

$$x^2 - 41{\cdot}2x + 308{\cdot}2 = 0.$$

The roots of this are $x = 9{\cdot}8$ and $x = 31{\cdot}4$. The smaller root is inadmissible as $9{\cdot}8$ is less than 20, and the equation is only true when $x > 20$. Hence the second point of inflexion is where $x = 31{\cdot}4$ ft.

The B.M. and S.F. diagrams are shown in Fig. 244.

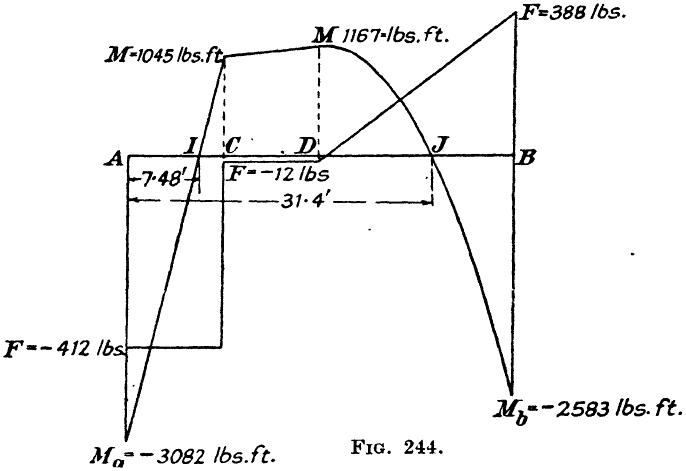

FIG. 244.

Finally to find the deflection under the concentrated load, i.e. where $x = 10$. We have above ($x = 10$),

$$EIy = -\frac{27{,}000M_a}{240} - \frac{1{,}000M_b}{240} - \frac{200{,}000}{3} + 10A + B$$

$$= -\frac{2{,}700}{24}M_a - \frac{100}{24}M_b - \frac{200{,}000}{3} - \frac{200}{3}M_a + \frac{200}{3}M_b$$

$$+ \frac{1{,}750{,}000}{3} + \frac{800}{3}M_a$$

$$= 87{\cdot}5M_a + 62{\cdot}5M_b + \frac{1{,}550{,}000}{3}$$

$$= 85{,}200 \text{ lbs. ft.}^3$$

Also $EI = 30 \times 3{\cdot}79 \times 10^6 = 113{\cdot}7 \times 10^6$ lbs. in.^{2}

$= 0{\cdot}79 \times 10^6$ lbs. ft.^{2}

Hence

$$y = \frac{0{\cdot}0852}{0{\cdot}79} = 0{\cdot}108 \text{ ft.} = 1{\cdot}3''.$$

EXAMPLES XV

1. A beam 25 ft. span is built-in at the ends, and carries a load of 6 tons at the centre, and loads of 3 tons 5 ft. from each end. Calculate the maximum bending moment and the positions of the points of inflexion.

2. A girder of span 20 ft. is built-in at each end and carries two loads of 8 and 12 tons respectively placed at 5 ft. and 12 ft. from the left end. Find the bending moments at the ends and centre, and the points of contrary flexure. (Birmingham, VI, B, 1910.)

3. A beam of 30 ft. span has its ends built into the supporting walls, and is loaded uniformly with 100 lbs. per linear foot. Find the maximum bending moment, and draw the B.M. diagram to scale. (Birmingham, VIII, 1911.)

4. A horizontal beam of moment of inertia I and length l is built-in at the ends, and a couple M is applied to the beam, about a horizontal axis at right angles to the beam at the middle point. Show that the slope at the middle is given by $Ml/16EI$. (Intercoll. Exam., Cambridge, 1922.)

5. A horizontal shaft of length l is subjected at its centre to a vertical load W. The shaft fits in bearings at its ends, and when the slope of the shaft at the ends is θ the bearings exert a bending moment on the shaft of magnitude $k\theta$. Prove that the central deflection of the shaft is

$$\frac{Wl^3}{192EI}\left(\frac{kl + 8EI}{kl + 2EI}\right)$$

(Mech. Sc. Trip., 1923.)

6. A built-in beam of 30 ft. span carries two loads of 24 tons, 10 ft. from each end. Obtain the values of the bending moments at the supports, and at the middle of the beam, also expressions for the deflection at the centre in terms of EI.

7. A beam of 20 ft. span is built-in at both ends, and carries a load 500 lbs. per ft. over the left-hand half of its length. Find the position and magnitude of the maximum deflection, if $E = 13{,}400$ tons/in.^{2} and $I = 225$ ins.^{4}

8. A girder is encastré at each end, and has a span of 40 ft. The distributed load varies uniformly from zero at one end to a maximum of 1 ton per ft. at the other end. Calculate the bending moments at the supports.

9. A built-in girder has a span of 36 ft., and $I = 400$ ins.^{4} A load of of 16 tons is on the girder at a distance 12 ft. from the left-hand end. Calculate the position and magnitude of the maximum deflection, taking $E = 13{,}500$ tons/in.^{2}

CHAPTER XVI

CONTINUOUS BEAMS

193. Fixing Moments at the Supports.—When the same beam runs across three or more supports it is spoken of as a continuous beam. Suppose we have three spans, as in Fig. 245, each bridged by a separate beam; the beams will bend independently in the manner shown. In order to make the axes of the three beams form a single continuous curve across the supports B and C, we shall have to apply to each beam couples acting as shown by the arrows. When the beam is one continuous girder these couples, on any bay such as BC, are supplied by the action of the contiguous bays. Thus AB and CD, bending downwards under their own loads, try to bend BC upwards, as shown by the dotted curve, thus applying the couples M_B and M_C to the bay BC. This upward bending is of course opposed by the down load on BC, and the general result is that the beam takes up a sinuous form, being, in general, concave upwards over the middle portion of each bay and convex upwards over the supports.

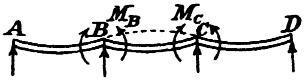

FIG. 245.

In order to draw the bending moment diagram for a continuous beam we must first find the couples such as M_B and M_C; these are usually referred to as the "fixing moments." In some cases there may also be external couples applied to the beam, at the supports, by the action of other members of the structure.

When the bending moments at the supports have been found, the Bending-Moment and Shearing-Force diagrams can be drawn for each bay according to § 107.

194. Theorem of Three Moments for Uniformly Distributed Load.—In Fig. 246, let A, B, C be the lines of support of any two neighbouring bays of a continuous beam.

Let I_1 and I_2 be the moments of inertia of the cross sections of AB and BC.
" w_1 and w_2 be the (constant) load per unit-length on AB and BC.

Let δ_A, δ_B, δ_C be the amount by which the corresponding supports sink under the load.
„ M_{AR} = the bending moment to the right of A.
„ M_{BL} = „ „ „ „ „ left of B, and so on.
„ $\mathfrak{M}_B$ = the external couple (if any) applied at B in a clockwise direction.

Then

$$M_{BL} + \mathfrak{M}_B = M_{BR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and there is a similar equation for each support. Now, for AB take the origin at the original position of the support A, measure x to the right and y downwards. The amount by which the support B sinks below the support A is $\delta_B - \delta_A$. Then, referring to equation (20), § 171, we see that the slope of the tangent at B is

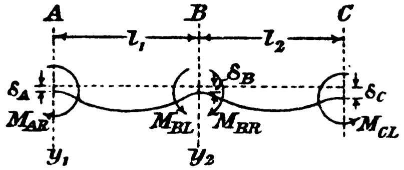

FIG. 246.

$$-\frac{l_1 M_{AR}}{6EI_1} - \frac{l_1 M_{BL}}{3EI_1} - \frac{w_1 l_1^3}{24EI_1} + \frac{\delta_B - \delta_A}{l_1} \quad . \quad . \quad . \quad \text{(i)}$$

Similarly, taking the origin at the original position of B for BC, the slope of the tangent at B, for the part BC, is

$$\frac{l_2 M_{BR}}{3EI_2} + \frac{l_2 M_{CL}}{6EI_2} + \frac{w_2 l_2^3}{24EI_2} + \frac{\delta_C - \delta_B}{l_2} \quad . \quad . \quad \text{(ii)}$$

Since the axis of the beam is continuous the expressions (i) and (ii) must be equal. Equating them we have, on rearranging and multiplying by 6:

$$\frac{l_1 M_{AR}}{I_1} + 2\left(\frac{l_1 M_{BL}}{I_1} + \frac{l_2 M_{BR}}{I_2}\right) + \frac{l_2 M_{CL}}{I_2} + \frac{1}{4}\left(\frac{w_1 l_1^3}{I_1} + \frac{w_2 l_2^3}{I_2}\right)$$
$$= \frac{6E}{l_1}(\delta_B - \delta_A) + \frac{6E}{l_2}(\delta_B - \delta_C) \quad . \quad . \quad . \quad . \quad (2)$$

Now, if there be altogether n supports and $n - 1$ bays, there will be $2n - 2$ bending moments to calculate, for at the first and last supports there will be only one bending moment. We shall have n equations of the type (1) and $n - 2$ equations of the type (2), i.e. $2n - 2$ equations altogether, so that we shall be able to find all the unknown bending moments.

The first and nth equations of type (1) will depend on the conditions which obtain at the first and last supports: if we take A as typical of the first support, we shall have, if A be quite free,

$$M_{AR} = 0;$$

if A be acted on by a clockwise couple $\mathfrak{M}$,

$$M_{AR} = \mathfrak{M}.$$

for example, if the beam overhangs beyond A by a length l carrying a uniformly distributed load w per unit length, and there is no other couple applied at A, $\mathfrak{M} = -wl^2/2$.

If the beam is encastré at A the slope there is zero, and we must have, from (17), p. 232,

$$2M_{AR} + M_{BL} + \frac{w_1 l_1^2}{4} = 0.$$

Again, if C be taken as the last support we must have:

If C is free, $M_{CL} = 0$.

If a clockwise couple $\mathfrak{M}$ be applied at C, $M_{CL} = -\mathfrak{M}$.

If C is encastré, from (18), p. 232,

$$M_{BR} + 2M_{CL} + \frac{w_2 l_2^2}{4} = 0.$$

Equation (2) above is a more general form of what is usually known as the "equation of three moments." The simplifications in particular cases are obvious:

(i) If there are no external couples applied at the supports, we can drop the second letter of the suffixes of the M's and write

$$\frac{l_1 M_A}{I_1} + 2\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right)M_B + \frac{l_2 M_C}{I_2} + \frac{1}{4}\left(\frac{w_1 l_1^3}{I_1} + \frac{w_2 l_2^3}{I_2}\right)$$
$$= \frac{6E}{l_1}(\delta_B - \delta_A) + \frac{6E}{l_2}(\delta_B - \delta_C) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(ii) If, in addition, the section be uniform throughout and $I_1 = I_2 = I$,

$$l_1 M_A + 2(l_1 + l_2)M_B + l_2 M_C + \tfrac{1}{4}(w_1 l_1^3 + w_2 l_2^3)$$
$$= 6EI\left(\frac{\delta_B - \delta_A}{l_1} + \frac{\delta_B - \delta_C}{l_2}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(iii) If there is no sinking of the supports:

$$\boldsymbol{l_1 \mathrm{M_A} + 2(l_1 + l_2)\mathrm{M_B} + l_2 \mathrm{M_C} + \tfrac{1}{4}(\mathrm{w}_1 l_1^3 + \mathrm{w}_2 l_2^3) = 0} \quad . \quad (4\text{A})$$

This is the simplest form of the equation of three moments, and the one most frequently required.

When the bending moments at the supports have been determined, the portions of the reactions due to each bay can be found from (16) and (17), § 107.

195. Theorem of Three Moments for Concentrated Loads.—Let there be a single concentrated load, W_1 and W_2 in each bay respectively, as shown in Fig. 247; let the notation be as in § 178 and § 194. Then considering AB, the slope of the tangent at B, with the origin at A, is from (28), p. 241,

$$-\frac{l_1 M_{AR}}{6EI_1} - \frac{l_1 M_{BL}}{3EI_1} + \frac{a_1 W_1(a_1^2 - l_1^2)}{6 l_1 E I_1} + \frac{\delta_B - \delta_A}{l_1} \quad . \quad (\text{i})$$

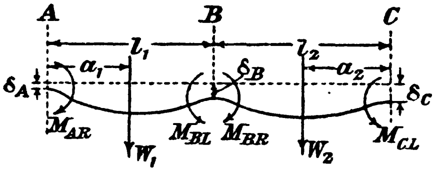

FIG. 247.

Now, considering BC, the slope of the tangent at B is, with the origin at B,*

$$\frac{l_2M_{BR}}{3EI_2}+\frac{l_2M_{CL}}{6EI_2}+\frac{a_2W_2(l_2^2-a_2^2)}{6l_2EI_2}+\frac{\delta_C-\delta_B}{l_2}\quad . \quad . \quad \text{(ii)}$$

Equating the expressions (i) and (ii) we have

$$\frac{l_1\mathrm{M_{AR}}}{\mathrm{I_1}}+2\left(\frac{l_1\mathrm{M_{BL}}}{\mathrm{I_1}}+\frac{l_2\mathrm{M_{BR}}}{\mathrm{I_2}}\right)+\frac{l_2\mathrm{M_{CL}}}{\mathrm{I_2}}+\frac{\mathrm{a_1W_1}}{l_1\mathrm{I_1}}(l_1^2-\mathrm{a_1}^2)+\frac{\mathrm{a_2W_2}}{l_2\mathrm{I_2}}(l_2^2-\mathrm{a_2}^2)$$
$$=\frac{6\mathrm{E}}{l_1}(\delta_\mathrm{B}-\delta_\mathrm{A})+\frac{6\mathrm{E}}{l_2}(\delta_\mathrm{B}-\delta_\mathrm{C})\quad . \quad . \quad . \quad . \quad (5)$$

Example 1.—In Fig. 248, AD is a continuous girder of uniform section, and of weight w per unit length, resting freely on four supports A, B, C, D,

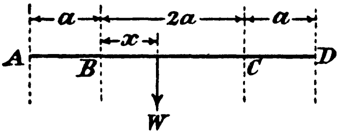

FIG. 248.

at the same level. The middle span carries a load W as shown. Determine the reactions on the supports A and D. (Mech. Sc. Trip., 1922, B.)

The "three-moment" equation for the supports A, B and C is obtained by combining (4A) and (5); the term depending on W is given by putting $W_1=0$, $l_1=a$, $a_2=2a-x$, $W_2=0$, and $l_2=2a$ in (5).

$$aM_A+6aM_B+2aM_C+\frac{w}{4}(a^3+8a^3)+\frac{W}{2a}(2a-x)\{4a^2-(2a-x)^2\}=0$$

or, since $M_A=0$,

$$6M_B+2M_C=-\frac{9wa^2}{4}-\frac{W}{2a^2}(2a-x)(4ax-x^2)\quad . \quad \text{(i)}$$

Similarly, for BCD, we have, since $M_D=0$,

$$2aM_B+6aM_C=-\frac{9wa^3}{4}-\frac{xW}{2a}(4a^2-x^2)$$

or

$$2M_B+6M_C=-\frac{9wa^2}{4}-\frac{Wx}{2a^2}(4a^2-x^2)\quad . \quad . \quad . \quad \text{(ii)}$$

Solving (i) and (ii) we find

$$\left.\begin{aligned}M_B&=-\frac{9}{32}wa^2-\frac{Wx}{16a^2}(2a-x)(5a-2x)\ .\ .\ .\ .\\M_C&=-\frac{9}{32}wa^2-\frac{Wx}{16a^2}(2a-x)(a+2x)\ \ .\ .\ .\ .\end{aligned}\right\}$$

---

* Note that here l_2-a_2 replaces a, and a_2 replaces $l-a$ in § 188.

Then (§ 107)

$$R_A = \frac{wa}{2} + \frac{M_B - M_A}{a}$$

$$= \frac{7}{32}wa - \frac{Wx}{16a^3}(2a - x)(5a - 2x).$$

$$R_D = \frac{wa}{2} + \frac{M_C - M_D}{a}$$

$$= \frac{7}{32}wa - \frac{Wx}{16a^3}(2a - x)(a + 2x).$$

Example 2.—A railway line for a single railway is constructed with two main girders, continuous over four supports, forming two side spans each 150 ft. long and a centre span 180 ft. long. The four supports are at the same level, the girders are of uniform section, and the ends are free from bending moment. The dead load on each main girder is 0·8 ton per ft. run, and the live load 0·4 ton per ft. run. Taking the case when a side span alone carries live load, determine the bending moments over the supports. Sketch the B.M. diagram and determine the pressures on the four supports. (Mech. Sc. Trip., 1910, B.)

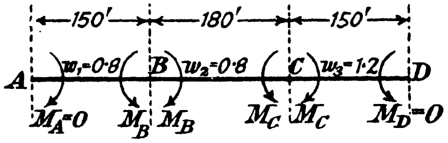

FIG. 249.

Let the right-hand span carry live load; then the conditions of the problem are shown in Fig. 249. The equations for finding the bending moments at A, B, C, D are given by (4A) § 194. Applying the equation first to the three supports A, B, C we have, in ton-feet units,

$$150M_A + 660M_B + 180M_C = -\tfrac{1}{4}(0{\cdot}8 \times 150^3 + 0{\cdot}8 \times 180^3)$$
$$= -1{,}842{,}000$$

and, since $M_A = 0$, this reduces to

$$M_B + 0{\cdot}273M_C = -2{,}790 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Again, applying the equation to B, C, D we have, remembering that $M_D = 0$,

$$180M_B + 660M_C = -\tfrac{1}{4}(0{\cdot}8 \times 180^3 + 1{\cdot}2 \times 150^3)$$
$$= -2{,}180{,}000$$

or

$$M_B + 3{\cdot}66M_C = -12{,}100 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Solving (i) and (ii) we find

$$\left.\begin{aligned} M_B &= -\ \mathbf{2{,}040\ tons.\ ft.} \\ M_C &= -\ \mathbf{2{,}745\ tons.\ ft.} \end{aligned}\right\}$$

The bending moment diagram can now be *drawn* for the whole girder by treating each bay as described in § 107, and particulars can be read from the drawing. The bending moment diagram for each bay, on account of the distributed load only, is a parabola of height $wl^2/8$, which works out to 2,250, 3,240, 3,375 tons. ft. for AB, BC, CD respectively. These parabolas are shown dotted. The bending moments due to the fixing moments are given by the straight lines AB', $B'C'$, $C'D$. The complete B.M. diagram

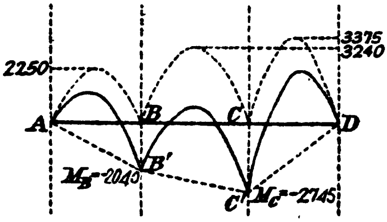

FIG. 250.

is then drawn by replotting the parabolas on the lines $AB'C'D$, as shown by the full curve, as explained on p. 130.

If we only require to sketch the B.M. diagram we can find the principal points on the curve by the formulæ given on pp. 130–131.

The calculations are most conveniently done in the form of a table.

Units: tons, feet.

Bay.	AB	BC	CD
M_1 (= B.M. at left hand end) . . .	0	− 2,040	− 2,745
M_2 (= „ right „) . . .	− 2,040	− 2,745	0
$M_1 - M_2$	2,040	695	− 2,745
$M_1 + M_2$	− 2,040	− 4,785	− 2,745
w	0·8	0·8	1·2
l	150	180	150
wl.	120	144	180
$(M_1 - M_2)/wl$	17·2	4·82	− 15·25
$l/2$	75	90	75
Dist. of M_{max} from left hand end of bay [equation (19) § 107.]	57·8	85·18	90·25
$\{(M_1 - M_2)/wl\}^2$	296	23·2	232
$\frac{w}{2}\{(M_1 - M_2)/wl\}^2$	118·4	9·28	139
$\frac{1}{2}(M_1 + M_2)$	− 1,020	− 2,392	− 1,372
$wl^2/8$	2,250	3,240	3,375
M_{max} [equation (20), § 107]	1,348	857	2,142
$(M_1 + M_2)/w$	− 2,555	− 5,980	− 2,285
$l^2/4$	5,625	8,100	5,625
$\frac{\lambda^2}{4} = l^2/4 + \left(\frac{M_1 - M_2}{wl}\right)^2 + \frac{M_1 + M_2}{w}\left(\begin{matrix}(23),\\ \S\ 107\end{matrix}\right)$	3,366	2,143	3,572
$\frac{\lambda}{2}$	58	46·2	59·8
Distance of points of inflexion from centre of bay (equation (24), § 107) = $\frac{\lambda}{2} + \frac{M_1 - M_2}{wl}$	75·20	51	44·6 (left)
$\frac{\lambda}{2} - \frac{M_1 - M_2}{wl}$	40·8	41·4	75·0 (right)
B.M. at middle of bay $= \frac{wl^2}{8} + \frac{M_1 + M_2}{2}$	1,230	848	2,003

The reactions at the supports are also given by (16) and (17), p. 130.

If R_{BL} and R_{BR} denote the reactions to the left and right of B, and so on, we have

$$\left.\begin{array}{ll}
R_{AR} = 60 - \frac{2{,}040}{150} = 60 - 13\cdot6 = 46\cdot4, & R_A = 46\cdot4 \\
\left.\begin{array}{l} R_{BL} = 60 + 13\cdot6 = 73\cdot6 \\ R_{BR} = 72 - \frac{695}{180} = 72 - 3\cdot9 = 68\cdot1 \end{array}\right\} & R_B = 141\cdot7 \\
\left.\begin{array}{l} R_{CL} = 72 + 3\cdot9 = 75\cdot9 \\ R_{CR} = 60 + \frac{2{,}745}{150} = 90 + 18\cdot3 = 108\cdot3 \end{array}\right\} & R_C = 184\cdot2 \\
R_{DL} = \quad = 90 - 18\cdot3 = 71\cdot7, & R_D = 71\cdot7
\end{array}\right\} \text{Total Reactions at the supports.}$$

Example 3.—A wooden beam 12″ × 12″ section, 12 ft. long, is supported at its ends A and C, and at its middle point B, on 3 girders. Each of these girders is such that a load of 10 tons applied at the points A, B, or C deflects the corresponding girder 1″. Loads of 5 tons are applied to the beam ABC at points distant 1 ft. and 5 ft. to the right and left of B. Find the magnitudes of the reactions at A, B and C, taking E for the beam as 600 tons/in.^{2} (Mech. Sc. Trip., 1918, B.)

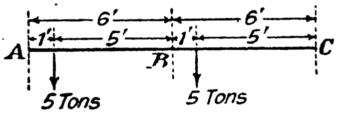

FIG. 251.

Let R_A, R_B and R_C denote the reactions at the supports.

δ_A, δ_B, δ_C denote the deflections of the supporting girders.

Then if R_A, R_B and R_C are expressed in tons,

$$\delta_A = \frac{R_A}{10}; \quad \delta_B = \frac{R_B}{10}; \quad \delta_C = \frac{R_C}{10} \text{ inches} \quad . \quad . \quad . \quad \text{(i)}$$

Also, if M_B = the B.M. at B in tons-feet,

$$\left.\begin{array}{l}
R_A = \frac{5 \times 5}{6} + \frac{M_B}{6} = \frac{25 + M_B}{6}, \\
R_C = \frac{1 \times 5}{6} + \frac{M_B}{6} = \frac{5 + M_B}{6} \\
R_B = 10 - (R_A + R_C) = \frac{30 - 2M_B}{6}
\end{array}\right\} \text{tons} \quad . \quad . \quad \text{(ii)}$$

Hence, from (i),

$$\delta_A = \frac{25 + M_B}{60}; \quad \delta_C = \frac{5 + M_B}{60}; \quad \delta_B = \frac{30 - 2M_B}{60} \quad . \quad \text{(iii)}$$

$$\left.\begin{array}{l}
\delta_B - \delta_A = \frac{5 - 3M_B}{60} \text{ ins.} = \frac{5 - 3M_B}{720} \text{ ft.} \\
\delta_B - \delta_C = \frac{-3M_B + 25}{60} \text{ ins.} = \frac{-3M_B + 25}{720} \text{ ft.}
\end{array}\right\} \quad . \quad \text{(iv)}$$

I for the beam $= \frac{1}{12}$ ft.^{4}

$E = 600 \times 144$ tons/ft.^{2}

$\therefore\ EI = 600 \times 144 \times \frac{1}{12} = 7{,}200$ tons. ft.^{2}

The equation of three moments is ($M_A = M_C = 0$),

$$24M_B = -\frac{1\times 5}{6}(6^2-1^2) - \frac{5\times 5}{6}(6^2-5^2) + \frac{6EI}{l_1}(\delta_B-\delta_A) + \frac{6EI}{l_2}(\delta_B-\delta_C)$$

Substituting for δ_A, δ_B, δ_C from (iv) this becomes

$$\begin{aligned} 24M_B &= -\frac{5\times 35}{6} - \frac{25\times 11}{6} + \frac{43{,}200}{6}\left(\frac{30-6M_B}{720}\right) \\ &= -75+300-60M_B \end{aligned}$$

Hence

$$\begin{aligned} 84M_B &= 225 \\ M_B &= 2{\cdot}68 \text{ tons. ft.} \end{aligned}$$

Substituting in (ii) we have

$$\begin{aligned} R_A &= \frac{27{\cdot}68}{6} &&= 4{\cdot}61 \text{ tons} \\ R_B &= \frac{30-5{\cdot}36}{6} &&= 4{\cdot}11 \quad ,, \\ R_C &= \frac{7{\cdot}68}{6} &&= 1{\cdot}28 \quad ,, \end{aligned}$$

Example 4.—One of the wing spars of a certain aeroplane is represented * in Fig. 252 by the straight line CC'; the supports are at A, B, C, A', B', C'. The loads are completely symmetrical about the centre line of the machine, and the supports are assumed not to move. Couples are applied to the spar as shown; the bending moments to the left of C and right of C' are $-$ 1,040 lbs. ins. It is required to find the bending moments at the supports.

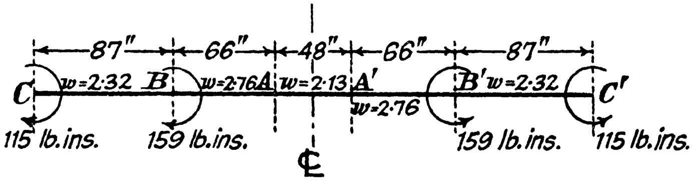

FIG. 252.

The units are pounds and inches throughout.

We have by symmetry

$$M_{CL} = M_{C'R} = -1{,}040\,;\; M_{BL} = M_{B'R}\,;\; M_{BR} = M_{B'L}\,;\; M_A = M_{A'} \qquad \text{(i)}$$

Also:

$$\begin{aligned} M_{CR} &= M_{CL} + 115 \;.\;.\;.\;.\;.\;.\;. && \text{(ii)} \\ M_{BR} &= M_{BL} + 159 \;.\;.\;.\;.\;.\;.\;. && \text{(iii)} \end{aligned}$$

For C, B, A the equation of moments is, from (2), § 194, putting

$$\delta_A = \delta_B = \delta_C = 0,$$

$$87M_{CR} + 174M_{BL} + 132M_{BR} + 66M_A + \tfrac{1}{4}(2{\cdot}32\times 87^3 + 2{\cdot}76\times 66^3) = 0 \quad \text{(iv)}$$

For B, A, A' it is

$$66M_{BR} + 228M_A + 48M_{A'} + \tfrac{1}{4}(2{\cdot}76\times 66^3 + 2{\cdot}13\times 48^3) = 0 \quad .\;\text{(v)}$$

From (i) and (ii) we have

$$M_{CR} = -1{,}040 + 115 = -925 \text{ lb. ins.} = M_{C'L}.$$

---

* In reality the loading along the spar acts upwards and the couples applied at the supports are counterclockwise; the spar is drawn upside down to agree with our conventions of sign.

From (iii) we substitute for M_{BR} in terms of M_{BL} in (iv) and (v), and in (v) we put $M_{A'} = M_A$. We thus obtain two equations for M_{BL} and M_A. Solving them we find

$$M_{BL} = -1{,}570 = M_{B'R}$$
$$M_A = -\ \ 597 = M_{A'}.$$

Then $\qquad M_{BR} = -1{,}411,$ (from (iii)) $= M_{B'L}$.

We have thus found all the bending moments at the supports. The bending moment diagram for the whole beam can then be drawn in the manner of Example 1, p. 261.

196. Theorem of Three Moments for Irregular Loading.—When the lateral loads are neither concentrated nor uniformly distributed, the equation connecting the fixing moments at three consecutive supports can be derived from the expressions for the slopes given in § 176, in the same way as above. In the notation of §§ 176 and 194, the slopes of AB and BC at B are given by equations (24) and (25) of § 176:

$$-\frac{l_1 M_{AR}}{6EI_1} - \frac{l_1 M_{BL}}{3EI_1} - \frac{A_1 \bar{x}_1}{EI_1 l_1}$$

and

$$+\frac{l_2 M_{BR}}{3EI_2} + \frac{l_2 M_{CL}}{6EI_2} + \frac{A_2 \bar{x}_2}{EI_2 l_2}$$

where A_1 and A_2 denote the areas of the B.M. diagrams for AB and BC on account of the lateral loads only, whilst $\bar{x}_1$ and $\bar{x}_2$ are the distances of their centroids from A and C respectively, there being no sinking of the supports.

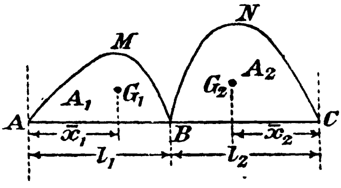

FIG. 253.

In Fig. 253 AMB and BNC are the B.M. diagrams for AB and BC, on account of lateral loads, as independent beams; G_1 and G_2 are the centroids of the areas AMB and BNC.

Equating the two expressions the moment equation becomes:

$$\frac{l_1 M_{AR}}{I_1} + 2\left(\frac{l_1 M_{BL}}{I_1} + \frac{l_2 M_{BR}}{I_2}\right) + \frac{l_2 M_{CL}}{I_2} + \frac{6A_1\bar{x}_1}{I_1 l_1} + \frac{6A_2\bar{x}_2}{I_2 l_2} = 0 \ . \quad (6)$$

The equation can be modified as in §§ 194, 195, when the section is uniform throughout, or when the supports sink, etc.

Example.—A continuous beam of uniform section is supported on three props at the same level. If the beam were not continuous over the centre prop, i.e. if the B.M. at the centre prop were zero, the loads on each segment are such as to give the triangular B.M. diagrams shown in Fig. 254. Find the B.M. on the centre prop, the beam being free over the end props. (Mech. Sc. Trip., 1904.)

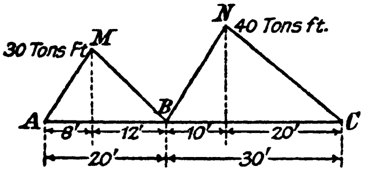

FIG. 254.

Let A_1 = area AMB.
A_2 = area BNC.
$\bar{x}_1$ = distance of centroid of AMB from A measured horizontally.
$\bar{x}_2$ = ,, ,, ,, BNC ,, C ,, ,,

Then

$$A_1\bar{x}_1 = \frac{30 \times 8}{2} \times \frac{16}{3} + \frac{30 \times 12}{2} \times 12 = 2{,}800 \text{ tons. ft.}^3$$

$$A_2\bar{x}_2 = \frac{40 \times 20}{2} \times \frac{40}{3} + \frac{40 \times 10}{2}\left(20 + \frac{10}{3}\right) = 10{,}000 \text{ tons. ft.}^3$$

The equation of moments is, then,

$$20M_A + 100M_B + 30M_C + \frac{6 \times 2{,}800}{20} + \frac{6 \times 10{,}000}{30} = 0$$

Also $M_A = M_C = 0$. Hence

$$100M_B = -840 - 2{,}000 = -2{,}840.$$

$$\therefore M_B = -28{\cdot}4 \text{ tons. ft.}$$

197. Irregular Loading and Varying Section.—In this case the equations giving the slopes at the ends of a bay are (26) and (27) of § 177, but the equations are made rather shorter if we take the origin at opposite ends of the two bays. For AB take the origin at A and measure x_1 to the right; for BC take the origin at C and measure x_2 to the left, as shown in Fig. 255.

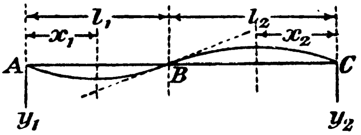

FIG. 255.

Then, from (27) of § 177, we easily deduce as the condition for continuity of slope at B, using the notation of §§ 194, 195:

$$\int_0^{l_1}\frac{x_1}{l_1} \cdot \frac{M_1}{I_1}dx_1 + \int_0^{l_2}\frac{x_2}{l_2} \cdot \frac{M_2}{I_2}dx_2 + M_{AR}\int_0^{l_1}\left\{\frac{x_1}{l_1} - \left(\frac{x_1}{l_1}\right)^2\right\}\frac{dx_1}{I_1}$$
$$+M_{BL}\int_0^{l_1}\left(\frac{x_1}{l_1}\right)^2\frac{dx_1}{I_1} + M_{BR}\int_0^{l_2}\left(\frac{x_2}{l_2}\right)^2\frac{dx_2}{I_2} + M_{CL}\int_0^{l_2}\left\{\frac{x_2}{l_2} - \left(\frac{x_2}{l_2}\right)^2\right\}\frac{dx_2}{I_2} = 0 \quad . (7)$$

where M_1 and M_2 denote the bending moments on AB and BC due to the lateral loads only, and the rest of the notation is as in §§ 194, 195, there being no sinking of the supports.

The various integrals must be evaluated by plotting appropriate curves and finding their areas.

198. Disadvantages of Continuous Beams.—The general remarks made in § 191 in reference to encastré beams apply for the most part to continuous beams: the greatest bending moments are less than those which would occur with the use of separate beams, but exact alignment of the supports is an essential condition for the realization of this, and any sinking of the supports may lead to unexpected and dangerous stresses. Furthermore, if the section is not uniform, the calculations become very laborious and consequently expensive in time and money.

199. Hinged Joints in the Spans.—Some of the disadvantages of continuous beams can be removed by the use of hinged joints between the supports, and this construction is sometimes adopted for other reasons. Since the bending moments at the supports are usually of opposite sign to those in the central portion of a bay, there will in general be two points between each pair of supports where the bending moment is zero, i.e. two points of inflexion. At these points hinged joints can be made in the beam without altering the bending moment diagram, but it is important that the joints *are at these points* and not elsewhere. Otherwise fresh calculations must be made, for the bending moment diagram may be very different from that of a continuous beam without joints. For instance, consider the case shown in Fig. 256, where $ABCD$ is part of a continuous beam with an overhanging end AB, and $AKLMN$ is the bending moment diagram. The points of inflexion for the bay BC are J_1 and J_2, and we can make hinged joints in the beam at these points without altering the bending moments anywhere. But suppose a hinge is placed at J_3: the presence of the hinge necessitates the bending moment there being zero; this bending moment is the algebraic sum of the B.M. due to the lateral load and that due to the fixing moments. In the figure BEC represents the B.M. diagram for BC for the lateral loads only, and the B.M. at J_3 is J_3E; therefore, the fixing moments must be such as to produce at J_3 a bending moment $= -J_3E$. We can find the fixing moment at C thus: make $BK' = BK$, join $K'E$ and produce it to meet MC at M', then CM'', $= -CM'$, is the fixing moment at C, and it can be seen that this may be considerably greater than when the beam is continuous. This shows the necessity of either placing the pin-joints at the points of inflexion of the continuous beam, or of making calculations which allow for the presence of the pin-joint, if placed elsewhere.

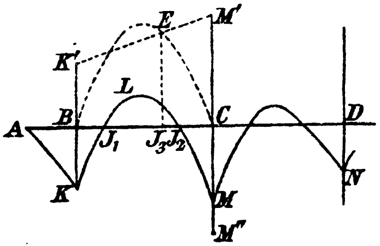

Fig. 256.

When there is a single pin-joint in one bay, such as E in Fig. 257, the calculations can be made as follows. We shall take the simple case where there is a uniformly distributed load on each bay. Let R be the action at E between BE and EC, as shown in the diagram. Let δ be

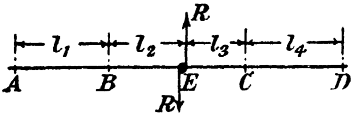

Fig. 257.

the deflection at E. We can regard ABE as a continuous beam for which $M_E = 0$ and the support at E (i.e. the rest of the beam) sinks a distance δ. We have then, if there are no external applied couples,

$$l_1M_A + 2(l_1 + l_2)M_B + \tfrac{1}{4}(w_1l_1^3 + w_2l_2^3) = -\frac{6EI\delta}{l_2} \quad . \quad \text{(i)}$$

the section of the beam being uniform throughout. Similarly for ECD we have

$$2(l_3 + l_4)M_C + l_4M_D + \tfrac{1}{4}(w_3l_3^3 + w_4l_4^3) = -\frac{6EI\delta}{l_3} \quad . \quad \text{(ii)}$$

Also

$$\left.\begin{aligned} -M_B &= \frac{w_2l_2^2}{2} + l_2R \\ -M_C &= \frac{w_3l_3^2}{2} - l_3R \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad \text{(iii)}$$

Eliminating δ from (i) and (ii) we have

$$l_2\{l_1M_A + 2(l_1 + l_2)M_B\} - l_3\{2(l_3 + l_4)M_C + l_4M_D\}$$
$$+ \frac{l_2}{4}(w_1l_1^3 + w_2l_2^3) - \frac{l_3}{4}(w_3l_3^3 + w_4l_4^3) = 0 \quad . \; . \quad (8)$$

Eliminating R from (iii) gives

$$\frac{M_B}{l_2} + \frac{M_C}{l_3} + \frac{w_2l_2}{2} + \frac{w_3l_3}{2} = 0 \quad . \; . \; . \; . \; . \; . \quad (9)$$

Equations (8) and (9) replace the two equations of "three moments" which we should have for A, B, C, D, if the beam were continuous. A similar treatment can be applied to the more general cases of non-uniform load, section, etc.

BEAMS RESTING ON COMPRESSIBLE GROUND

200. General Equations.—We propose now to consider the following problem: a beam AB (Fig. 258) rests on a compressible base, which

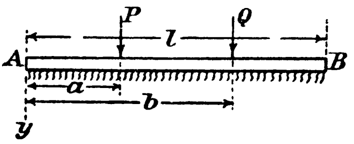

FIG. 258.

for purposes of description may be referred to as the ground. Vertical thrusts, such as P and Q, are applied to the upper side of the beam. It is required to find the bending moment at any point of the beam on the assumption that the pressure, at any point, between the beam and the ground is proportional to the deflection of the beam at that point. The problem finds immediate application in such cases as railway sleepers, where the thrusts P and Q would arise from the loads on the rails; bolts in timber beams, where the loads P and Q would be forces applied across the ends of the bolts, and other cases of a similar nature.

Let p = the upward pressure of the ground, per unit length of beam
= ky, where
y = the downward deflection of the beam.
M = the sagging bending moment at a distance x from A.

Then we shall have

$$EI\frac{d^2y}{dx^2} = -M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and, from § 96, p. 117,

$$\frac{d^2M}{dx^2} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where the positive sign is taken, as p is here acting on the beam in the opposite direction to y. Hence, differentiating twice, we have

$$EI\frac{d^4y}{dx^4} = -\frac{d^2M}{dx^2} = -p = -ky.$$

$$\therefore \frac{d^4y}{dx^4} + 4m^4y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$m = \sqrt[4]{\frac{k}{4EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The general solution of (15) is

$$y = A \cosh mx . \cos mx + B \sinh mx . \sin mx$$
$$+ C \cosh mx . \sin mx + D \sinh mx . \cos mx \quad . \quad . \quad . \quad (17)$$

Then, at a distance x from the origin the bending moment is given by (13), and the shearing force, F, is given by (§ 96)

$$F = -\frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

201. Single Load at the Centre of a Long Beam.*—If a load W rests on the top of a long beam which itself rests on a horizontal compressible bed, the central part of the beam will be depressed, but the extremities may rise off the ground. Let $2l$ be the length of beam in contact with the ground, and measure x from one end of this part of the beam towards the other end. Referring to equation (17) of § 200, when $x = 0$ we must have $y = 0$, $\frac{d^2y}{dx^2} = 0$, and $\frac{d^3y}{dx^3} = 0$. Hence we require

$$A = B = 0 ; \quad C = D.$$

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* § 201 is due to A. Saldanha, to whom I am indebted for permission to give it here. The importance of his treatment lies in his recognizing that the ends of the beams may leave the ground, a fact which other writers have overlooked.

At the centre we must have, from symmetry, $\frac{dy}{dx} = 0$, so that we get

$$2C \cos ml. \cosh ml = 0.$$

$$\therefore ml = \frac{\pi}{2}.$$

Hence the length in contact with the ground is $\frac{\pi}{2m}$. For equilibrium we must have

$$W = 2\int_0^l ky.dx = 2k\frac{C}{m} \sinh ml. \sin ml.$$

$$= \frac{2kC}{m} \sinh \frac{\pi}{2}.$$

$$\therefore C = \frac{mW}{2k \sinh \frac{\pi}{2}}.$$

The greatest deflection occurs when $x = l$, and is given by

$$y_{max} = \frac{mW}{2k}(\cot ml + \coth ml) = \frac{mW}{2k} \coth \frac{\pi}{2}$$

EXAMPLES XVI

1. A continuous girder of uniform section rests upon four supports at the same level, forming three equal spans of 100 ft. The girder carries a load of 2 tons per ft. uniformly distributed. Draw to scale the B.M. diagram for the whole girder, and calculate the loads carried by each support. (Mech. Sc. Trip., 1906.)

2. A continuous girder of two equal spans, each 75 ft. long, is carried on three supports, all at the same level. Find the maximum bending moments and shearing forces—(*a*) when one span is loaded with 1,000 lb. per lin. ft.; (*b*) when both spans are loaded with the same intensity of load. The beam cannot rise above any of the supports.

Draw the B.M. and S.F. diagrams in each case.

What distributions of the uniform load will give maximum positive and negative shearing forces at the centre of either span. (Birmingham University.)

3. A thin steel strip is laced in and out through a row of stiff pegs. The pegs are all $\frac{1}{2}''$ diameter, and their centres are spaced 20″ apart in a straight line. There are five pegs in all, and the section of the strip is a rectangle $1'' \times \frac{1}{16}''$. Find the thrusts on the several pegs, taking $E = 30 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip., 1914, B.)

4. A straight elastic beam of uniform section rests on four similar elastic supports which are spaced l feet apart. The supports are such that they compress d for each unit of load upon them. Show that, when a uniformly distributed load of total amount W comes on the beam, the reactions at the centre props are each

$$\frac{\frac{11}{6} + \frac{3EId}{l^3}}{5 + \frac{12EId}{l^3}}$$ (Mech. Sc. Trip., 1919, B.)

5. A continuous girder, carrying a uniformly distributed load of 1·5 tons per foot, covers three spans. The end spans are each 20 ft. in length and the central span is 30 ft. Calculate the greatest bending moment and shearing force, and the positions of the points of contraflexure. (Mech. Sc. Trip., 1919.)

6. Fig. 259 shows diagrammatically in plan the arrangement of girders carrying a floor. The load is carried by the cross-girders and these are supported on two main continuous girders resting on stanchions at their ends and at the middle.

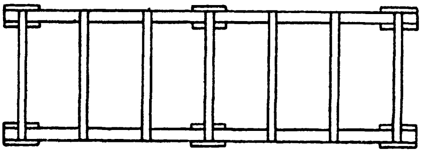

FIG. 259.

Draw the B.M. and S.F. diagrams for one of the main girders when there is a uniformly distributed load of 200 lbs./ft.^{2} on the first three panels and 100 lbs./ft.^{2} on the second three panels. Each panel is 16′ × 8′. (Mech. Sc. Trip., 1921.)

7. Four parallel girders, with the same cross section, are laid across two parallel rigid supports. The distance between the supports is $3a$, and the distance between each girder is c. The girders support two longitudinal beams, laid on top of them parallel to the rigid supports, and dividing each of the four girders into three equal spans of length a. The length of each longitudinal beam is $3c$, and each carries a load W uniformly distributed. Show that, at the point where a longitudinal beam crosses one of the two central girders, the pressure between them is $W(15a^3I_2 + 11c^3I_1)/3(20a^3I_2 + 10c^3I_1)$, where I_1 and I_2 refer to the girders and beams respectively. (Mech. Sc. Trip. B., 1913.)

8. A 12″ × 12″ timber beam, 24 ft. long and initially straight, is supported at the two ends and at the middle. When a load of 8 tons is placed on the middle of the beam, the central support sinks 1·5″ below the line joining the two end supports. Find the pressure exerted on each support, if $E = 1{\cdot}5 \times 10^6$ lbs./in.^{2} (Intercoll. Exam., Cambridge, 1919.)

9. A uniform continuous girder ABC rests upon three similar floating supports, situated at each end and at the middle point B. The buoyancy of each float is such that every additional ton of load increases its immersion by h. Initially all the floats are equally immersed. If a load W tons be placed on the girder at B, show that the proportion carried by the central float is $W(1 + 3hEI/a^3)/(1 + 9hEI/a^3)$, where $2a$ is the length of the girder. (Mech. Sc. Trip., 1922.)

10. A uniform beam of length l rests on a compressible ground, and carries a uniformly distributed load w per unit length. Assuming that the beam makes contact for the whole of its length, show that the deflection at a distance x from one end is given by

$$y = \frac{w}{2k}\Big[2 - (\cosh mx - \sinh mx)\cos mx$$
$$- (\cosh m.\overline{l-x} - \sinh m.\overline{l-x})\cos m.\overline{l-x}\Big].$$

CHAPTER XVII

RIGID ARCHES

202. General Discussion.*—Beams which are curved are frequently used in engineering, particularly in roof work and bridges ; in such cases they are usually called arches or " arched ribs." In general the applied loads may have any direction, but they are most commonly vertical, the arch having a vertical axis of symmetry. We shall limit ourselves to the cases where the loads are vertical, and assume that the dimensions are such that the Bernoulli-Euler theory of bending is applicable. Such an arch is shown in Fig. 260, A and B being the abutments, W_1, W_2 ... W_n a series of vertical loads.

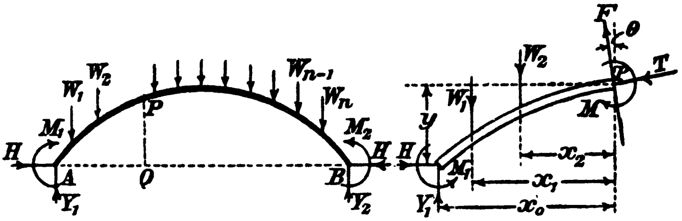

FIG. 260.

The reactions at A and B can be resolved into vertical and horizontal components as shown, and since the external loads are only vertical the horizontal components of the reactions at A and B must be equal.

The action at any section P will consist of a shearing force F normal to the axis, a thrust T tangential to the axis, and bending moment M. These three quantities can all be determined when the reactions H, Y_1, Y_2 (and the bending moments at A and B if these are not pin-joints) have been found.

If A and B are not pin-joints, let M_1 and M_2 be the fixing moments, then considering the equilibrium of AP, we have, taking M positive when it increases the curvature,

$$M = M_1 - x_0 Y_1 + x_1 W_1 + x_2 W_2 + Hy = M_0 + Hy \quad . \quad . \quad . \quad (1)$$

where M_0 is the bending moment at Q for an imaginary horizontal beam AB carrying the same loads.

---

* A considerable investigation of the theory of arches was carried out by Bresse. See Todhunter and Pearson's *History*, Vol. II, Pt. 1, p. 352. For masonry arches see a paper, *The Mechanics of the Voussoir Arch*, by Pippard, Tranter, and Chitty, Inst. Civ. Eng., 1937.

Also $$F = (-Y_1 + W_1 + W_2)\cos\theta + H\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $$T = (Y_1 - W_1 - W_2)\sin\theta + H\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus, if we can find H, M_1 and Y, the resultant action on any cross section can be determined. The ease with which these three quantities can be found depends on the nature of the arch. There are three main types of arch to consider: arches with hinges at the abutments and the crown, arches with hinges only at the abutments, and arches with no hinges at all. The first type can be dealt with by the methods of statics and call for no special treatment here.

GENERAL THEORY

203. Arch Hinged only at the Abutments (Fig. 261).—In this case we can find Y_1 and Y_2 by simple resolution of forces, and the fixing moments at A and B are zero, but we cannot find H by the methods of pure statics. The difficulty is overcome by working out the deformation of the arch and writing down the condition that the distance AB is unchanged.

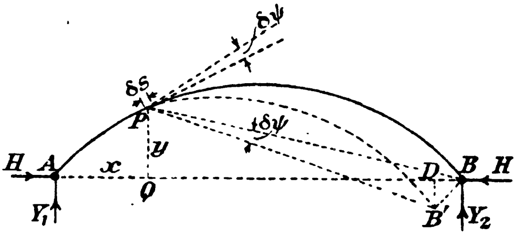

FIG. 261.

Consider an element δs of the axis, at P, and suppose it is rotated through an angle $\delta\psi$. Then if AP remains fixed and PB remains rigid, B would move to a position such as B' if it were free, and

$$BB' = PB.\delta\psi$$

The horizontal displacement of B would be

$$\begin{aligned} BD &= BB' \cos B'BD = BB' \cos QPB = \delta\psi.PB \cos QPB \\ &= y.\delta\psi \end{aligned}$$

But $\delta\psi = \delta s\left(\dfrac{1}{R} - \dfrac{1}{R_0}\right)$ where $\dfrac{1}{R} - \dfrac{1}{R_0}$ denotes the change of curvature at P. Therefore

$$\begin{aligned} BD &= y\delta s.\left(\frac{1}{R} - \frac{1}{R_0}\right) \\ &= \frac{My\delta s}{EI}, \text{ by § 122,} \end{aligned}$$

where M is the bending moment, and I the moment of inertia of the cross section, at P.

Hence the total horizontal displacement of B towards A, due to a change of curvature all along the arch, if free, would be

$$\int_A^B \frac{M.y.ds}{EI}.$$

But this must be zero. Hence, since we have $M = M_0 + Hy$ as in § 202, we must have

$$\int_A^B \frac{M_0 y + Hy^2}{EI} ds = 0$$

or

$$\mathrm{H} = \frac{-\int_A^B \mathrm{M_0 y ds/I}}{\int_A^B \mathrm{y^2 ds/I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If I be constant this becomes

$$H = \frac{-\int_A^B M_0 y.ds}{\int_A^B y^2 ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When the arch is sufficiently flat * we can write dx instead of ds. In other cases, such as circular and elliptic arches y and ds can be expressed in terms of a single parameter and the integrations effected if M_0 takes a suitable shape. More usually graphical integration must be employed.

204. Arch Built-in at Both Ends (Fig. 262).—In addition to the horizontal reaction H we have the fixing moments M_1 and M_2 at A and B to find.

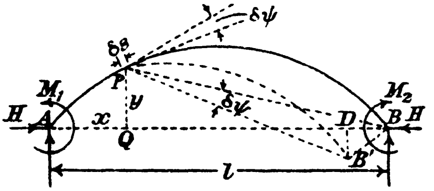

FIG. 262.

Let M_0 denote the bending moment due to the vertical loads only, as before.

We have three quantities to find: H, M_1 and M_2. The conditions for finding them are (i) the horizontal and vertical displacements of B are both zero, and (ii) the total change of slope from A to B is zero.

The bending moment at P is

$$M = M_0 + Hy + M_1 - \frac{x}{l}(M_1 - M_2) \quad . \quad . \quad . \quad (i)$$

M and M_0 being taken positive when they tend to increase the curvature, as before.

---

* When rise is not more than $\frac{1}{10}$ of the span.

The notation of Fig. 262 being the same as that of Fig. 261, we have

$$\delta\psi = \delta s\left(\frac{1}{R} - \frac{1}{R_0}\right) = \frac{M.\delta s}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

$$BD = \frac{My\delta s}{EI}, \text{ as in § 203} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

and

$$\begin{aligned} B'D &= PB.\delta\psi.\ \sin B'BD = PB.\delta\psi.\ \sin BPQ \\ &= (l - x)\delta\psi \\ &= \frac{M(l - x)\delta s}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)} \end{aligned}$$

The conditions enumerated above require, then,

$$\int_A^B \frac{M.ds}{EI} = 0, \qquad \int_A^B \frac{Myds}{EI} = 0, \qquad \int_A^B \frac{M(l - x)ds}{EI} = 0.$$

Substituting for M from (i), these become:

$$\int_A^B \frac{M_0ds}{I} + H\int_A^B \frac{yds}{I} + M_1\int_A^B \frac{ds}{I} - \frac{M_1 - M_2}{l}\int_A^B \frac{xds}{I} = 0 \quad . \quad (6)$$

$$\int_A^B \frac{M_0yds}{I} + H\int_A^B \frac{y^2ds}{I} + M_1\int_A^B \frac{yds}{I} - \frac{M_1 - M_2}{l}\int_A^B \frac{xy.ds}{I} = 0 \quad (7)$$

$$\int_A^B \frac{(l - x)M_0ds}{I} + H\int_A^B \frac{(l - x)yds}{I} + M_1\int_A^B \frac{(l - x)ds}{I} - \frac{M_1 - M_2}{l}\int_A^B \frac{(l - x)xds}{I} = 0$$

Multiplying the first of these by l and subtracting from the last we have for the third equation:

$$\int_A^B \frac{M_0xds}{I} + H\int_A^B \frac{xyds}{I} + M_1\int_A^B \frac{xds}{I} - \frac{M_1 - M_2}{l}\int_A^B \frac{x^2ds}{I} = 0 \quad . \quad (8)$$

When the integrations have been effected, graphically or otherwise, these three equations suffice to find M_1, M_2 and H. The bending moment, etc., at any point can then be determined as in § 202.

It must be noticed that here and in § 203 we have neglected the shortening due to direct thrust. In the case of a fairly flat arch we can assume that the axial thrust is everywhere equal to H, so that the shortening of the span due to H will be Hl/ES. When this is to be taken into account we must add l/S to the denominator of equation (4) and lI/S to the denominator of (5), and Hl/ES should be added to the left-hand side of equation (7), where S is the (constant) area of the cross section. If the cross section is not constant a suitable expression is easily worked out.

205. Deflection of Arched Ribs.—Suppose first that the slope at A is changed by an amount $\delta\theta$, as shown in Fig. 263, while AB remains rigid. Then P will be displaced to P', the angle PAP' being $\delta\theta$.

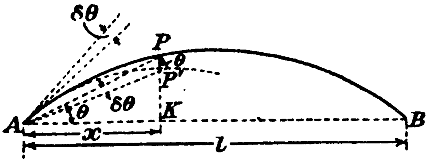

FIG. 263.

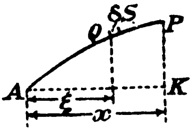

FIG. 264.

$$\text{The vertical displacement of } P = PP' \cos\theta = AP.\delta\theta.\cos\theta \quad = x.\delta\theta.$$

We must next consider the deflection at P due to changes of curvature between A and P. Let Q be an element δs between A and P, the bending moment at P being M. Then we have seen that the change in angle between the ends of δs is given by $d\psi = M.\delta s./EI$.

From similar reasoning to the above it can be seen that this will give a vertical deflection at P equal to $(x - \xi).\delta\psi$ (Fig. 264). Hence the total vertical deflection of P is

$$d = x.\delta\theta + \int_A^P (x - \xi).d\psi$$

$$= x.\delta\theta + \int_A^P \frac{(x - \xi)M}{EI}.ds \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

At B we must have $d = 0$,

$$\therefore \quad 0 = l.\delta\theta + \int_A^B \frac{(l - \xi)M}{EI}.ds \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii), we have, eliminating $\delta\theta$,

$$d = \int_A^P \frac{(x - \xi)M}{EI}.ds - \frac{x}{l}\int_A^B \frac{(l - \xi)M}{EI} ds$$

or

$$d = \frac{1}{E}\left\{x\int_B^P \frac{Mds}{I} - \int_A^P \frac{M\xi.ds}{I} + \frac{x}{l}\int_A^B \frac{M\xi}{I} ds\right\} \quad . \quad . \quad . \quad (9)$$

If the arch is fairly flat we can write $d\xi$ for ds; the expression for the deflection then becomes

$$d = \frac{1}{E}\left\{x\int_l^x \frac{Md\xi}{I} - \int_0^x \frac{M\xi d\xi}{I} + \frac{x}{l}\int_0^l \frac{M\xi d\xi}{I}\right\} \quad . \quad . \quad . \quad (10)$$

If the section is uniform I comes outside the integrals in both (9) and (10). It will be seen that, when we can make the approximation $ds = d\xi$, the shape of the beam does not directly influence the expression for d. Therefore the deflection at P is the same as the deflection of the corresponding point K of a horizontal beam AB subject to the same horizontal distribution of bending moment, the section of the hypothetical beam at K being the same as the normal section of the arch at P.

If the ends are built in, d can be expressed in the more convenient form

$$\frac{1}{E}\left\{x\int_A^P \frac{Mds}{I} - \int_A^P \frac{M\xi ds}{I}\right\}.$$

206. Temperature Stresses.—Except in the case of the three-hinged arch, a change of temperature produces changes of stress: if the temperature rises, the span of the arch would increase if it were not prevented by the abutments. The resistance offered by the abutments introduces extra forces and moments on any section; in general, there will be an extra horizontal thrust H', extra vertical reactions Y_1' and Y_2', and extra fixing moments M_1' and M_2'.

Let a = the coefficient of linear expansion,
t = the rise of temperature,
l = the span at the initial temperature.

Regarding A (Fig. 265) as fixed, the horizontal displacement of B away from A, if free, would be atl.

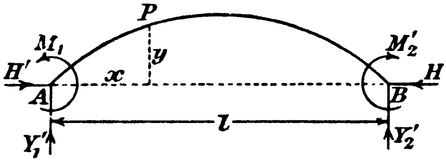

FIG. 265.

The extra bending moment at any point P will be

$$M' = H'y + M_1' - \frac{x}{l}(M_1' - M_2').$$

The horizontal displacement of B towards A will be (as in § 204)

$$\int_A^B \frac{M'yds}{EI} + \frac{H'l}{ES}$$

Since this must neutralize the displacement arising from the increase of temperature we must have

$$atl = \int_A^B \frac{M'yds}{EI} + \frac{H'l}{ES} \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Also, as in § 204, we must have

$$\int_A^B \frac{M'ds}{EI} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

$$\int_A^B \frac{M' \ (l - x) \ ds}{EI} = 0 \quad . \quad . \quad . \quad (iii)$$

These three equations will enable us to find H', M_1' and M_2'. The ordinary statical equations of equilibrium will then give Y_1' and Y_2'. The corresponding stresses can then be estimated.

When the arch is symmetrical $M_2' = M_1'$, and equations (ii) and (iii) become the same.

In the case of the two hinged arch M_1' and M_2' vanish, and Y_1' and Y_2' also vanish. The equations (ii) and (iii) are not true as they do not allow for the change of slope of the arch at the hinges. We have in this case

$$M' = H'y.$$

$$\therefore\ atl = \int_A^B \frac{H'y^2ds}{EI} + \frac{H'l}{ES}, \text{ from (i)}$$

$$\therefore\ H' = \frac{atlE}{\int_A^B \frac{y^2ds}{I} + \frac{l}{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If I is constant this becomes

$$H' = \frac{atlEI}{\int_A^B y^2ds + \frac{lI}{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It should be noticed that throughout this chapter we have found it convenient to reverse our convention with regard to the sign of bending moment; we here take a "hogging" bending moment as positive, whereas, in dealing with straight beams we took a "sagging" bending moment as positive. But in both cases a positive bending moment gives a positive increase of curvature, which is the reason for the change in dealing with arches.

Example.—A circular two-pin arch, with rigid abutments and a uniform cross section, has a span of 150 ft. and a rise of 10 ft. Find the horizontal thrust due to a concentrated load of 20 tons at the middle of each quarter span. If the depth of the section is 30″, find the greatest change of bending stress due to a rise of temperature of 20° C. $E = 13{,}000$ tons/in.^{2}, coeff. of expansion $= 11 \times 10^{-6}$. (Mech. Sc. Trip. B., 1919.)

Also draw the B.M. diagram for the arch at the original temperature.

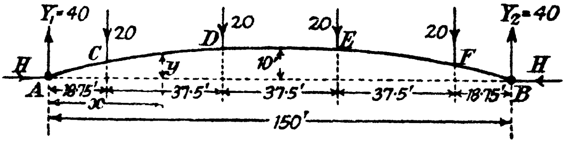

FIG. 266.

The arrangement is shown in Fig. 266.

This example could be worked out by direct integration, but the process leads to troublesome arithmetic. The rise of the arch being only $\frac{1}{15}$th of the span, we shall not make any appreciable error if we take $ds = dx$.

The bending moment diagram for the vertical loads only is easily drawn and is shown for half the arch in Fig. 267, where the centre-line of the arch is also drawn. Reading values of M_0 and y from these two curves, values of M_0y are found and plotted, as shown. A curve of y^2 is also drawn. From the areas of the curves we find, for half the arch,

$$-\int M_0ydx = 620{,}000 \text{ tons. ft.}^3$$

$$\int y^2dx = 4{,}000 \text{ ft.}^3$$

On account of symmetry it is sufficient to consider only one-half of the arch. Then, from equation (5) we have

$$H = \frac{620,000}{4,000} = 155 \text{ tons.}$$

The bending moment diagram for the arch can now be drawn from the equation

$$M = Hy + M_0.$$

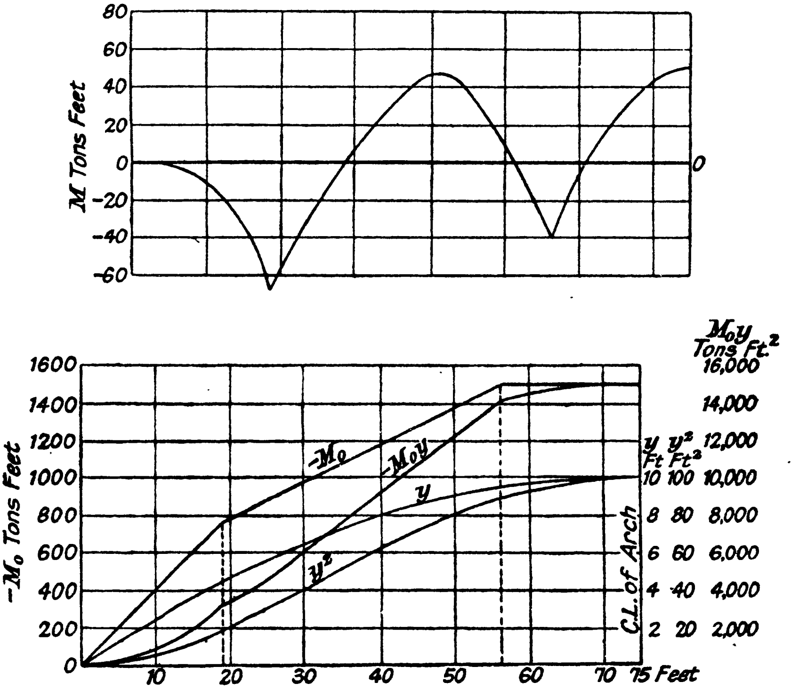

FIG. 267.

The curve has been plotted in the upper part of Fig. 267, on a horizontal base.

The additional thrust due to a rise of temperature of 20° C. is given by (12). We have

$$a = 11 \times 10^{-6}\,;\ t = 20^\circ\,;\ l = 150 \text{ ft.}$$
$$E = 13,000 \text{ tons/in.}^2$$

Hence the additional thrust H' is given by

$$H' = \frac{11 \times 10^{-6} \times 20 \times 150 \text{ ft.}}{8,000 \text{ ft.}^3} \times 13,000 \frac{\text{tons}}{\text{in.}^2} \times I$$

$$= 0{\cdot}0536 I \frac{\text{tons}}{\text{ft.}^2 \text{ in.}^2} = 0{\cdot}0536 \times 144\, I \text{ tons/ft.}^4$$

$$= 7{\cdot}72\, I \times \text{tons/ft.}^4$$

If H' is to be in tons I must be measured in ft.^{4}

The change of bending moment is $H'y$, which is a maximum at the crown.

The depth of the section is 30″, hence the greatest change of bending stress is

$$\frac{H' \text{ tons} \times 10 \text{ ft.} \times 15 \text{ ins.}}{I \text{ ft.}^4}$$

$$= 7{\cdot}72\, I \text{ ft.}^4 \frac{\text{tons}}{\text{ft.}^4} \times \frac{10 \text{ ft.} \times 15 \text{ ins.}}{I \text{ ft.}^4}$$

$$= 7{\cdot}72 \times 150 \frac{\text{tons} \times \text{ins.}}{\text{ft.}^3}$$

$$= \frac{7{\cdot}72 \times 150}{1{,}728} \frac{\text{tons}}{\text{in.}^2}$$

$$= 0{\cdot}67 \text{ tons/in.}^2$$

PARABOLIC ARCHES

207. Two-hinged Parabolic Arch with Uniformly Distributed Load.

—In Fig. 268 let ACB be a parabolic arch of uniform section hinged at A and B, and carrying a load uniformly distributed *along the span.*

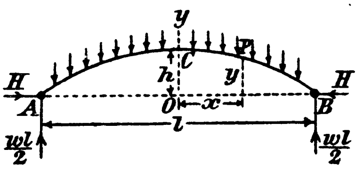

FIG. 268.

Let l = the span of the arch = AB.
h = the rise of the arch = OC.
w = the load per unit horizontal length.

If we take O as origin, the equation of the centre line of the arch is

$$y = \frac{4h}{l^2}\left(\frac{l^2}{4} - x^2\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

From this we have
$$\frac{dy}{dx} = -\frac{8hx}{l^2}.$$

Hence an element of arc at (x, y) is

$$ds = \sqrt{dy^2 + dx^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{64h^2x^2}{l^4}}\,.\,dx \quad \text{(ii)}$$

The bending moment at any point P due to the vertical forces only is

$$M_0 = \frac{w}{2}\left(x^2 - \frac{l^2}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Hence, from (i) and (iii),

$$\int_A^B M_0 y ds = -\frac{2wh}{l^2}\int_A^B\left(x^2 - \frac{l^2}{4}\right)^2 . ds \quad . \quad . \quad \text{(iv)}$$

and

$$\int_A^B y^2 ds = \frac{16h^2}{l^4}\int_A^B\left(\frac{l^2}{4} - x^2\right)^2 ds. \quad . \quad . \quad . \quad \text{(v)}$$

From (5) H is given by the division of (iv) by (v),

$$\therefore H = \frac{wl^2}{8h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The total bending moment at any point (x, y) on the centre line of the arch is

$$M = M_0 + Hy = \frac{w}{2}\left(x^2 - \frac{l^2}{4}\right) + \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) = 0$$

Thus the arch is everywhere free from bending moment.

208. Two-hinged Parabolic Arch with Concentrated Load.—In Fig. 269 ACB is a parabolic arch of uniform section and carrying a single vertical load W at a horizontal distance a from A.

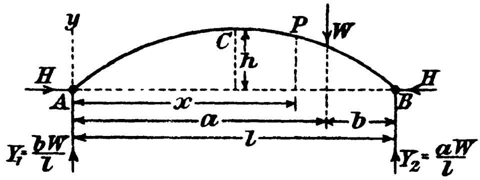

FIG. 269.

Taking the origin at A the equation of the centre line is

$$y = \frac{4h}{l^2}(lx - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

from which

$$\frac{dy}{dx} = \frac{4h}{l^2}(l - 2x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

The bending moment at P due to the vertical forces only is

$$M_0 = -\frac{aW}{l}(l - x) + W\{a - x\} \quad . \quad . \quad . \quad . \quad (\text{iii})$$

where the term { } is omitted if $x > a$.

We have then

$$\int_A^B M_0 y\,ds = \int_A^P M_0 y\,ds + \int_P^B M_0 y\,ds$$

$$= -\frac{4ahW}{l^3}\int_A^P x(l-x)^2 ds + \frac{4Wh}{l^2}\int_P^B x(l-x)\,(a-x)ds \quad . \quad (\text{iv})$$

Also

$$\int_A^B y^2 ds = \frac{16h^2}{l^4}\int_A^B x^2(l - x)^2 ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v})$$

The evaluation of the various integrals depends on whether we can take $\delta s = \delta x$ with sufficient accuracy or not, i.e. on whether $\frac{h}{l}$ is small enough for h^2/l^2 to be neglected. In this case the integrals are easily evaluated * and we find from (iv) and (v):

$$\int_A^B M_0 y dx = -\frac{ahlW}{3}\left(1 - 2\frac{a^2}{l^2} + \frac{a^3}{l^3}\right)$$

$$\int_A^B y^2 dx = \frac{8h^2 l}{15}$$

Hence

$$H = \frac{5}{8}\frac{a}{h}\left(1 - 2\frac{a^2}{l^2} + \frac{a^3}{l^3}\right)W \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The total bending moment at any point is then

$$M = M_0 + Hy$$

$$= \frac{5}{2}aW \cdot \frac{x}{l}\left(1 - \frac{x}{l}\right)\left(1 - 2\frac{a^2}{l^2} + \frac{a^3}{l^3}\right) - aW\left(1 - \frac{x}{l}\right) + W\{a - x\}$$

or

$$M = aW\left(1 - \frac{x}{l}\right)\left[\frac{5}{2} \cdot \frac{x}{l}\left(1 - 2\frac{a^2}{l^2} + \frac{a^3}{l^3}\right) - 1\right] + W\{a - x\} \quad . \quad . \quad (15)$$

the last term being omitted if $x > a$.

209. Built-in Parabolic Arch with Uniformly Distributed Load.—We have seen in § 207 that when the arch is hinged at both ends there is no bending moment anywhere, consequently no extra constraint is introduced if the ends are built-in: the fixing-moments will be zero and the horizontal thrust will have the same value, viz. $wl^2/8h$.

210. Built-in Parabolic Arch with Concentrated Load.—On account of the complexity of the work in the general case we shall limit ourselves to the treatment of the case when the rise is small and we can write $ds = dx$. The section is assumed uniform as before.

Referring to Fig. 269, let M_1 and M_2 be the fixing moments at A and B. We find M_1 and M_2 from the equations of § 204; the results will readily be found to be:

$$\left.\begin{aligned} M_1 &= aW\left(1 - \frac{9}{2}\frac{a}{l} + 8\frac{a^2}{l^2} - \frac{5}{2}\frac{a^3}{l^3}\right) \quad . \quad . \quad . \\ M_2 &= \frac{aW}{2}\left(-3\frac{a}{l} + 4\frac{a^2}{l^2} - 5\frac{a^3}{l^3}\right) \quad . \quad . \quad . \quad . \\ H &= \frac{15a^2W}{4hl}\left(1 - \frac{a}{l}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

---

* If the rise is not small in the above sense the best procedure is to write

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.dx = \sqrt{1 + \frac{16h^2}{l^2}\left(1 - \frac{2x}{l}\right)^2}.dx.$$

Expanding this by the Binomial Theorem and neglecting $\frac{h^4}{l^4}$ we get

$$ds = \left(1 + 8\frac{h^2}{l^2} - 32\frac{h^2}{l^2} \cdot \frac{x}{l} + 32\frac{h^2}{l^2} \cdot \frac{x^2}{l^2}\right)dx.$$

Substitute this value of ds and evaluate the integrals.

211. Piston Rings.—The two features which a piston ring should possess are: first, that the outside should be truly circular; and, secondly, that it should exert a uniform pressure on the walls of the cylinder. A common method of manufacture is to turn a ring of uniform section, whose outside diameter is slightly larger than the bore of the cylinder, and then to cut out a small piece to allow the ring to be sprung into the cylinder; we shall see presently that this is an inferior method as the ring will not be circular when in the cylinder, and will not bear on the cylinder uniformly round the circumference.

The problem to be considered is this: an incomplete ring of rectangular cross section, with a constant depth at right angles to the plane of the ring, is to be circular on the outside in the unstrained state, and circular when it is loaded with a uniform radial pressure; how must the radial thickness of the ring vary?

Let R_0 = the initial radius of the outside.
,, w = the radial pressure per unit length of arc.
,, R = the radius of the outside under the action of w.

In Fig. 270, AOC is the axis of symmetry of the ring, the thickest part being at C, and the thinnest at A where the ring is cut. Referring to the right-hand part of the figure, the bending moment at B due to the pressure on an element $Rd\varphi$ is

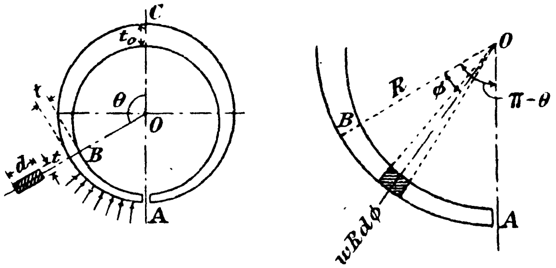

FIG. 270.

$$w.Rd\varphi \times R \sin \varphi.$$

Hence the total bending moment at B is

$$M = wR^2 \int_0^{\pi-\theta} \sin \varphi . d\varphi = wR^2(1 + \cos \theta)$$

$$\therefore \frac{1}{R} - \frac{1}{R_0} = \frac{M}{EI} = \frac{wR^2(1 + \cos \theta)}{EI} \quad . \quad . \quad . \quad (i)$$

Since R and R_0 are constant, the right-hand side of (i) must be constant.

Let t_0 = the thickness of the ring at C,
t = ,, ,, ,, ,, ,, B,
d = the depth of the ring perpendicular to the plane of the paper.

Then $I = \frac{t^3 d}{12}$. At C we have $\theta = 0$ and $I = \frac{t_0^3 d}{12}$. Therefore the condition that the right-hand side of (i) is constant requires

$$\frac{12wR^2(1 + \cos\theta)}{Et^3 d} = \frac{24wR^2}{Et_0^3 d}$$

$$\therefore \frac{t^3}{t_0^3} = \tfrac{1}{2}(1 + \cos\theta),$$

or

$$\frac{t}{t_0} = \sqrt[3]{\frac{1}{2}(1 + \cos\theta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This gives the law of thickness of the ring.

The maximum bending moment occurs at C, and is equal to $2wR^2$. If p = the maximum permissible stress, we must have

$$p = \frac{12wR^2}{t_0^2 d}$$

$$\therefore t_0 = 2\sqrt{3}.R\sqrt{\frac{w}{pd}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

R being the radius of the cylinder.

The initial radius of the outside of the ring, R_0, is given by

$$\frac{1}{R} - \frac{1}{R_0} = \frac{M}{EI} = \frac{24wR^2}{Et_0^3 d}, \text{ at } C.$$

Substituting the above value of t_0 this gives

$$\frac{1}{R_0} = \frac{1}{R}\left[1 - \sqrt{\frac{p^3 d}{3E^2 w}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Thus R_0 and t_0 are both expressed in terms of the fundamental quantities R, p, d, E, w.

It will be evident that if t is made constant, the condition of constant curvature after springing into the cylinder will not be fulfilled, with the consequence that the pressure between the ring and the cylinder will not be uniform round the circumference.

EXAMPLES XVII

1. Fig. 271 shows the centre line of an arch pin-jointed at A, B, and C. Draw the line of pressure * for the given system of loading and determining the Bending Moment at the point D. (Birmingham, 1911.)

2. The span of an arch is 160 ft. and the rise 20 ft. The arch is hinged at the abutments and at the crown. It carries loads of 20 and 40 tons concentrated at points distant 30 ft. and 60 ft. from the two ends respectively. Determine the horizontal and vertical components of the three hinge reactions. If the height of the centre line of the arch at points 40 ft. from the abutments is 15 ft., determine the bending moments in the arch at these two sections. (Intercoll. Exam., Cambridge, 1911.)

3. A three-hinged metal arch is parabolic in form, the centre hinge being

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* See note on p. 287.

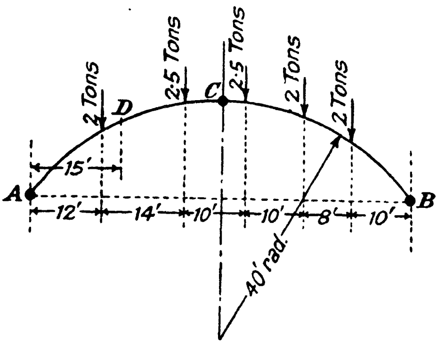

FIG. 271.

the vertex of the parabola. The dead load on the arch is uniformly distributed horizontally, and in addition there is a concentrated load W acting at a distance x, measured horizontally, from the vertex. Show that the maximum bending moment set up in the arch is $Wx(l^2 - x^2)/2l^2$, where $2l$ is the span between the lower hinges. (Mech. Sc. Trip., 1906.)

4. The centre line of an arch is parabolic in form. The arch is hinged at the crown and at the springings. The span is 140 ft. and the rise is 30 ft. Vertical loads of 20, 40, 60, 60, 40, 20 tons act at points distant 20, 40, 60, 80, 100, 120 ft. respectively from the left hinge. Draw the line of resultant thrust in the arch and determine the horizontal thrust at the hinges. (Intercoll. Exam., Cambridge, 1910.)

5. A parabolic three-hinged arch having a span $2l$, and a rise h, carries a dead load W uniformly distributed, and also a live load P concentrated at a point distant x from the centre. Prove that the horizontal abutment thrust is $\frac{1}{4h}\{Wl + 2P(l - x)\}$. Calculate the B.M. at the section where the load P is applied, and deduce that it has its maximum value when $x = l/\sqrt{3}$, and that this maximum value is $lP/3\sqrt{3}$. (Mech. Sc. Trip., 1922.)

6. A three-hinged arch is parabolic in form. The span is 120 ft. and the rise 15 ft. The left half is subjected to a uniformly distributed load w tons per ft. run. Prove that M, the "sagging" bending-moment in the arch, at a distance x to the right of the left hinge can be expressed in the form

$$M = w\left[\frac{x}{4}(60 - x) + \frac{1}{2}\{x - 60\}^2\right]$$

where the term $\{\}$ is omitted if $x < 60$.

Prove that the deflection due to the change in curvature in the arch is given by

$$y = \frac{w}{48EI}[x(x - 60)(x^2 - 60x - 3{,}600) - 2\{x - 60\}^4],$$

and, assuming that the thrust in the arch has the constant value $60w$, show that the additional deflection due to the shortening of the two halves of the arch is given by

$$y = \frac{17}{ES}[x - 2\{x - 60\}].$$

S is the area and I the moment of inertia of the cross section. (Mech. Sc. Trip. B., 1923.)

7. One span of a railway bridge, built of circular arched ribs of cast iron,

is 62 ft. span with a vertical rise of 8 ft. After erection, it was found that, due to settlement, the span had increased, and it was estimated that this reduced the horizontal thrust, per rib, at the abutments by 15·9 tons. Find the change in the stress at the top and bottom of the middle cross section, which is of I form with top flange $10\frac{1}{2}'' \times 1\frac{3}{8}''$, web $27\frac{1}{4}'' \times \frac{1}{2}''$, and bottom flange $12'' \times 1\frac{3}{8}''$. (Intercoll. Exam., Cambridge, 1921.)

8. The central line of a two-pin arch has a span of 150 ft. At a distance of x feet measured from one pin, along the horizontal line joining the pins, the ordinate of the centre line is y inches, and the B.M. due to the vertical load only is M_0 tons. ft. The values of these at equal intervals along the span given in the table :

x .	7·5	22·5	37·5	52·5	67·5	82·5	97·5	112·5	127·5	142·5
y .	26	72	106	128	136	136	128	106	72	26
M_0 .	350	980	1,490	1,820	1,900	1,740	1,500	1,160	740	280

Find the value of the horizontal thrust, and tabulate the values of the B.M. in the rib and the ordinates of the linear arch, at the points given. Also plot the B.M. curve, the centre line, and the linear arch.

If the temperature rise 45° C. above the setting-up temperature, what horizontal thrust is produced, if the coeff. of expansion be $1{\cdot}25 \times 10^{-5}$? Mech. Sc. Trip. B., 1909.)

9. A two-pin arch with hinges at the same level is subjected to a given distribution of vertical load. If the form of the arch is similar to the form of the B.M. diagram for a horizontal beam, supported at its ends, having the same span and load distribution as the arch, and if the horizontal thrust is suitably adjusted, prove that the arch can be rendered entirely free from bending moment.

To specify the form and load distribution of such an arch, the point midway between hinges is taken as origin, x is measured horizontally and y is measured vertically upwards. The span is $2l$, and the rise at the centre is h. The load distribution is

$$w = w_0 + \frac{w_1 - w_0}{l^2}x^2.$$

Prove that for freedom from bending moment the form of the arch is given by

$$y = \frac{(l^2 - x^2)h}{(5w_0 + w_1)l^4}\{6l^2w_0 + (l^2 + x^2)(w_1 - w_0)\},$$

and the horizontal thrust must be adjusted to have the value

$$(5w_0 + w_1)l^2/12h.$$

(Mech. Sc. Trip B., 1923.)

10. A cast-iron piston ring $\frac{1}{4}''$ wide works in a cylinder $5''$ diameter. The ring is so shaped that it exerts a uniform pressure of $2\frac{1}{2}$ lbs./in.^{2} against the walls of the cylinder. Show by means of diagrams, plotted on a horizontal base, the values of the bending moment and shearing force at each section of the ring.

If the tensile stress in the iron is limited to 3,000 lbs./in.^{2}, show that the radial thickness of the ring at the section diametrically opposite the slit is $\frac{1}{4}''$ approximately. (Mech. Sc. Trip., 1907.)

11. If the radial thickness of a piston ring be constant and equal to t, show that the initial radius of curvature, R_0, of the ring should be made according to the formula

$$R_0 = \frac{Et^3R}{Et^3 - 24wR^2\cos^2\frac{\theta}{2}}$$

the notation being the same as in § 211.

12. A piston ring of cast iron is finished to a circle 12½" outside diameter, its section being 1" deep and ¾" thick. A piece 1⅝" long is then cut out of the ring and the ring is forced into a cylinder 12" bore. Find the distribution of pressure between the ring and the cylinder walls, taking $E = 8{,}000$ tons/in.^{2} (Mech. Sc. Trip., 1905.)

13. A two-hinge arch is of circular form, the rise of the centre line being h, and the span l. The load is w (constant) per unit horizontal length. Prove that the horizontal thrust at the abutments is given by

$$H = \frac{wR}{8} \cdot \frac{\frac{3}{2}\sin a + a\cos a + a\cos 3a - \frac{7}{6}\sin 3a}{a - \frac{3}{4}\sin 2a + \frac{a}{2}\cos 2a},$$

where $2a$ is the angle subtended by the span at the centre of the arc, the radius of which is R.

14. If the arch of Ex. 13 be built-in at the ends, and M_1 denote the fixing moments at either end, show that H and M_1 are given by

$$HR(\sin a - a\cos a) + aM_1 = \frac{wR^4}{4}\left(\frac{1}{2}\sin 2a - a\cos a\right)$$

$$HR\left(a - \frac{3}{4}\sin 2a + \frac{a}{2}\cos 2a\right) + (\sin a - a\cos a)M_1$$
$$= \frac{wR^2}{8}\left(\frac{3}{2}\sin a + a\cos a + a\cos 3a - \frac{7}{6}\sin 3a\right)$$

15. In a two-pin arch bridge, the two supporting steel ribs are uniform in section, parabolic in form and rise to a height of 15 ft. at the centre of a span of 120 ft. Calculate the decrease in the horizontal thrust on the abutments if the distance between them increases by 1 in. Determine also the corresponding alteration in the bending moment at the centre of a rib.

The cross-section of a rib has a moment of inertia $0{\cdot}75 \times 12^4$ in.^{4}, and E for the material is 30×10^6 lbs. per square in.

It may be assumed that the length of the rib, measured along its centre line, remains unaltered. (Mecn. Sc. Trip., B., 1924.)

16. A span of 50 ft. is bridged by a parabolic arched rib, 10 ft. high at the centre. If the load be 2 tons per ft. run of span, find the horizontal thrust at the abutments.

If an additional load of 1 ton per ft. run be distributed along one half of the span, from one end to the centre, find the value of the horizontal thrust and show how to determine the bending moment at any section of the rib. (Intercoll. Exam., Cambridge, 1905.)

Note on Ex. 1, *p.* 284.—On AB as base (Fig. 271) draw the B.M. diagram for a horizontal beam AB, carrying the same vertical loads as the arch, and draw it to such a scale that it passes through the hinge C. This B.M. diagram is then called the line of pressure, or line of resultant thrust. Its slope at any point P gives the direction of the resultant force acting on a cross section of the arch at a point Q, vertically below or above P. Furthermore, the bending moment at Q is $H.PQ$, where H is the horizontal thrust at the abutments. We leave the proof of this as an exercise for the student.

CHAPTER XVIII

STRUTS OF UNIFORM SECTION

212. Statement of the Problem.—A member, of a structure or machine, which is subjected only to end thrust is usually called a *strut*, and the consideration of the strength of struts is of great importance in all branches of engineering. On account of various imperfections which must occur in practice, we can say that the line of action of the resultant thrust will never coincide with the geometrical axis of the strut. The result of this is an inevitable flexure of the strut. Thus, in Fig. 272, AB is a member subjected to a thrust P, the line of action of which is distant h from the axis of the member, so that there is an applied bending moment Ph. The effect of this will be to cause the axis AB to deflect in the manner shown by the dotted line, so that, if δ be the deflection of the axis at any point, the bending moment is increased to $P(\delta + h)$.

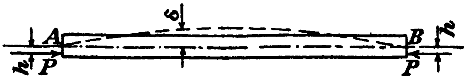

FIG. 272.

This extra bending moment will cause a greater deflection, which in turn will increase the bending moment, and the effect is cumulative until equilibrium is established. The problem we have to consider is the determination of the value of the bending moment at any point in the member.

When a strut is very short and stumpy, the effects of possible deflection are of no importance, but when the member is long and slender these effects are the prime consideration. We have dealt with strength of members of the former type in Chapter XII; in the present chapter we shall consider members of the latter kind.

THE EULER THEORY OF STRUTS

213. Strut Pin-jointed at Both Ends.—A *perfectly* straight rod of uniform cross section and homogeneous structure is subjected to an end thrust P, the line of action of which lies truly along the axis of the rod in its unstrained position. We seek to discover what value of the end thrust will be able to hold the rod with its axis deformed away from the straight. The ends of the strut are fixed in position but quite free to change their direction if the strut bends.

In Fig. 273, OA is the unstrained axis of the rod. If possible let forces P, in the directions OA and AO, hold the rod in the deflected

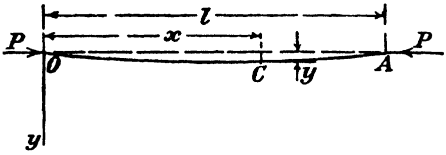

FIG. 273.

position OCA, the deflection at C being y. Then the bending moment at C is Py, and for small deflections we have in the usual notation,

$$-EI\frac{d^2y}{dx^2}=Py,$$

where I is the smallest principal moment of inertia of the cross section.

It is convenient to write this in the form

$$\frac{d^2y}{dx^2}+a^2y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where

$$a^2=\frac{P}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The solution of (i) is

$$y=A\cos ax+B\sin ax. \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

where A and B are constants.

Now, y must vanish with x, and therefore $A=0$; y must also vanish when $x=l$, which requires

$$B\sin al=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

This is satisfied if $B=0$, but then there is no deflection at all; if B is not zero we must have

$$\sin al=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

that is,

$$al=0,\ \pi,\ 2\pi\ \ldots\ldots$$

or

$$a^2=0,\ \frac{\pi^2}{l^2},\ \frac{4\pi^2}{l^2}\ \ldots\ldots$$

or

$$P=0,\ \frac{\pi^2EI}{l^2},\ \frac{4\pi^2EI}{l^2}\ \ldots\ldots$$

The value $P=0$ obviously need not be considered.

The value $P=\pi^2EI/l^2$ will satisfy all the conditions of the problem, so that this load will sustain the strut in a deflected state, and we need not consider the higher values of P.

We must now consider precisely what has been established: if P

is less than π^2EI/l^2 the rod will remain straight and the only stresses will be those due to direct compression. If $P = \pi^2EI/l^2$, it will suffice to maintain any deflection which may be given to the rod, and the above equations fail to tell us the deflection under a load equal to or exceeding this value. This is not to say that the deflection will become indefinite. We must recognize that we have here another instance of that universal quality of the world whereby a continuous quantitative change results in a sudden qualitative change. We have already found an instance of this in the case of yield. If we have a strut under load, it remains perfectly stiff and stable under increasing load until a certain critical load is reached. At once the strut becomes flabby, and will stay deflected if pushed aside. The first phase has ended, and, by a sudden transition, a new phase begins. The theory which was adequate for the first phase will not suffice for the second phase, which differs from the first in a qualitative way. Any further increase in load will result in further deflection,* with increasing stresses, until the material yields under the compressive stress, and the second phase is ended.

It is evident that a strut which has become *unstable* or flabby in the manner indicated has ceased to have any value as a structural member. Consequently, the value of P found above is taken as the limiting load that a *perfectly straight* strut could bear *when the line of action of the thrust coincides with the axis of the strut.* The first deflection referred to above, when P reaches this value, is called buckling or crippling, and the value

$$\mathrm{P_e} = \frac{\pi^2\mathrm{EI}}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is called the Euler buckling load.

214. Limitations of Euler's Formula.—The limitations of this formula are of the utmost importance, and we shall now deal with them. In the first place, Euler's theory assumes

(i) that the axis of the strut is perfectly straight when unloaded;
(ii) that the line of thrust coincides exactly with the unstrained axis of the strut.

Neither of these assumptions will usually be realized in practice, so that the theory refers to an ideal strut and not a real strut, and the effect of the imperfections is to convert the strut problem from a problem in stability into a problem of stresses, of which the Euler theory takes no account.

We must then consider the questions: When can we apply the Euler formula to a real strut? And, when the Euler formula is inapplicable, how can we estimate the strength of a strut?

---

* When the deflection is small compared with $l/2$, its amount is

$$4\left\{\frac{l}{\pi} - \left(\frac{EI}{P}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left(\frac{EI}{P}\right)^{\frac{1}{4}}.$$

(Love, *Theory of Elasticity*, 3rd Ed., p. 412.)

If S denote the area of the cross section, and k the least radius of gyration, (1) can be written

$$p_e = \frac{P_e}{S} = \frac{\pi^2 E}{(l/k)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that p_e is the stress corresponding with the Euler failing load. According to this formula the stress required to cause failure will increase indefinitely as l/k is decreased, as shown by the curve ABC in Fig. 274. But evidently failure *must* take place when the compressive stress is equal to the stress of the material at the Yield Point. Let OY represent the yield stress of the material, then it is clear that the Euler formula cannot possibly apply if l/k is less than OD.

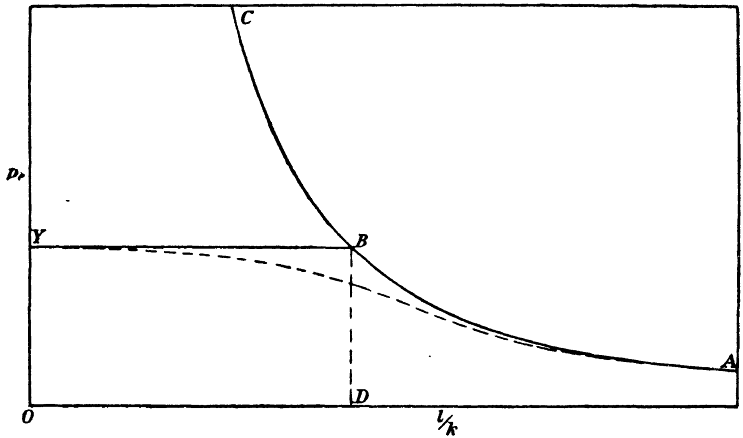

FIG. 274.

Thus, suppose the yield point is 20 tons/in.^{2}, with E = 13,500 tons/in.^{2} we must have

$$\frac{l^2}{k^2} \not< \frac{\pi^2 \times 13{,}500}{20},$$

i.e. $l/k > 81{\cdot}5$, if the Euler formula is to be applicable. For a solid strut of circular cross section this corresponds with a length of about 20 diameters.

So far, then, we should expect the failing load to be given by the graph ABY, but we have not allowed for the fact that assumptions (i) and (ii) are not usually realized in practice. When this allowance is made we shall find that the law is more nearly expressed by a curve such as that shown dotted.

A. Robertson has shown,* by experiments in which particular care was taken to obtain perfectly axial loading, that Euler's formula is correct, for really axially loaded struts, for all lengths down to those giving stresses equal to the elastic limit of the material. When the elastic limit is exceeded certain modifications must be made, even with truly axial loading. The necessary theoretical modification has been given by

* Report to Royal Society Sub-Committee on Struts.

Southwell,* and the correctness of this has been confirmed experimentally by Robertson.

We must now investigate the effects of non-axial loading and initial crookedness in the strut (§ 215 *et seq.*).

Example 1.—The cross section of an aeroplane inter-plane strut is such that $I = 0{\cdot}178b^4$, where b is the thickness. Calculate the thickness of a parallel strut, of length 80″ between pin-centres, to take an end load of 2,000 lbs. with a factor of safety of 5, assuming it can be treated by the Euler formula, and taking $E = 1{\cdot}6 \times 10^6$ lbs./in.^{2}

We must design for a thrust of 5 × 2,000 lbs. = 10,000 lbs.

The Euler formula gives

$$10{,}000 \text{ lbs.} = \frac{\pi^2 \times 1{\cdot}6 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}178b^4}{6{,}400 \text{ in.}^2}$$

$$\therefore\ b^4 = \frac{64 \text{ in.}^4 \times \text{lbs.}}{\pi^2 \times 1{\cdot}6 \text{ lbs.} \times 0{\cdot}178} = 22{\cdot}8 \text{ in.}^4$$

whence

$$b = 2{\cdot}19''.$$

Example 2.—What thrust will a round steel rod take without buckling if it is ½″ diameter, 8′ 0″ long, perfectly straight, and pin-jointed at the ends, the load being applied exactly along the axis of the rod ?

$$I = \frac{\pi \times (0{\cdot}5 \text{ ins.})^4}{64} = \frac{\pi}{1{,}024}\text{ins.}^4$$

$$l = 96''.$$

Taking $E = 30 \times 10^6$ lbs./in.^{2}, we have

$$P = \frac{\pi^2 \times 30 \times 10^6 \text{ lbs./in.}^2}{96 \times 96 \text{ in.}^2} \times \frac{\pi}{1{,}024}\text{ins.}^4$$

$$= 98{\cdot}5 \text{ lbs.}$$

215. Strut with Eccentric End-Load.—In Fig. 275 let ON be the line of action of P, at a distance h from the unstrained axis of the strut AB. Let y be the deflection of any point C on the axis, measured from the line of action of P. Then, as before, the bending moment at C is Py, and the equation for y will be

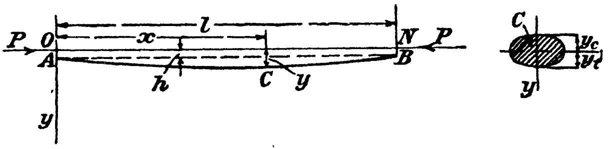

FIG. 275.

$$y = A \cos ax + B \sin ax \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where $\qquad a^2 = P/EI.$

The conditions to be satisfied are $y = h$ when $x = 0$ and when $x = l$. Hence we must have

$$h = A$$
$$h = A \cos al + B \sin al,$$

whence

$$B = h(1 - \cos al)/\sin al$$

---

* *Engineering*, Aug. 23, 1912.

Substituting for A and B in (i) gives

$$y = h \cos ax + h\left(\frac{1 - \cos al}{\sin al}\right) \sin ax.$$

$$= h(\cos ax + \tan \frac{al}{2} \sin ax)$$

$$= h \cos a\left(\frac{l}{2} - x\right) \sec \frac{al}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

At the centre $x = l/2$ and y is a maximum ; denoting the maximum deflection by y_0 we have

$$y_0 = h \sec \frac{al}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Then the maximum bending moment is

$$\mathbf{M = Ph \sec \frac{\alpha l}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If y_c and y_t denote the distances from the centroid of the section to the compression and tension faces, as shown in Fig. 275, the maximum compressive and tensile stresses set up by M will be

$$\frac{y_c \,.\, Ph \sec \frac{al}{2}}{I} \text{ and } \frac{y_t \,.\, Ph \sec \frac{al}{2}}{I}$$

If S denote the area of the cross section, the direct compressive stress will be P/S. Hence the maximum compressive stress in the strut will be

$$p_c = \frac{P}{S} + \frac{Phy_c}{I} \sec \frac{al}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If f denote the maximum permissible compressive stress in the material, we have

$$P = \frac{f}{\frac{1}{S} + \frac{hy_c}{I} \sec \frac{al}{2}}$$

or, writing $I = Sk^2$,

$$\mathbf{P} = \frac{\mathbf{fS}}{\mathbf{1} + \frac{\mathbf{hy_c}}{\mathbf{k^2}} \sec \frac{\boldsymbol{\alpha l}}{\mathbf{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This equation is inconvenient for finding P, since a itself contains P, and in arithmetical cases a solution can only be found by trial. In such cases the following method will be found very useful.

Let $P_e = \frac{\pi^2 EI}{l^2}$, i.e. the Euler failing load of the same strut axially loaded. Then we can write

$$\boldsymbol{\theta} = \frac{\boldsymbol{\alpha l}}{\mathbf{2}} = \frac{l}{2}\sqrt{\frac{P}{EI}} = \frac{\pi}{2}\sqrt{\frac{l^2 P}{\pi^2 EI}} = \frac{\pi}{\mathbf{2}}\sqrt{\frac{\mathbf{P}}{\mathbf{P_e}}} \quad . \quad . \quad . \quad (6)$$

Now, an extremely close approximation to the value of sec θ, for all values of θ between 0 and $\pi/2$, is *

$$\sec\theta \doteq \frac{1+\dfrac{4\theta^2}{\pi^2}\times 0{\cdot}26}{1-\dfrac{4\theta^2}{\pi^2}} = \frac{P_e+0{\cdot}26\,P}{P_e-P}$$

Substituting this value of sec θ, i.e. sec $\frac{al}{2}$, in (5) and rearranging the equation gives the following quadratic for P:

$$P^2\left(1-0{\cdot}26\,\frac{hy_c}{k^2}\right)-P\left\{P_e\left(1+\frac{hy_c}{k^2}\right)+f.S\right\}+fS\,.\,P_e=0\ . \qquad (7)$$

which is readily solved in any given case.

If $h = 0$, this reduces to

$$P^2-P(P_e+fS)+fSP_e=0\ .\ .\ .\ .\ .\ . \qquad (8)$$

which gives $P = P_e$ or fS. The former is the ordinary Euler value, the latter refers to a strut so short that bending is negligible : it is the load which will cause the direct stress to reach the limiting value f.

It will be seen from (ii) that if $al = \pi$, i.e. $P = P_e$, y becomes infinite, i.e. the strut will buckle, but this is of no practical interest since a lower value of P will always cause stress failure.

Example.—The 3″ × 3″ angle shown in Fig. 276 has a sectional area 2·75 in.² $AG = 1{\cdot}32''$, $OG = 0{\cdot}25''$. The radius of gyration about xx is 0·573″. The angle is acting as a strut having an unsupported length of 60″. The actions at the ends amount to a longitudinal thrust of magnitude P acting through O. G is the centroid of the section. Consider the mid-cross-section of the strut, and show that, if the compressive stress is not to exceed 8 tons/in.², P must not exceed 9 tons approximately. (Mech. Sc. Trip. B, 1923.)

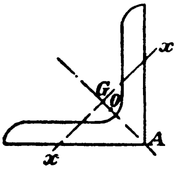

FIG. 276.

We have

$S = 2{\cdot}75$ in.² ; $l = 60''$.

$k = 0{\cdot}573''$; $k^2 = 0{\cdot}328$ in.²

$EI = 14{,}000 \times 2{\cdot}75 \times 0{\cdot}328 = 12{,}600$ tons. in.²

$$a^2 = \frac{P}{12{,}600}\text{ins.}^{-2}$$

$$a^2l^2 = \frac{3{,}600P}{12{,}600} = 0{\cdot}286P$$

$$\therefore\ al = 0{\cdot}536\sqrt{P}$$

$$\frac{al}{2} = 0{\cdot}268\sqrt{P}$$

---

* This expression was given by H. A. Webb in connection with another problem and is correct within 0·5 per cent., which is considerably more accurate than that given by Perry, particularly as $\theta \longrightarrow \frac{\pi}{2}$.

In the notation of § 215 we have

$$h = 0{\cdot}25'' ; \quad y_c = 1{\cdot}32''.$$

$$f = \frac{P}{S} + \frac{Phy_c}{Sk^2} \sec. \frac{al}{2}$$

$$= \frac{P}{2{\cdot}75}\left\{1 + \frac{0{\cdot}25 \times 1{\cdot}32}{0{\cdot}328} \sec. (\cdot 268\sqrt{P})\right\} \text{ tons/in.}^2$$

$$= \frac{P}{2{\cdot}75}\{1 + 1{\cdot}005 \sec. (\cdot 268\sqrt{P})\}\text{tons/in.}^2$$

When $P = 9$ tons.

$$f = 3{\cdot}28\ (1 + 1{\cdot}005 \sec. 0{\cdot}804)$$
$$= 8 \text{ tons/in.}^2 \text{ approximately.}$$

Alternatively, using equation (7), we have $\frac{hy_c}{k^2} = 1{\cdot}005$,

$$P_e = \frac{\pi^2 \times 12{,}600}{3{,}600} = 34{\cdot}5 \text{ tons} ; \ fS = 22 \text{ tons.}$$

Hence P is given by $0{\cdot}74P^2 - 91P + 759 = 0$

or $$P^2 - 123P \times 1{,}026 = 0$$

The roots of this are $P = 9$ or -114, the latter of which is out of the question. Hence the value of P required is $P = 9$ tons, as before.

216. The Effect of Initial Crookedness.—Ayrton and Perry published in 1886 * an analysis on the following lines, from which they concluded that an initial eccentricity of loading could be taken as equivalent to an initial curvature of the central line.

Referring to Fig. 273, let y_0 denote the distance of any point on the axis of the strut from the line of action of the load, in the unstrained state. Then y_0 can be expressed as a function of x. For simplicity and convenience let us take the initial shape of the axis to be given by

$$y_0 = c \sin \frac{\pi x}{l} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

so that c represents the maximum distance of the axis from the line OA in the unstrained state. Then in the strained state we must have

$$EI\left(\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2}\right) = -Py \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) we have

$$\frac{d^2y_0}{dx^2} = -\frac{c\pi^2}{l^2} \sin \frac{\pi x}{l}.$$

Substituting in (ii), and writing $a^2 = \frac{P}{EI}$ as before, gives

$$\frac{d^2y}{dx^2} + a^2y = -\frac{c\pi^2}{l^2} \sin \frac{\pi x}{l}.$$

The solution of this is

$$y = A \cos ax + B \sin ax + \frac{c\pi^2}{\pi^2 - a^2l^2} \sin \frac{\pi x}{l}.$$

---

* *The Engineer*, Dec. 10, 1886.

We must have $y = 0$ when $x = 0$; therefore $A = 0$. We must also have $y = 0$ when $x = l$, and therefore $B \sin al = 0$. This requires either $B = 0$ or $al = \pi$. In the latter case y becomes infinite. Let us then consider the state of affairs when $al < \pi$, so that $B = 0$. Then

$$y = \frac{c\pi^2}{\pi^2 - a^2l^2} \sin \frac{\pi x}{l}.$$

The maximum deflection is

$$y_{max} = \frac{c\pi^2}{\pi^2 - a^2l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now, we have seen in § 215 that, provided $al < \pi$, the maximum deflection of a strut with an initial eccentricity of loading h is given by

$$y_{max} = h \sec \frac{al}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

By comparing the values of the coefficients of c and h in (iii) and (9) Perry and Ayrton came to the conclusion mentioned above.

If we write, as before (§ 215), $al = \pi\sqrt{\frac{P}{P_e}}$, (9) shows that the effect of the end load is to make the central deflection

$$\frac{P_e}{P_e - P}$$

times its initial value. Also, taking the very good approximation to $\sec \frac{al}{2}$ given in § 215, we see that if there is originally an eccentricity h, the central deflection becomes

$$\frac{P_e + 0{\cdot}26\,P}{P_e - P}$$

times the eccentricity. Hence we can write

$$\frac{\text{max. B.M. per unit initial deflection}}{\text{max. B.M. per unit eccentricity}} = \frac{P_e}{P_e + 0{\cdot}26\,P} = \frac{1}{1 + 0{\cdot}26\frac{P}{P_e}}.$$

The maximum value that this can have is $\frac{1}{1{\cdot}26}$, or roughly $\frac{4}{5}$, in the worst case when P approaches P_e. Thus we can regard an initial crookedness as equivalent to a certain initial eccentricity of loading, and vice versa.

If the maximum deviation of the strut from the straight be c_1, and the eccentricity of loading be h_1, then we can take an "equivalent eccentricity" h given by

$$h = h_1 + c_1\Big/\left(1 + 0{\cdot}26\frac{P}{P_e}\right)$$

or

$$h = h_1 + \tfrac{4}{5}c_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

approximately.

Alternatively we might take an "equivalent initial curvature."

$$c = c_1 + \tfrac{5}{4}h_1 \text{ (approximately)} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We have arrived at the above results by assuming the initial shape of the axis of the strut to have the form of a sine curve, but actually it makes very little difference whether we take a sine curve, a circular arc or a parabolic arc, as the following figures * will show :

P/P_e	Max. deflection/initial deflection. Parabola or circle.	Sine curve.
0	1·0	1·0
0·4	1·687	1·667
0·6	2·546	2·500
0·8	5·126	5·000
0·9	10·29	10·0
1·0	∞	∞

217. Strut with One End Encastré, the other End being Free to Rotate.—In Fig. 277 the end O of the strut OA is encastré in the direction OA ; the end A is constrained to remain on the line OA but is free to rotate round A. A bending moment must be incurred at the encastrément, and this involves the presence of a force R at A at right angles to OA to maintain equilibrium, with the corresponding reaction at O. The equation for small deflections is then

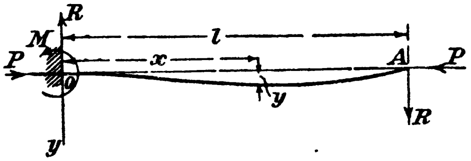

FIG. 277.

$$-EI\frac{d^2y}{dx^2} = Py - R(l - x)$$

or

$$\frac{d^2y}{dx^2} + a^2y = a^2\frac{R}{P}(l - x)$$

where $a^2 = P/EI$ as before. The solution of this equation is

$$y = A\cos ax + B\sin ax + \frac{R}{P}(l - x) \quad . \quad . \quad . \quad \text{(i)}$$

where A and B are constants to be determined. We must have $y = 0$ when $x = 0$ or l, hence

$$A + \frac{Rl}{P} = 0$$

$$A\cos al + B\sin al = 0.$$

These give

$$A = -\frac{Rl}{P},\ B = \frac{Rl}{P}\cot al \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

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* E. H. Salmon, *Columns*, p. 29.

We must also have $\frac{dy}{dx} = 0$ when $x = 0$. From (i)

$$\frac{dy}{dx} = -aA \sin ax + aB \cos ax - \frac{R}{P}$$

Therefore we must have

$$aB = \frac{R}{P}$$

or

$$\frac{alR}{P} \cot al = \frac{R}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

i.e.

$$al = \tan al \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iv)$$

This is the condition that the thrust P may be able to hold the strut in a deflected position. At the same time the force R, and therefore A and B, become indeterminate. The smallest root of (iv) which is greater than zero is

$$al = 4{\cdot}49$$

or $a^2l^2 = 20 = 2\pi^2$ approximately.

Hence, in this case the smallest thrust which can hold the strut deflected is given approximately by

$$\mathbf{P} = \frac{\mathbf{2\pi^2EI}}{\boldsymbol{l^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It will be seen that the critical load is double that of the strut with both ends free to rotate.

218. Strut with One End Encastré, the Other End being Free to take up any Position (Fig. 278).—Let OA be the axis of the strut in the undeflected state. Suppose A moves to A' under the action of the thrust P. Then a fixing moment $M = P \times AA'$ is brought into play at O. Let $AA' = h$, then the deflection is given by

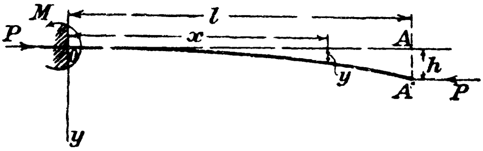

FIG. 278.

$$-EI\frac{d^2y}{dx^2} = P(y - h)$$

or

$$\frac{d^2y}{dx^2} + a^2y = a^2h$$

where $a^2 = P/EI$. The solution of this equation is

$$y = A \cos ax + B \sin ax + h \quad . \quad . \quad . \quad . \quad (i)$$

giving

$$\frac{dy}{dx} = -aA \sin ax + aB \cos ax \quad . \quad . \quad . \quad . \quad (ii)$$

The conditions to be satisfied are:

$$y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = 0,$$

$$y = h \text{ when } x = l.$$

Hence from (i) we must have

$$A = -h$$

$$A \cos al + B \sin al = 0$$

$$\therefore B = -A \cot al = h \cot al \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

From (ii) we must have $B = 0$. Therefore (iii) shows that either $h = 0$, in which case there is no deflection, or $\cot al = 0$, and in this case h is indeterminate. We have then

$$\cot al = 0.$$

$$al = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots$$

Hence the smallest value of P which will cause the strut to be unstable is given by $a^2l^2 = \pi^2/4$, or

$$\mathrm{P} = \frac{\pi^2 \mathrm{EI}}{4l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It should be noticed that this is the same as for a strut of length $\frac{l}{2}$ pin-jointed at both ends.

219. Strut with Both Ends Encastré.—Referring to Fig. 279, the ends O and A are both encastré in the direction OA, so that fixing moments M_0 are introduced at the ends, these moments being equal by symmetry. The equation for the deflections is

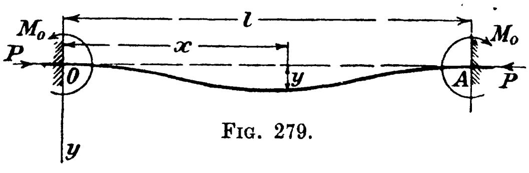

FIG. 279.

$$-EI\frac{d^2y}{dx^2} = Py - M_0$$

or

$$\frac{d^2y}{dx^2} + a^2y = a^2\frac{M_0}{P},$$

where $a^2 = P/EI$.

The solution of this is

$$y = A \cos ax + B \sin ax + \frac{M_0}{P} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

giving

$$\frac{dy}{dx} = -aA \sin ax + aB \cos ax \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The conditions to be satisfied are :

$$y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = 0 \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Also, for symmetry,

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The conditions (iii) require $A = -\frac{M_0}{P}$ and $B = 0$. Then (ii) and (iv) give

$$\frac{M_0}{P} \sin \frac{al}{2} = 0.$$

Since M_0 is not zero we must have $\sin \frac{al}{2} = 0$. The smallest value a which satisfies this is $\frac{2\pi}{l}$, which gives

$$\mathrm{P} = \frac{4\pi^2 \mathrm{EI}}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is the smallest end-thrust which can hold the encastré strut deflected, and is four times the corresponding load for a strut the ends of which are free to turn. In practice such a large factor is not used since perfect encastrément is not obtainable. It must also be noted that this formula does not in any way apply to a strut fixed to other members having a flexural rigidity comparable with that of the strut itself, however rigid the connection may be.

220. The Imperfections of Real Struts.—In all the theoretical investigations above we have made, tacitly or otherwise, certain assumptions which cannot usually be realized in practice, and attention was called to this in §§ 213–4. We shall now consider the matter in greater detail, but space will not permit an exhaustive discussion.* We have mentioned above, and dealt with theoretically, two important departures of the real strut from ideal conditions : crookedness and eccentricity of loading. In addition to these defects there are several uncertainties which combine to make the exact design of struts an almost impossible achievement, and among them the following may be mentioned as examples : Variations in the elastic properties of the material and in the dimensions of supposedly uniform sections, past history including the effects of manufacturing processes, flaws and local defects, uncertainty as to the exact conditions obtaining at the ends, etc.

It can be shown that a variation in the value of E through the material of the strut is equivalent in effect to a certain fictitious eccentricity of loading, or a certain initial curvature, or both. A want of uniformity of area in rolled or drawn sections may also be represented by an eccentricity of loading, and often the effect of end fixings may be so represented. The effects of manufacturing processes, flaws, etc., can be allowed for by a certain reduction of the permissible stresses. Thus we can regard all the defects of a real strut as represented by initial curvature and eccentricity of loading, and if we could be sure of the value of these we could calculate the stresses with very little uncertainty, whilst the reduction in the strength of the material could be allowed for by a suitable increase in the factor of safety.

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* The reader is referred to E. H. Salmon's *Columns* (Oxford University Press), where he will find an exhaustive bibliography of the literature of struts, and a lengthy discussion of the matter considered above, with a review of all the most important experimental work. The third and fourth chapters of this work should be studied by all who deal with the design of struts.

The trouble is that in practical cases we do not know exactly the degree of fictitious eccentricity or initial curvature to allow for, and all the empirical formulæ are simply efforts to dodge this difficulty: in fact, they do little more than call an " uncertainty " a " doubt " and deceive the unwary into thinking that by so doing they have got round the trouble!

Let us see to what extent this eccentricity may be estimated. Fortunately a small error in calculating the exact value of the eccentricity does not greatly affect the stresses, for we shall find that the estimate can only be a rough approximation.

221. Eccentricity of Loading.—It is to be expected that the accidental eccentricity of the line of action of the resultant force will vary in some manner with the dimensions of the strut, but presumably not in any definite regular manner. Some writers would make it a function of the dimensions of the cross section only; others suggest a function of the length; others again make it depend on l/k. Salmon, who has investigated the matter in great detail, suggests taking the eccentricity of load as $l/1{,}000$. In addition to this he makes an allowance for possible variations of Young's Modulus by an extra eccentricity which he estimates may be calculated from the formula $\bar{y}/10$, where $\bar{y}$ is the distance, from the neutral axis, of the centroid of that portion of the cross section which is on the convex side of the strut. For lattice girder struts he takes 1/20th of the width of the strut in the plane of bending. Salmon also adds a further eccentricity to allow for unequal areas of the opposite members (intended to be equal) of a built-up strut, and proposes it should be calculated as 1/160th of the width. Thus he would give

$$h = \frac{l}{1{,}000} + \frac{\bar{y}}{10} \text{ for ordinary columns, } \quad . \quad . \quad . \quad . \quad (15)$$

$$h = \frac{l}{1{,}000} + \frac{B}{20} + \frac{B}{160}$$

$$= \frac{l}{1{,}000} + \frac{9B}{160} \text{ for lattice braced columns, } \quad . \quad . \quad . \quad (16)$$

B being the width in the plane of bending.

222. Initial Curvature.—It has already been pointed out that no real strut can be expected to be perfectly straight, but to apply this axiom to practical design we must be able to estimate what is a reasonable initial deflection to assume. Some authorities propose to take a certain fraction of the length, others to assume it proportional to l/k. Salmon (*loc. cit.*, pp. 152–3) has plotted the initial deflections observed by various workers, mostly under laboratory conditions, and finds that practically all the points fall within the limit $l/750$ or $0{\cdot}0023\ l/k$; in the former case the maximum deflection is considered whatever its direction happens to be, whilst in the second it is the deflection in the direction of the k taken to evaluate l/k. He concludes that at least this amount of initial deflection must be reckoned on, or it may be twice as much if the columns are not straightened. Robertson * gives the value $0{\cdot}003\ l/k$.

In § 221 variations of E were taken as equivalent to an eccentricity of loading, but they might be regarded as equivalent to an initial curvature, or partly one and partly the other. In a pin-jointed strut it is safer to take it as entirely eccentricity of loading, so that in this case Salmon takes

$$\left.\begin{array}{ll} h = l/1{,}000 + \bar{y}/10 & \text{together with a deflection } c = l/750 \\ \text{or } h = l/1{,}000 + 9B/160 & \quad ,, \qquad ,, \qquad ,, \qquad c = l/750 \end{array}\right\} \quad . \quad . \quad (17)$$

With an encastré strut it is the initial curvature which is the important factor, since eccentricity only modifies the fixing moments, and Salmon divides the allowance equally between h and c, thus

$$\left.\begin{array}{ll} h = l/1{,}000 + \bar{y}/20, & \text{together with a deflection } c = l/750 + \bar{y}/20 \\ \text{or } h = l/1{,}000 + 5B/160 & \quad ,, \qquad ,, \qquad ,, \qquad c = l/750 + B/40 \end{array}\right\} \quad . \quad (18)$$

223. Equivalent Eccentricity.**—If we examine the above values given by Salmon for various shapes, we shall find that the equivalent eccentricity calculated as in § 216 is not very different from that given by (B = thickness of strut in direction of bending).

In the case of solid struts $$\mathbf{h} = \frac{l}{500} + \frac{\mathrm{B}}{50} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and for lattice struts, $$\mathbf{h} = \frac{l}{500} + \frac{\mathrm{B}}{20} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Furthermore, these formulæ for h do not differ greatly from that given by the Air Ministry for commercial solid drawn steel tubes,† and give results which will be safe in practice.

224. Reduction in Strength.—The effects of the past history of the material, including cold straightening, manufacturing processes, annealing, etc., are very difficult to estimate, but their importance is becoming more and more realized. Baker (1888), Considère,‡ Howard,§ Moncrieff,|| Lilly ¶ and others, have all called attention to the matter, but at present we can do little more than increase the factor of safety or decrease the working stress by some arbitrary amount. Salmon recommends reducing

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* *Aeroplane Structures*, Pippard and Pritchard.

† Ditto: $h = l/600 + D/40$ where D is the internal diameter.

‡ *Congrès Int. des Procédés de Const. Exposition Univ. Int. de* 1889, Vol. III.

§ *Proc. Amer. Soc. for Testing Materials*, Vol. VIII (1908).

|| *Proc. Amer. Soc. Civil Engineers*, Vol. XLV (1900).

¶ *The Design of Columns and Struts*, 1908.

** A useful and instructive paper is *Column Eccentricity*. C. N. Ross, obtainable from Royal Society House, 5 Elizabeth Street, Sydney.

the working stress at least 10 per cent., and preferably 20 per cent., below that allowed for tension in the same material.

225. End Conditions.—In most theoretical work the strut is usually supposed perfectly free in direction, or perfectly fixed in direction, at the ends. In practice all struts are imperfectly fixed in direction at the ends. If the strut is pin-jointed to the rest of the structure of which it is a member, friction between the pin and its bearing must exert some constraint against flexure, which may vary in amount from almost perfect fixing if the pin be relatively large and a tight fit, to practically zero for a relatively small, loose-fitting, well-lubricated pin. If the strut is supposed to be rigidly attached to the rest of the structure we ought not to begin to think of it as an encastré strut unless the members to which it is attached are so relatively rigid that their flexure by the strut is inconceivable; if the strut and the neighbouring members are of comparable stiffness the strut may actually fail under a smaller load than if it were pin-jointed. But even when these considerations allow us to regard the strut as encastré, it must be remembered that no fixings can be absolutely rigid in practice, and this is particularly important when we notice what a small angular movement of the ends precedes the failure of a pin-jointed strut: a slope of the order of 1/1,000 for loads just under the failing load is what may be expected in many cases.* This being so, it is evident that a very slight want of rigidity in the fixings will convert an encastré strut into a pin-jointed strut for all practical purposes. A common method of dealing with so-called encastré struts is to regard them as pin-jointed struts having a length of 0·6 to 0·8 of the actual length When attached to flexible members the strength of the structure as a whole should be considered by estimating the bending moments at the joints.

DESIGN OF STRUTS

226. Range of the Euler Formula.—Let us first examine the range within which the Euler formula may be expected to apply to real struts. The notation here is the same as that of § 215, where we have seen that the failing load of an eccentrically loaded strut is given by

$$P^2\left(1-0{\cdot}26\frac{hy_c}{k^2}\right)-P\left\{P_e\left(1+\frac{hy_c}{k^2}\right)+fS\right\}+fS\,.\,P_e=0 \quad \text{(i)}$$

where

$$P_e=\frac{\pi^2EI}{l^2}=\frac{\pi^2ESk^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Let us investigate the extent by which a small eccentricity makes the failing load depart from the Euler value given by (ii). Regarding P and h as variables, (i) gives on differentiation

$$2P\left(1-0{\cdot}26\frac{hy_c}{k^2}\right)dP-0{\cdot}26\frac{P^2y_c}{k^2}dh-dP\left\{P_e\left(1+\frac{hy_c}{k^2}\right)+fS\right\}$$
$$-\frac{PP_ey_c}{k^2}\,.\,dh=0\,. \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

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* 1909–1910 Watertown Arsenal experiments; see Salmon (*loc. cit.*), p. 166.

For a given shape of cross section, k will be proportional to y_c and we can put

$$k^2 = \gamma y_c^2,$$

γ being a constant depending only on the shape of the cross section. Making this substitution in (iii) and dividing through by P^2 we have

$$2\frac{dP}{P}\left(1 - 0{\cdot}26\frac{h}{\gamma y_c}\right) - 0{\cdot}26\frac{dh}{\gamma y_c} - \frac{dP}{P}\left\{\frac{P_e}{P}\left(1 + \frac{h}{\gamma y_c}\right) + \frac{fS}{P}\right\} - \frac{P_e}{P}\cdot\frac{dh}{\gamma y_c} = 0.$$

When $h = 0$, P is equal to P_e; if we put these values in this equation we shall have a relation between the amount by which P departs from P_e on account of a small eccentricity dh; we get

$$\frac{2dP}{P_e} - 0{\cdot}26\frac{dh}{\gamma y_c} - \frac{dP}{P_e}\left(1 + \frac{fS}{P_e}\right) - \frac{dh}{\gamma y_c} = 0$$

or

$$\frac{dP}{P_e} = \frac{1{\cdot}26}{1 - \dfrac{fS}{P_e}}\cdot\frac{dh}{\gamma y_c}. \qquad (21)$$

In this equation dh/y_c represents the eccentricity of the end load expressed as a fraction of y_c, and dP/P_e is the corresponding change of failing load expressed as a fraction of the Euler load P_e. fS is the load which would cause failure by direct stress.

Substituting for P_e from (ii) in the right-hand side of (21) it becomes

$$\frac{dP}{P_e} = \frac{1{\cdot}26}{1 - \dfrac{f}{\pi^2 E}\left(\dfrac{l}{k}\right)^2}\cdot\frac{dh}{\gamma y_c} \qquad (22)$$

Equations (21) and (22) enable us to express in two ways the conditions which must be fulfilled if a given equivalent total eccentricity is not to cause the load which the strut will bear to depart from the Euler load by more than a prescribed percentage. From equation (21) we can set a limit to $\dfrac{P_e}{fS}$, and from (22) we can set a limit to l/k. To illustrate this we shall consider some numerical examples.

Let us take $dh = 0{\cdot}02y_c$, and stipulate that the failing load must not be less than 0·95 P_e, i.e. $dP = -0{\cdot}05P_e$. Then we have from (21)

$$\frac{fS}{P_e} = 1 + \frac{0{\cdot}504}{\gamma}.$$

For a circular section $\gamma = 1/4$, so that

$$\frac{fS}{P_e} = 3{\cdot}016,$$

or $\dfrac{P_e}{fS} = \dfrac{1}{3}$, approximately,

that is, the stress intensity under the Euler load must not exceed one-third of the limiting stress allowable. For a rectangular section of least width a, γ is 1/3, and we find that the ratio is 1/2·5 instead of 1/3. If the eccentricity is doubled, the ratio is 1/5 for the round section, and 1/4 for the rectangular section, approximately.

Again, from (22) we have, for the same conditions,

$$\frac{l^2}{k^2} = \frac{\pi^2 E}{f}\left(1 + \frac{0{\cdot}504}{\gamma}\right)$$

For the circular section we find that we must have

$$\frac{l^2}{k^2} \not< \frac{3{\cdot}016\pi^2 E}{f},$$

and for the rectangular section:

$$\frac{l^2}{k^2} \not< \frac{2{\cdot}512\pi^2 E}{f}.$$

For a steel strut having $E = 13{,}500$ tons/in.^{2} and $f = 20$ tons/in.^{2}, these correspond with $l/k \not< 140$ and 130 respectively.

For a steel having a yield point of 40 tons/in.^{2}, the least values of l/k will be 100 for the circular section and 93 for the rectangular section.

For a timber having $E = 1{\cdot}5 \times 10^6$ lbs./in.^{2}, and $p_c = 5{,}000$ lbs./in.^{2}, the values are 95 and 87 for round and rectangular sections respectively.

The above remarks and figures should be sufficient to show the reader the limitations which must be considered to restrict the use of the Euler formula, and before applying it in any practical case he should make perfectly sure that he will be justified in doing so.

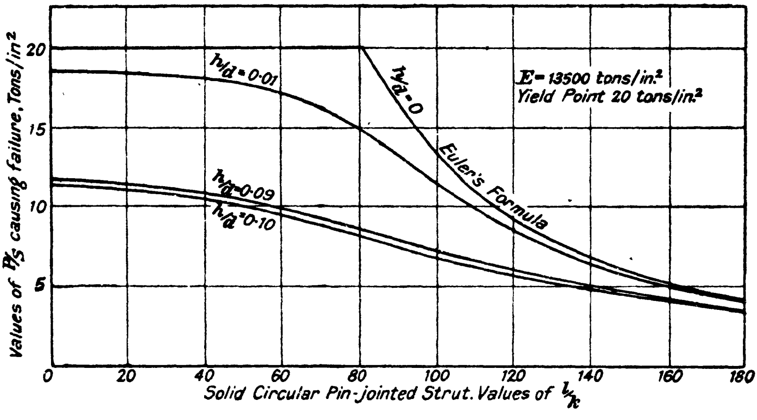

FIG. 280.

The curves shown in Fig. 280 will indicate how the permissible load, per unit area of cross section, on a strut of given material depends on the eccentricity and on the value of l/k. The curves refer to pin-jointed struts of circular cross section of diameter d, the eccentricity of the load being h.

227. Empirical Formulæ.—We have seen above that when l/k is less than about 90–150 for steel, or 80–100 for timber, the Euler theory must be expected to give values for the failing load which will be too high, even though the load per unit area be then less than the yield point, on account of the imperfections of real struts. Consequently there is a need for a reliable formula upon which to base the design of struts, and many formulæ have been given by various writers. Yet, from the very nature of the problem, it seems that a reliable formula covering all conditions must ever be sought in vain. We have seen how an eccentricity of end load affects the failing load: in practice it will hardly ever be zero and the amount of eccentricity cannot be foretold with any degree of accuracy. In view of such facts as these it seems unreasonable to look for a formula which will yield accurate results in all cases, and the best we can hope for is an empirical formula which will be applicable with moderate accuracy over a limited range of circumstances. In recent years the subject has excited considerable controversy,* but at present the choice of a formula is largely a matter of taste and convenience, whilst the selection of suitable values for the empirical constants in any particular formula is a matter for judgment based on experience. The reader cannot be too emphatically warned against using any particular values of empirical constants without ascertaining the conditions in which they were found, and they should never be used outside the range of the experiments from which they were found. In most specifications for struts it is usually stipulated that l/k shall not exceed 100, or 70 in the case of bridge construction, thus practically ruling out Euler's formula.

We shall now consider the more important empirical formulæ.

228. The Rankine-Gordon Formula.—This formula appears to have originated from Tredgold in analysing the results of Hodgkinson's experiments, and seems to be based on the following argument: If y denote the maximum deflection, the bending moment due to an end load P is Py, which will give rise to a stress $Pyb/2I$ where b is the thickness of the (symmetrical) section parallel to the plane of bending. At the same time the stress due to direct compression is P/S, where S is the area of the cross section. Hence, if $f =$ the maximum permissible stress we must have, as in § 214,

$$f = \frac{P}{S} + \frac{Pyb}{2I}$$

from which we derive

$$\frac{P}{S} = \frac{f}{1 + \dfrac{yb}{2k^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

This formula will be perfectly true for any strut provided we know y, which is the real crux of the matter. Tredgold assumed the curvature to be circular and arrived at formulæ for struts of rectangular and circular cross sections. This was modified by Gordon who put it in the form

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* Reference may be made to the following:—H. Basquin in the *Journal of the Society of Western Engineers*, June, 1913, 1914; C. P. Buchanan in *Engineering News*, Dec. 26, 1907; J. Kübler, C. J. Kriemler, and L. Prandtl in *Zeitschr. d. Deutschen Ingenieure*, Bd. 44 (1900), also Kübler and Kriemler in *Zeitschr. f. Math. u. Phys.*, Bde. 45–47 (1900–1902); a paper by Timoschenko in *Ann. des ponts et chaussées*, 1913; Southwell in *Phil. Trans. R.S.* (Ser. A), vol. 213 (1913); T. Strand in *Teknisk Ukeblad* (Sept. 27, 1918); Salmon's *Columns*.

$$\frac{P}{S} = \frac{f}{1 + c\left(\frac{l}{b}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

c being a constant depending on the cross section and material. Rankine further modified it to

$$\frac{\mathbf{P}}{\mathbf{S}} = \frac{\mathbf{f}}{1 + \mathbf{a}\left(\frac{l}{\mathbf{k}}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

a being a similar constant to Gordon's c.

From the above it will be seen that these formulæ do not really avoid the difficulty of finding y; they simply dodge it by substituting the unknown a or c for the unknown y. The constants a and c are then regarded as empirical numbers to be found by experiment. The underlying error is that y is really a function of P, so that a and c should also depend on P, whereas they are treated as constants.*

The following values of a and f are those generally accepted when this formula is used:

	Mild Steel.	Wrought Iron.	Cast Iron.	Ash or Spruce.
f . . .	21	16	36	2·25 tons/in.^{2}
$1/a$ . . .	7,500	9,000	1,600	3,000

For encastré struts $1/a$ is usually taken as four times the above.

Example.—A cast-iron column, 8″ external diameter and 6″ internal diameter, is 20′ long. Applying the Rankine-Gordon formula (24) and assuming $a = 1/6400$, find the compressive stress produced by an axial load of 50 tons. (R.N.E.C., Keyham, 1920.)

We have

$$S = \frac{\pi}{4}(8^2 - 6^2) \text{ in.}^2 = 7\pi \text{ in.}^2 = 22 \text{ in.}^2$$

$$I = \frac{\pi}{64}(8^4 - 6^4) \text{ ins.}^4 = \frac{700\pi}{16} \text{ ins.}^4$$

$$\therefore\ k^2 = \frac{I}{S} = \frac{100}{16} \text{ in.}^2 \quad k = 2{\cdot}5''$$

$$l = 240''; \quad \frac{l}{k} = 96; \quad \left(\frac{l}{k}\right)^2 = 9{,}216.$$

$$1 + a\left(\frac{l}{k}\right)^2 = 2{\cdot}44$$

$$f = \frac{P}{S}\left\{1 + a\left(\frac{l}{k}\right)^2\right\} = \frac{50 \text{ tons}}{22 \text{ in.}^2} \times 2{\cdot}44 = 5{\cdot}5 \text{ tons/in.}^2$$

229. Straight Line Formulæ.—On account of the great variation of the experimental results of tests on struts under conditions approaching those of actual practice, any empirical formula can only be expected to give a more or less rough prediction of failing loads for actual struts. For this reason there is a great deal in favour of the simplest possible formula, namely, one following a straight line, and such formulæ are in common use in America. We have in these cases

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* Sometimes an attempt is made to connect these formulæ with Euler's formula by starting with the empirical relation $1/P = 1/fS + l^2/\pi^2Ek^2$ on the ground that it satisfies the conditions at both ends of the scale, when l is very large or very small. But this seems rather specious; it gives the value $a = f/\pi^2E$.

$$\mathrm{P} = \mathrm{S}\left(\mathrm{f} - \mathrm{n}\frac{l}{\mathrm{k}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where n is a constant. A frequent specification for mild steel is

$$P = S\left(16{,}000 - 80\frac{l}{k}\right), \text{ with free ends} \quad . \quad . \quad . \quad . \quad (26)$$

$$P = S\left(16{,}000 - 60\frac{l}{k}\right), \text{ with fixed ends} \quad . \quad . \quad . \quad (27)$$

in pound-inch units, with the stipulation that in no case shall l/k exceed 100, or 70 in bridge work.* This formula, though simple in itself, is not so convenient for the direct calculation of the area required as the parabolic formula of the next article.

230. Johnson's Parabolic Formula.—Johnson has given the following formula

$$\mathrm{P} = \mathrm{S}\left(\mathrm{f} - \mathrm{r}\frac{l^2}{\mathrm{k}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where r is a constant. If P/S be plotted against l/k, this equation will give a parabola, and r is chosen to make the parabola touch the Euler curve $P/S = \pi^2 Ek^2 l^2$. Johnson, as the result of experiments, gives the following figures:

For mild steel, $f = 42{,}000$ lbs./in.^{2}
For wrought iron, $f = 34{,}000$ lbs./in.^{2}

$$\text{For mild steel} \quad \left\{\begin{array}{l} r = 0{\cdot}97 \text{ if } l/k < 150, \text{ ends pivoted.} \\ r = 0{\cdot}62 \text{ if } l/k < 190, \text{ ends fixed.} \end{array}\right.$$

$$\text{For wrought iron} \left\{\begin{array}{l} r = 0{\cdot}67 \text{ if } l/k < 170, \text{ ends pivoted.} \\ r = 0{\cdot}43 \text{ if } l/k < 210, \text{ ends fixed.} \end{array}\right.$$

For cast iron the Rankine-Gordon formula gives better results.

For practical struts it is usually only the parabolic part of the curve which is required.

The area of cross section required can be found directly by the following device which is due to Asimont: †

Let $S = sk^2$, s depending only on the shape of the section.
,, S_0 = the area required to resist direct crushing.

$$= \frac{P}{f}.$$

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* Taken from *The Strength and Elasticity of Structural Members* by R. J. Woods.

† *Zeit. des Bayer Arch.-u.-Ing.-Ver.*, München., Bd. viii, Heft 6. The application is due to Ostenfeld (*Zeits. d. Ver. deu. Ing.*, Dec. 31, 1898).

The parabolic formula may be written

$$\frac{P}{S} = f - r\frac{l^2 s}{S}$$

or

$$\frac{fS_0}{S} = f - \frac{l^2 rs}{S}$$

$$\therefore S = S_0 + \frac{l^2 rs}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

231. Fidler's Formula.—The following formula is given in Fidler's *Bridge Construction*:

$$\frac{P}{S} = \frac{1}{\mu}\{f + p_e - \sqrt{(f + p_e)^2 - 2\mu f p_e}\} \quad . \quad . \quad . \qquad (30)$$

where f = ultimate strength of material in compression.

p_e = the Euler stress $= \pi^2 E k^2 / l^2$.

μ = a constant of average value 1·2.

232. Perry's Formula.—The following three formulæ are less empirical in their nature than the foregoing, although all depend on at least one empirical constant. They represent, in various forms, attempts to allow for such imperfections as initial crookedness and eccentricity of load. Perry gave the following formula:

$$p = \tfrac{1}{2}\{f + p_e(1 + \eta)\} - \tfrac{1}{2}\sqrt{\{f + p_e(1 + \eta)^2 - 4fp_e\}} \quad . \quad . \qquad (31)$$

where $p = P/S$.

f = yield stress in compression.

p_e = Euler stress $= \pi^2 E k^2 / l^2$.

$\eta = \frac{b}{2k^2}\left(c + \frac{6}{5}h\right)$

c = maximum initial deflection.

h = the eccentricity.

b = the thickness in the direction of the smallest radius of gyration.

233. Robertson's Formula.—Prof. Andrew Robertson modifies Perry's formula by taking $\eta = 0{\cdot}003\, l/k$, which he finds fits a large number of experimental results extraordinarily well.

234. Author's Formula.—Solving equation (7) in § 215 we have

$$\mathrm{P/S} = \frac{1}{2\mathrm{a}}\{(\mathrm{f} + \mathrm{bp_e}) - \sqrt{(\mathrm{f} + \mathrm{bp_e})^2 - 4\mathrm{f.ap_e}}\} \quad . \quad . \qquad (32)$$

where $a = 1 - 0{\cdot}26\,\frac{hy_c}{k^2}$, and $b = 1 + \frac{hy_c}{k^2}$.

where h is to be calculated from the formula (19) or (20) of § 223.

The chief reason for the difference of form between this and Perry's formula is the more accurate approximation taken for sec θ in § 215. In (32) p, p_e, f and k^2 have the same meaning as above; for struts of symmetrical section y_c in (32) will equal $b/2$ in (31). If the constants can be accurately known in both cases (32) may be expected to yield better results than (31).

235. Stress Determining Strut Failure.[*]—Robertson has shown that, for ductile materials such as mild steel, high tensile steel, and wrought iron, it is the value of the stress at *yield* in compression, and not the ultimate strength, which determines failure. The reason for this is the reduction of stress which occurs when the yield point is reached: the strain of the material has to be many times the elastic strain before the yield stress can be supported. Thus the slightest increase of load will bring about a reduction of stress in that portion of the strut which has reached the yield point, since the rest of the strut prevents the strain being sufficient to support the yield stress. The effect is to shift the line of action of the resultant stress farther away from the axis of the strut, which is accompanied by considerable strain in the material, resulting finally in the complete collapse of the strut.

It is, then, the value of the yield stress in compression which we require for determining the strength of struts, and Robertson has shown that, provided care is taken that the load is uniformly distributed during test, the elastic limit, yield point, and Young's Modulus are practically the same in compression and tension; also that the yield point and elastic limit are generally identical for steels which have a well-defined drop of stress at yield.[†] Some of his experimental results are given in the following table and in the table on p. 321.

TESTS ON SOLID ROUND STEEL SPECIMENS

	Tension (tons/in.^{2}).			Compression (tons/in.^{2}).		
	Elastic Limit.	Yield Point.	E	Elastic Limit.	Yield Point.	E
Mild steel . . .	18·3	18·3	13,300	19·2	19·2	13,300
,, ,, . . .	20·0 [‡]	20·2 [‡]	13,300 [‡]	—	—	—
"36-ton" steel . .	21·5	22·8	13,200	24·0	24·1	13,500
,, ,, . .	23·0 [‡]	23·0 [‡]	13,210 [‡]	—	—	—

236. Factors of Safety for Struts.—In dealing with members which are in tension, shear, or bending unaccompanied by thrust, or compression without bending we can write

$$\text{factor of safety} = \frac{\text{load at failure}}{\text{working load}} = \frac{\text{stress at failure}}{\text{working stress}}.$$

In the case of struts this is not strictly true, although the empirical formulæ of § 228 would make it appear to be. We know that the failure of a strut is due to stresses set up by thrust and flexure combined, the flexure being due to a real or fictitious eccentricity of the end load, and increasing as the latter increases but not proportionally.

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[*] See a paper by Westergaard and Osgood on struts stressed beyond the elastic limit, published in the *Proceedings of the American Society of Mechanical Engineers*.

[†] See B.A. report on stresses in overstrained materials, 1931.

[‡] Special loading shackles to ensure uniform distribution of strain.

A consideration of the equations obtained in § 215 for an eccentrically loaded strut will explain matters. Taking the approximate equation (7) and dividing by S^2 we can write it in the form

$$ap^2 - (b + p_c)p + p_c p_e = 0 \qquad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

where p = the average stress $= P/S$,
p_c = the maximum compressive stress,
p_e = the Euler stress $= \pi^2 E k^2/l^2$,
and a and b are constants depending on the dimensions of the strut.

Now let f = the yield stress in compression.
n = the factor of safety.
p_c = the maximum working stress $= \dfrac{f}{n}$.

From (i) we can find p and hence P, obtaining

$$P = \frac{S}{2a}\{b + p_c - \sqrt{(b + p_c)^2 - 4ap_c p_e}\}$$

If we insert in this f or np_c instead of p_c we shall obtain the average stress at failure and hence the load P' which will cause breakdown:

$$P' = \frac{S}{2a}\{b + np_c - \sqrt{(b + np_c)^2 - 4anp_c p_e}\},$$

evidently $P' \neq nP$.

To find the true factor of safety of a given strut under a given end load, using equation (i) we should proceed thus: we have first $p = P/S$, so that from (i) we can find the maximum stress $p_c = p(b - ap)/(p_e - p)$, then the factor of safety is f/p_c.

Conversely, if we wish to design a strut by the same equation, to have a given factor of safety with a given load, we first calculate $p_c = f/n$, and then, taking a trial section, find p from (i) and hence $P = pS$. If this does not give the correct value of P we must try again.

We have here interpreted the factor of safety as meaning stress at failure ÷ working stress; if we take it to mean the multiple of the working load which will cause failure we must design the strut so that $p_c = f$ when $p = nP/S$.

237. Shearing Forces in Struts.—In any strut the bending moment depends on the displacement of the axis and varies along the strut. Consequently there must be shearing forces. When the data are sufficient these can be calculated from the exact formulæ given above for the bending moment. For instance, in the case of strut pin-jointed at each end, with an eccentric load, we have from equation (ii), § 215,

$$M = Py = Ph \cos a\left(\frac{l}{2} - x\right)\sec\frac{al}{2}.$$

Hence, if F denote the shearing force at any section,

$$F = \frac{dM}{dx} = aPh \sin a\left(\frac{l}{2} - x\right)\sec\frac{al}{2}.$$

This is numerically a maximum when $x = 0$ or l, which gives

$$F_{max} = aPh \tan \frac{al}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Other cases can be calculated in the same way. When an empirical formula is used to calculate the safe load on a strut or to design a strut, we can form an approximate idea of the shearing force thus :

Let p = the average stress on a section = P/S.
p_b = the stress due to bending.
f = the working stress taken to calculate the strut.
δ = the deflection at the centre.
Z = the modulus of the cross section.

Then
$$f = p + p_b = p + \frac{P\delta}{Z}$$

from which
$$\delta = \frac{Z}{P}(f - p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

In the case of a pin-jointed strut the Euler theory suggests that we should assume a sine curve for the shape of the deflected axis. Let us take, with the notation of Fig. 273,

$$y = \delta \sin \frac{\pi x}{l}$$

Then the bending moment is

$$M = P\delta \sin \frac{\pi x}{l},$$

and the shearing force is given by

$$F = \frac{dM}{dx} = \frac{\pi P\delta}{l} \cos \frac{\pi x}{l}.$$

This is a maximum at the ends of the strut, where $x = 0$ or l, giving

$$\mathbf{F}_{\mathbf{max}} = \pm \frac{\pi P\delta}{l} = \pm \frac{\pi \mathbf{Z}}{l}(\mathbf{f} - \mathbf{p}) \quad . \quad . \quad . \quad . \quad (35)$$

With an encastré strut, we may assume for the deflection curve,

$$y = \frac{\delta}{2}\left(1 - \cos \frac{2\pi x}{l}\right)$$

and
$$M = Py + M_0 = \frac{P\delta}{2}\left(l - \cos \frac{2\pi x}{l}\right) + M_0,$$

where M_0 represents the fixing moments at the ends. Then

$$F = \frac{dM}{dx} = \frac{\pi P\delta}{l} \sin \frac{2\pi x}{l}.$$

This is a maximum when $x = \frac{l}{4}$ or $\frac{3l}{4}$, and

$$\mathbf{F}_{\mathbf{max}} = \pm \frac{\pi P\delta}{l} = \pm \frac{\pi \mathbf{Z}}{l}(\mathbf{f} - \mathbf{p}) \quad . \quad . \quad . \quad . \quad (36)$$

BRACED STRUTS*

238. Braced Struts.—Large struts are frequently of built-up lattice girder construction. In these cases the deflections, bending moments, etc., may be calculated in the same manner as for solid struts, the value of the moment of inertia being calculated for the whole cross section of the strut. This statement, of course, infers that the eccentricity of loading, etc., can be calculated, so that the same uncertainties must be faced as in the design of solid struts, and the problem can be dealt with in the same way, that is by trying to estimate the eccentricity or by empirical formulæ. But with reasonable care, whatever method of design be adopted, the chance of the strut failing as a whole is not large : the most important feature of the design of built-up struts is the consideration of local strength, that is the strength of the individual members between the panel points, the strength of the bracing, the failure of the separate plates of the flanges between the rivets, and so on. The flanges, between the panel points, must be considered as individual struts, pin-jointed at their ends, either eccentrically loaded or by empirical formulæ, the former for preference.

Let M denote the bending moment on any section, and let $b =$ the breadth between the centroids of the flanges. Then the thrust in each flange is $\pm M/b$. If the total area of the cross section of the flanges be S, and the load on the strut be P, the direct thrust on each flange is P/S. Hence the total thrusts in the two flanges are $P/S \pm M/b$. Thus the greatest load which has to be taken by a single flange between the panel points is $P/S + M/b$. We are here assuming that the two flanges are equal; if they are not a similar expression is easily worked out. If the eccentricity of loading on the strut is known, M can be calculated by the exact methods ; if an empirical formula is used, it can be estimated approximately in the manner of § 237.

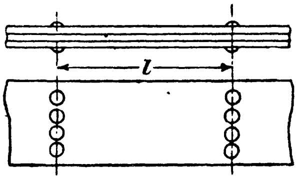

FIG. 281.

When the flanges are built of several plates riveted together, as in Fig. 281, the possibility of the outside plate buckling between the rows of rivets must also be considered. The plate may be treated as an imperfectly encastré strut, by taking a free length of $0{\cdot}7l$ to $0{\cdot}8l$.

The strength of the bracing to resist the shearing forces must also be considered, and we have shown above how to estimate the total shearing force to be dealt with. When double lattice bracing is used as in Fig. 282 (a) the thickness of the lattice plates should not be less than $\frac{1}{60}$th of the distance between the nearest end rivets, whilst with single bracing (Fig. 282 b) it must not be less than $\frac{1}{40}$th. The stress in compression should not be greater than **(8,600 − 63 l/t)** lbs./in.^{2} for mild steel, where l is the length of the bracing member and t the thickness. When nickel steel is used an increase of 40 per cent. on this may be allowed. If F is the total shearing force, the thrust in single lattice bracing is $F \sec \theta$ (Fig. 282 (*b*)) ; with double lattice bracing the shear is taken by thrust in one bar and tension in the other, and if these are assumed equal each is $\frac{1}{2}F \sec \theta$ (Fig. 282 (*a*)).

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* This subject is treated in detail by Salmon (*loc. cit.*) and Timoshenko, *Strength of Materials*, p. 594.

Of the various types of bracing in use the double lattice system (Fig. 282 (*a*)) is probably the best, particularly if the bars are riveted together where they cross. Experiments made with built-up beams * indicate that such a type has a considerable superiority over the single system (Fig. 282 (*b*)), and that there is considerable gain from riveting together the crossing bars ; the same experiments also showed the system (*b*) to be vastly superior to the system (*c*). On the other hand, other experiments † showed very little difference between the merits of the single and double systems, even when the ends of the former are independent as in (*c*). A tie bar between opposite points, such as A and B in Fig. 282 (*a*), is bad as it causes the diagonals to take part of the axial load on the column.

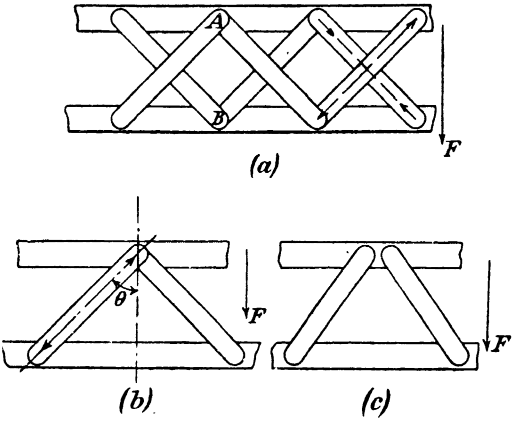

FIG. 282.

Example 1.—Design a lattice braced strut of length 49′ 0″ to take an axial thrust of 320 tons, with the specification that the load is given by $P/S = 16{,}000 - 80l/k$ lbs./in.^{2}, and that l/k is not to exceed 70, also that the flanges are to be single 1″ plates joined to the bracing by 4″ × 4″ × ⅝″ angles, as shown in Fig. 283. The bracing is to be of the double lattice type, the centroid of the section of each angle iron is 1·22″ from each face, the moment of inertia about an axis parallel to either side is 6·56 in.^{4}, and the area is 4·61 in.^{2}

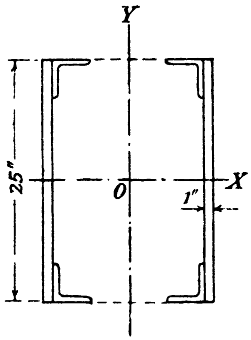

FIG. 283.

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* Basquin, *Journal of Western Society of Engineers*, Chicago, 1913.

† Watertown Arsenal, 1909–10.

If $l/k = 70$, we must have $k = \dfrac{588}{70} = 8\cdot43''$, and

$$\frac{P}{S} = 16{,}000 - 5{,}600 = 10{,}400 \text{ lbs./in.}^2$$

$$\therefore\ S \doteq \frac{320 \times 2{,}240}{10{,}400} = 68\cdot9 \text{ in.}^2$$

The area of the four angles is 18·44 in.², so that the total area of the plates must be 50·46 in.² Make them 25″ deep, find k and check the stress.

The moment of inertia about OX will be found to be 4,974 ins.⁴, whilst the area is 68·44 in.² This gives $k = 8\cdot53''$, which is slightly larger than the value we laid down. We must have

$$\frac{P}{S} < 16{,}000 - 80 \times \frac{588}{8\cdot53}$$
$$< 10{,}470 \text{ lbs./in.}^2$$

With 25″ flanges $P/S = 10{,}470$ lbs./in.², and so is just within the limit.

We must now make the moment of inertia about OY right.

Let D = the distance between the centres of the flanges. Then the moment of inertia about OY is

$$2\left(\frac{25 \times 1^3}{12} + 25 \times \frac{D^2}{4}\right) + 4\left\{6\cdot56 + 4\cdot61\left(\frac{D}{2} - 1\cdot72\right)^2\right\}$$
$$= 4\cdot1 + 12\cdot5D^2 + 26\cdot24 + 18\cdot44\left(\frac{D^2}{4} - 1\cdot72D + 2\cdot96\right)$$
$$= 17\cdot11D^2 - 31\cdot7D + 84\cdot9.$$

Equating this to the moment of inertia about OX and solving for D we get $D = 17\cdot85''$, so take $D = 18''$. This gives $I_{OY} = 5{,}025$ ins.⁴

We must next consider what is the greatest permissible free length of each flange by itself. Taking one flange plate with its two angles we find that the area is 34·22 in.², and the moment of inertia 35·1 ins.⁴ about an axis, through the centroid of the whole flange, parallel to the plates.

Hence $k^2 = 35\cdot1/34\cdot22 = 1\cdot026$ in.², or $k = 1\cdot012''$.

We must have

$$\frac{P_1}{34\cdot22} < 16{,}000 - 80 \times \frac{l_1}{1\cdot012}$$
$$\text{i.e.} < 16{,}000 - 79l_1, \quad \ldots\ldots \quad \text{(i)}$$

where P_1 is the maximum load in one flange, and l_1 is the unsupported length.

Now P_1 will be greater than half the total load on account of bending.

If f_b denote the stress due to bending,

f_c ,, ,, ,, ,, thrust,

we have

$$f_b + f_c = 16{,}000 \text{ lbs./in.}^2$$

$$\therefore\ f_b = 16{,}000 - \frac{320 \times 2{,}240}{68\cdot44} = 5{,}530 \text{ lbs./in.}^2$$

The bending moment required to produce this is

$$M = f_b Z = \frac{5{,}530}{2{,}240} \times \frac{5{,}025}{9\cdot5} \times 1{,}306 \text{ tons. ins.}$$

This produces a thrust in one flange and tension in the other equal to $\dfrac{1{,}306}{18} = 72\cdot6$ tons. Hence the maximum total thrust in one flange is $160 + 72\cdot6 = 232\cdot6$ tons.

Hence, from (i) we must have

$$\frac{232{\cdot}6 \times 2{,}240}{34{\cdot}22} < 16{,}000 - 79l_1$$

$$15{,}230 < 16{,}000 - 79l_1$$

which gives $l_1 \not> 9{\cdot}75''$.

This means that the panel points of the lattice bracing should not be more than $9\frac{1}{2}''$ apart.

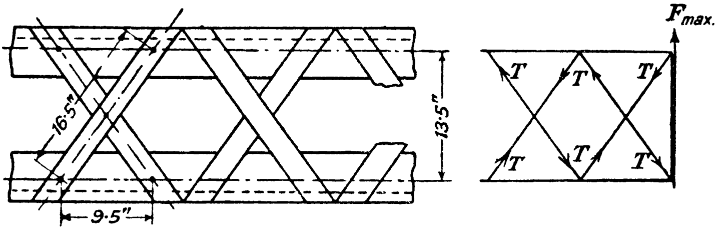

FIG. 284.

The centre lines of the angles are 13·5″ apart, and we shall take these lines as the line of the rivets attaching the bracing to the angles, as shown in Fig. 284.

We must next estimate the probable shearing force.

From equation (35) § 237, we have

$$F_{max} = \frac{\pi Z}{l}(f - p).$$

In the present case $f - p = 5{,}530$ lbs./in.^{2} (see above). $l = 588''$, Z for bending in the plane OX is $5025/9{\cdot}5 = 530$ ins.^{3} Hence

$$F_{max} = \frac{\pi \times 530}{588} \times 5{,}530 = 15{,}600 \text{ lbs.}$$

This is for both sides of the strut, for one side it will be 7,800 lbs. If we assume that half this is taken by thrust in one lattice bar, and half by tension in the other lattice bar, we see that the forces T are (Fig. 284)

$$\frac{16{\cdot}5}{13{\cdot}5} \times 3{,}900 = 4{,}760 \text{ lbs.}$$

The thickness of the lattice plates must not be less than $\frac{16{\cdot}5}{60} = 0{\cdot}26''$; say $\frac{3}{8}''$.

Let b = the width of the lattice bars, then for those in compression we must have, according to § 238,

$$\frac{4{,}760}{0{\cdot}375b} \not> 8{,}600 - 63 \times \frac{16{\cdot}5}{0{\cdot}375}$$

$$\not> 5{,}910$$

or

$$b \not< \frac{4{,}760}{0{\cdot}375 \times 5{,}910} \not< 2{\cdot}15''.$$

This is perhaps rather wide; if we take the thickness $\frac{1}{2}''$ instead of $\frac{3}{8}''$ we find $b = 1{\cdot}61''$, or say $1\frac{5}{8}''$. A single rivet $\frac{7}{8}''$ diameter will take the loads in the lattice bars, with a shearing stress of 4·53 tons/in.^{2}; the nett area across the rivet hole to take tension will be 0·438 in.^{2}, giving a tensile stress of 4·8 tons/in.^{2}

Example 2.—Check the above design by exact methods on the assumption that the equivalent eccentricity is that given by § 223.

The equivalent eccentricity is

$$h = \frac{l}{500} + \frac{B}{20}.$$

The smaller moment of inertia is about OX, so that we must consider bending about that axis, and we must take $B = 25''$. Hence, since $l = 488''$, we have

$$h = 0{\cdot}98 + 1{\cdot}25 = 2{\cdot}23''.$$

We also have

$$I = 4{,}974 \text{ ins.}^4$$
$$P = 320 \text{ tons.}$$

Take $E = 13{,}500 \text{ tons/in.}^2$, then

$$a^2 = \frac{P}{EI} = \frac{320}{13{,}500 \times 4{,}974} = \frac{1}{210{,}000} \text{ ins.}^{-2}$$

$$a = \frac{1}{460} \text{ ins.}^{-1}$$

$$\frac{al}{2} = \frac{244}{460} = 0{\cdot}533 \text{ radian} = 30{\cdot}5^\circ$$

$$\sec. \frac{al}{2} = 1{\cdot}16$$

The area of the cross section is 68·44 in.^{2} Hence, from equation (4) § 215, the maximum compressive stress is

$$\frac{320}{68{\cdot}44} + \frac{320 \times 2{\cdot}23 \times 12{\cdot}5}{4{,}974} \times 1{\cdot}16 \text{ tons/in.}^2$$
$$= 4{\cdot}68 + 2{\cdot}09 = 6{\cdot}77 \text{ tons/in.}^2 = 15{,}200 \text{ lbs./in.}^2$$

This is below the 16,000 lbs./in.^{2} which is allowed.*

We must next consider bending in the other plane, for which we have $B = 19''$, $h = 1{\cdot}93''$, $I = 5{,}025$ ins.^{4}

$$a^2 = \frac{1}{212{,}000},$$

which is sensibly the same as before. Hence the maximum bending moment in this plane is $Ph \sec. \frac{al}{2} = 320 \times 1{\cdot}93 \times 1{\cdot}16 = 716$ tons. ins.

This requires a thrust in one flange, and a tension in the other, equal to 39·8 tons. Thus the total maximum thrust is 160 + 40 = 200 tons, nearly, in one flange. This is less than our previous calculation gave.

Next consider one flange between the panel points:

$$l = 9{\cdot}5'',\ B = 5''.$$
$$h = 0{\cdot}0195 + 0{\cdot}1 = 0{\cdot}12'' \text{ nearly.}$$
$$I = 35{\cdot}1 \text{ ins.}^4;\ P = 200 \text{ tons.}$$

$$a^2 = \frac{200}{13{,}500 \times 35{\cdot}1} = \frac{1}{2{,}370}$$

$$a = \frac{1}{48{\cdot}7} \text{ ins.}^{-1}$$

$$\frac{al}{2} = \frac{4{\cdot}75}{48{\cdot}7} = 0{\cdot}0975 \text{ radian} = 5{\cdot}7^\circ.$$

$$\sec. \frac{al}{2} = 1{\cdot}005.$$

---

* The eccentricity h would have to be 2·64″ to bring the stress up to the maximum permissible value.

The area is 34·2 in.^{2} Hence the maximum stress is

$$\frac{200}{34{\cdot}2} + \frac{200 \times 0{\cdot}12 \times 3{\cdot}74}{35{\cdot}1} \times 1{\cdot}005$$

$$= 5{\cdot}85 + 2{\cdot}58 = 8{\cdot}43 \text{ tons/in.}^2 = 18{,}900 \text{ lbs/in.}^2,$$

which is greater than that allowed.

Finally we shall calculate the maximum shearing force in the plane OX. It is given by equation (33), p. 312:

$$F_{max} = aPh \tan\frac{al}{2}$$

$$= \frac{320 \times 1{\cdot}93}{460} \times 0{\cdot}5890$$

$$= 0{\cdot}79 \text{ ton} = 1{,}770 \text{ lbs.},$$

which is considerably less than before.

STEEL TUBULAR STRUTS

239. Equivalent Eccentricity.—The remarks which have been made above concerning struts in general apply of course to steel tubular struts in particular, but, on account of the researches carried on during the war in connection with struts for aeroplanes,* this particular class of strut stands in a more satisfactory position than other classes. The result is that there is really no necessity for falling back on rough empirical formulæ for design, such as those of Rankine or Gordon. Research shows that with ordinarily well-manufactured solid drawn steel tubes the equivalent eccentricity of load due to initial crookedness or eccentricity of bore is very unlikely to exceed a value given by

$$h = \frac{\text{length}}{600} + \frac{\text{diameter}}{40} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

We may therefore use with considerable confidence the formulæ for eccentrically loaded struts, either the accurate formula (5), p. 293, or the more convenient approximate quadratic (7). In this case the quantity hy_c/k^2 which enters into the equations can conveniently be written thus:

$$\frac{hy_c}{k^2} = \frac{1}{5}\left(1 + \frac{1}{60}\frac{l}{k}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In designing steel tubular struts attention must be paid to two points: crinkling and heat treatment. The yield point of mild steel may be considerably reduced by any heat treatment which it receives, such as welding or brazing. On this account precautions should be taken to prevent such treatment, or the strength must be taken as that of the annealed material.

240. Crinkling.—When a tubular steel strut is under compression the tube may "crinkle"—i.e. the walls of the tube may cave in and form folds, after the manner of a concertina. These folds may be circular, oval or polygonal, and they may occur after or before the longitudinal stress reaches the yield point. The matter was investigated [*] analytically by R. V. Southwell, who deduced the formula

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* Useful curves for design of tubular steel struts were published by the Air Ministry and are reproduced in Pippard and Pritchard's *Aeroplane Structures* (Longmans).

$$p = E\frac{t}{R}\sqrt{\frac{m^2}{3(m^2-1)}} \quad \ldots\ldots \quad (39)$$

where p = the stress causing collapse, t = thickness of tube, R = mean radius of tube, $1/m$ = Poisson's ration. The stress which will produce lobed failure is $(k^2 - 1)/(k^2 + 1)$ times that given by (39), where k = the number of lobes formed. These formulæ will only apply if $p <$ the yield point. Thus, for a mild steel tube, having a yield point of 20 tons per square inch, the formula will only apply when t/R is less than about 1/400—i.e. it will not apply to many tubes used in practice.

The phenomenon has been examined experimentally by Mason,[†] Popplewell and Carrington,[‡] and Andrew Robertson.[§] In Mason's experiments the load was applied through conical cups; in 31 tests of tubes 3 in. × 14 S.W.G., having $t/R = 0{\cdot}052$, there were nine cases in which signs of failure were noticed before the longitudinal stress reached the yield point; with tubes 2¾ in. × 10 S.W.G., no signs of collapse were observed before this happened.

Popplewell and Carrington experimented with nickel-chromium steel tubes, both in the annealed and in the unannealed states. The composition of the steel was: Carbon, 0·2 to 0·3 per cent.; nickel, 4 to 5 per cent.; chromium, 1 to 1·5 per cent. In tension the yield point of the annealed tubes varied from 36·4 to 38·5 tons per square inch. In the compression tests the load was transmitted through hardened and ground parallel steel plates; the ends of the tubes were either sunk into the plates or plugged, or both. Extensometer readings taken on opposite sides of the tubes showed whether there was any bending. The authors of the paper referred to concluded that "(1) in the case of the annealed tubes, there is a definite crinkling stress which varies, approximately, with t/R, so long as this does not exceed 0·1; (2) for values of t/R greater than 0·1 the elastic breakdown is identical with the elastic limit of the material"; for the unannealed tubes they conclude that "special attention is necessary when t/R is greater than 0·1." The results of the experiments are shown in Fig. 285. In connection with this, the authors state that "The results for the hard tubes were complicated by the fact that the thicker ones appeared to be softer than the thinner ones. . . . To test this further, a thick tube ($t/R = 0{\cdot}172$) was machined down inside and outside until $t/R = 0{\cdot}055$, and tested." The machined tube collapsed at 80,000 lb. per square inch. For a full account of these experiments the reader is referred to the original paper, where stress strain diagrams and full experimental results are given.

In 1914 experiments were made by W. H. Barling [||] on annealed mild steel tubes, and he came to similar conclusions—namely, that, below a certain critical value of t/R, the crinkling stress is less than the yield stress, and proportional to t/R. The results of his experiments are included in Fig. 285.

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[*] *Phil. Trans. Royal Society*, Series A, vol. 213.

[†] *Proc. Mech. Engineers*, 1909.

[‡] *Proc. Civil Engineers*, 1916–17, Pt. I.

[§] *Proc. R.S.*, Series A, 1928, vol. 121.

[||] See *Aeronautics*, Dec. 4, 1918, or *Royal Aeronautical Society*, Reprint No. 9.

Robertson's experiments do not confirm the results of Popplewell, Carrington and Barling. The experiments were made on $1\frac{1}{4}$ in. × 18 S.W.G. tubes, $3\frac{1}{4}$ in. long, the ends being faced up on a mandril. Mild steel and air-hardening high tensile steel tubes were tested, both annealed and unannealed. The tubes were tested in a special jig to ensure parallelism of the hardened and ground end plates transmitting the load, and every possible precaution was taken to obtain a uniform distribution of strain. Robertson concludes from his experiments that:

(i) For tubes of mild steel having a yield in compression of 22 tons/in.^{2}, the strength depends on the $Y.P.$, not on the wrinkling stress, provided $t/R \nless 0{\cdot}022$.

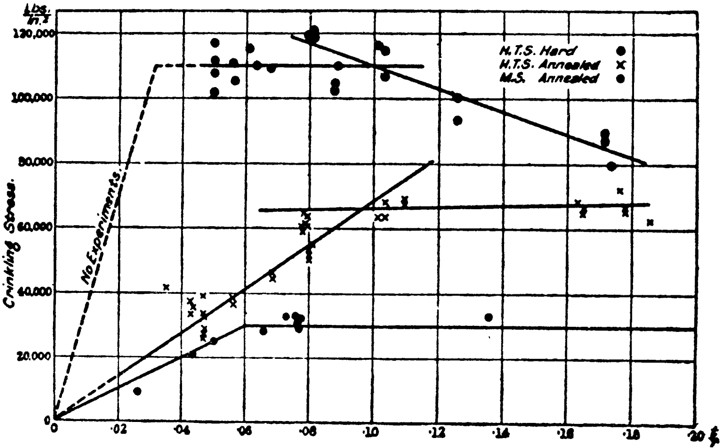

Fig. 285.

(ii) For short specimens of thin tubes, failure will occur either at the yield or at a stress which is some fraction of the Southwell value, varying between 0·4 and 0·6.

(iii) For tubes having t/R greater than the value determined by (ii), yield precedes wrinkling.

(iv) A reasonable estimate of the strength of a steel tubular strut can be made by using the eccentricity formula, and inserting as the stress term the lower of two values—i.e. the yield or an appropriate fraction of the Southwell value. To avoid elastic instability as a primary cause of failure, the value of t/R for steel tubes should be greater than $0{\cdot}4 \times$ Southwell's value of p.

Robertson states in his paper that Southwell's formula for lobed failure is inaccurate, a statement which has been criticized by Carrington,* and Robertson has agreed that Carrington's criticism is sound. For Robertson's criticism of Barling, Webb, Carrington and Popplewell's experiments the reader is referred to the original paper (*Proc. R.S.*, 1928); it is mainly based on the fact that "they used tubes

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* *Phil. Mag.*, June, 1929.

COMPRESSION TESTS ON TUBES (mean radius R, thickness t).

Specimens 3¼ in. long.	t/R.	Elastic Limit.	Yield Point.	Young's Modulus.	Collapse.
Mild steel	·083	29·3	33·3	12,580	38·4
,, ...	·083	28·4	31·2	12,930	36·3
Mild steel annealed at 900° C., cooled in air	·083	21·0	21·4	12,730	27·2
,, annealed at 900° C., cooled in air	·083	22·6	22·6	12,580	—
,, annealed at 900° C., cooled in furnace . . .	·083	17·14	17·14	13,380	23·8
,, annealed at 900° C., cooled in furnace . . .	·083	16·78	16·78	13,280	22·7
Annealed *high tensile* steel* .	·083	33·4	38·5	13,350	43·3
,, ,, ,, .	·083	27·0	29·5	13,070	40·0
,, ,, ,, .	·083	24·6	28·0	12,660	38·6
,, ,, ,, .	·083	28·0	29·5	12,570	—
,, ,, ,, .	·083	30·3	32·4	13,380	—
,, ,, ,, .	·083	29·5	30·8	12,880	—
Hardened *high tensile* steel* .	·083	16·3	No definite yield point	12,340	No collapse below 72 tons /in.^{2}
,, ,, ,, .	·083	34·0		11,620	
,, ,, ,, .	·083	26·0		12,900	
,, ,, ,, .	·083	22·0		12,400	
,, ,, ,, .	·083	22·0		12,780	
,, ,, ,, .	·083	20·9		12,850	

of different diameters and thicknesses and assumed that they were dealing with the same material": they should all have been turned down from thick ones.

REINFORCED CONCRETE STRUTS

241. Short Struts where Bending is Negligible.—If the column be reinforced with rods surrounded by hoops at relatively large distances apart, the calculations can be done in the manner indicated on pp. 14–16. With the Considère system, however, the method given below may be used. The reinforcement consists of steel rods spaced equally round the column, a short distance, inwards, from the surface. The rods are surrounded by a wire taking the form of a helix of small pitch, as shown in Fig. 286.

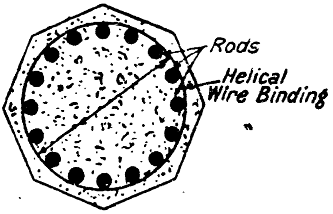

FIG. 286.

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* The steel referred to here is air-hardening steel, and the annealed tube shows no marked drop of stress at yield, such as occurs with mild steel, but there is always an elastic limit (limit of proportionality of stress and strain), and a point where the slope of the stress strain curve becomes very markedly less, and this latter point is taken as the yield. This explains the apparent contradiction by the table of the statement in § 235 that the elastic limit and yield point are generally identical.

Let P = the total load on the column.
,, S = ,, ,, area of the cross section of the column.
,, S_s = ,, ,, ,, ,, ,, ,, ,, rods.
,, p_s = the longitudinal stress in the steel.
,, p_c = ,, ,, ,, ,, ,, concrete.
,, E_s = Young's Modulus for the steel.
,, E_c = ,, ,, ,, ,, concrete.

Then, from § 16, p. 15,

$$p_c = \frac{E_c P}{E_c(S - S_s) + E_s S_s}$$

$$\therefore P = p_c\left\{S + \left(\frac{E_s}{E_c} - 1\right)S_s\right\} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Now, the helical reinforcement can be regarded as behaving as a thin tube; suppose the diameter of the cross section of this tube is d, and the radial thickness of the spiral binding is t. The hoop stress in the tube being p_s, the radial pressure, p_r, between the concrete and the tube, is given by

$$p_s = \frac{2p_r t}{d}$$

$$\therefore p_r = \frac{p_s d}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The procedure now depends on what theory of elastic failure we adopt. Suppose we adopt the maximum shearing-stress theory: then $p_c - p_r$ must not exceed a certain value f. Hence

$$\begin{aligned} p_c &= p_r + f \\ &= \frac{p_s d}{2t} + f \end{aligned}$$

Hence, from (i)

$$\mathbf{P} = \left(\frac{\mathbf{p_s d}}{\mathbf{2t}} + \mathbf{f}\right)\left\{\mathbf{S} + \left(\frac{\mathbf{E_s}}{\mathbf{E_c}} - \mathbf{1}\right)\mathbf{S_s}\right\} \quad . \quad . \quad . \quad . \quad (42)$$

This gives the safe load on the column when the maximum stress in the steel, and the maximum stress-difference in the concrete, are prescribed, and the dimensions are known.

242. Long Struts.—Consider two separate struts: one consisting of the concrete without the reinforcing rods, and the other consisting only of the reinforcing rods braced together in their proper relative positions by imaginary shear bracing. Let I_c and I_s denote the moments of inertia of the two cross sections. In the composite strut the imaginary shear bracing between the steel rods is supplied by the adhesion between the steel and concrete.

Let P_c and P_s denote the load taken by the concrete and steel respectively. Then, if the concrete strut and steel strut, imagined separate, bend to the same shape, which must be the case when they form one composite strut, we must have

$$\frac{P_c}{E_c I_c} = \frac{P_s}{E_s I_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

If this condition is satisfied, the Euler failing load of the combination (§ 213) is given by

$$\mathrm{P_e} = \frac{\pi^2(\mathrm{E_c I_c} + \mathrm{E_s I_s})}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where l is the length of the strut.

If there is no slipping between the steel and concrete, which is a necessary condition if we apply the formula (44), we require

$$\frac{P_c}{P_s} = \frac{E_c S_c}{E_s S_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where S_c is the nett cross section of the concrete. For both (43) and (45) to be true *the radii of gyration of the two imaginary separate struts must be equal.* Hence this is the condition to be aimed at in design.

EXAMPLES XVIII

1. Calculate the Euler failing load of a pin-jointed strut made of round steel rod $\frac{3}{4}''$ diameter and 12 ft. long.

2. An aeroplane wing strut is of streamline section and made of silver spruce. It is 72″ long between pin-joints. The maximum axis of the section is 4·5″ and the minimum 1·5″, and it is parallel throughout its length. Calculate the axial load under which it will fail, given that $I = BD^3/24$, where B and D are lengths of the maximum and minimum axes respectively, and that $E = 1{\cdot}6 \times 10^6$ lbs./in.^{2} (A.F.R.Ae.S. Exam., 1922.)

3. A mild steel column has a cross section such that $S = 50$ in.^{2} and $k = 4''$. If both ends are encastré, and the length is 20 ft., find the load which the column can take, according to the Rankine-Gordon formula. Use the constants given in § 228.

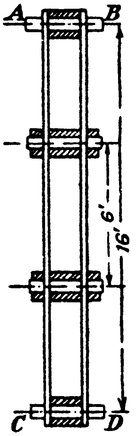
4. In question 2, if the load is eccentric by an amount $h = 0{\cdot}03''$, calculate the load which will produce a maximum compressive stress of 5,000 lbs./in.^{2}

5. Find thickness of a round steel tubular strut 1·5″ external diameter, 6 ft. long, pin-jointed at the ends, to take a load of 2,000 lbs.

6. An encastré strut is built of steel with the section shown in Fig. 288, and the length is 15 ft. In erection the ends are given an initial slope of 1 in 100. If $E = 13{,}000$ tons/in.^{2}, and the permissible stress is 10 tons/in.^{2} calculate the maximum end load. (See Ex. 21.)

7. Calculate the Euler crippling load for a strut encastré at both ends, the cross section being a square $\frac{1}{2}'' \times \frac{1}{2}''$, and the length 5 ft. Take

$$E = 30 \times 10^6 \text{ lb./in.}^2$$



8. Fig. 287 shows a strut composed of two parallel plates each $\frac{3}{4}''$ × 4" kept apart by four distance pieces. The strut is fitted with pin joints *AB, CD,* at its extremities. The ends are maintained vertically above one another and the axes of the pins are kept parallel. Taking E for the material as 30×10^6 lbs./in.^{2}, calculate what vertical load the strut will carry without buckling in a plane perpendicular to the axes of the pins. Assuming that the ends of the struts remain vertical in direction, examine whether the plates in the six-foot length will buckle under a less load. (Mech. Sc. Trip., 1910.)

Consider any other possible modes of failure, assuming that the overall width across the two plates is 4".

FIG. 287.

9. By means of Rankine's formula the buckling load for a hollow cylindrical pillar 20' long, 8" external diameter, 6" internal diameter, is found to be 324·5 tons. The direct crushing strength of the material was taken as 36 tons/in.^{2} and the ends considered fixed. What would be the buckling load of the same pillar, shortened by 5 ft., under the same conditions? (R.N.E.C., Keyham, 1922.)

10. For a strut with riveted ends, axially loaded and having a free length l, it is specified that p the safe load measured in tons per square inch is given by $p = 8\left(1 - \frac{1}{300}\,\frac{l}{k}\right)$ where k is the least radius of gyration of the section. For values of $\frac{l}{k}$ not exceeding 120 and for a material which has an elastic limit of 15 tons per square inch, how does this specification compare with one in which it is laid down that the intensity of stress must not exceed half the elastic limit or one-third the Euler limit for a strut with hinged ends?

Take E as 14,000 tons per sq. inch.

The cross section of a strut is represented by Fig. 288.

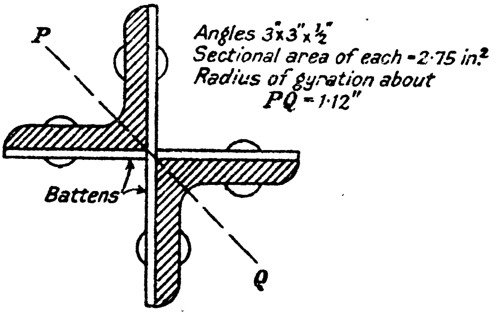

P
Angles 3"x3"x½"
Sectional area of each = 2·75 in.²
Radius of gyration about PQ = 1·12"
Battens
Q

FIG. 288.

The angles are joined by horizontal battens which alternate at intervals down the strut. Using the straight line formula stated above and taking l = 10 ft., find what axial load this strut can safely support.

If a strut is constructed having a section obtained by increasing or decreasing all the linear dimensions given in Fig. 288 in the same ratio, prove that the sectional area S to carry a load P for this same length is given by

$$S = \frac{P}{8} + \cdot 35 + \sqrt{\cdot 123 + \cdot 088P}.$$

(Mech. Sc. Trip., 1923.)

11. Consider a solid steel strut of square cross section, pin-jointed at the ends, with an equivalent eccentricity equal to 0·01 of the side of the cross section. Let the yield point be 15 tons/in.^{2}, and E = 13,500 tons/in.^{2} On a base of P/S plot curves showing the maximum and minimum stress on the central cross section for values of l/k = 30; 80; 150; 200.

12. A strut is made of a steel tube of 2 inches external diameter and $\frac{1}{8}$ inch thickness, and is loaded between pivots at its ends. Draw a graph to show to scale how the limiting load depends on the length—as measured between the pivots. Assume that E=13,000 ton/inch^2 and that plastic yield starts at 16 ton/inch^2. On the same base sketch also a corresponding graph for the loads allowed by the Rankine-Gordon formula. Explain what precautions are required to justify the higher of the two graphs. (R.N.E. College, Keyham, 1937.)

13. A steel scaffolding pole acts as a strut, but the load is applied eccentrically—at 3 in. distance from the centre line with leverages in the same direction at top and bottom. The pole is tubular, 2 in. external diameter and $\frac{1}{4}$ in. thick, 8 feet in length between its ends—which are not fixed in direction. If the steel has yield point 20 ton/inch^2 and E=12,000 ton/inch^2, estimate approximately the load required to buckle the strut. (R.N.E. College, Keyham, 1936.)

14. Two similar members of the same dimensions are connected together at their ends by two equal rigid links as shown in Fig. 289, the links being pin-jointed to the members. At the middle the members are rigidly connected by a distance piece. Equal couples are applied to the links, the axes of the couples being parallel to the pins of the joints. Show that buckling will occur in the top member if the couples M exceed a value given by $\tan\frac{al}{2} = \tanh\frac{al}{2}$, where $a^2 = M/EId$. (Mech. Sc. Trip. B., 1921.)

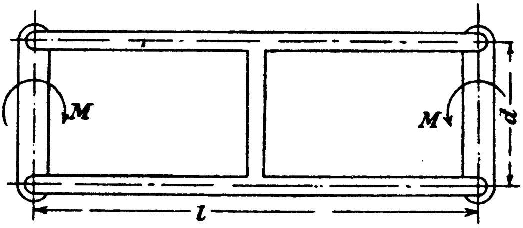


FIG. 289.

15. The ends of a strut are capable of a limited small angular movement to the extent of φ radians, the load is applied axially and the strut is initially straight. Show that, provided the end load is greater than the Euler value of the strut when pin-jointed at its ends, the fixing moments at the ends are given by

$$M_0 = \frac{\varphi P}{a}\cot\frac{al}{2},$$

and that the maximum deflection is

$$\frac{\varphi}{a}\tan\frac{al}{4}$$

in the usual notation.

16. If in erecting a strut with fixed ends, the ends have to be forced into position, with the result that they are acted upon by equal fixing moments M_1, the axis of the strut is bent into a circular arc. The maximum deflection of the axis is then h. Prove that when the load is applied, the fixing moments will be increased by

$$\frac{8Ph}{a^2l^2}\left(\frac{al}{2}\cot\frac{al}{2}-1\right),$$

and that the central deflection becomes

$$\frac{4Ph}{al^2}\tan\frac{al}{4}$$

if h^2/l^2 is negligible.

17. In the case of a strut with one end encastré but the other end fixed in position only (as in § 217), if there is initial curvature such that the unstrained shape of the axis is given by $y_0 = c\sin(\pi x/l)$, show that the bending moment at the encastré end is given by

$$\frac{\pi a^2l^2Pc}{(\pi^2-a^2l^2)(1-al\cot al)}.$$

18. A uniform rod is encastré at one end in a vertical position, and free at the upper end. It is acted on by a vertical load w per unit length. Show that instability will occur when $l = 1{\cdot}99\sqrt[3]{EI/w}$.

19. A uniform rod is pin-jointed at both ends and acted on by a distributed axial load, the thrust at a distance x from either end being $w\left(\frac{l}{2}-x\right)$, where l is the length of the rod. Show that the critical load is given by $wl^2/8 = \pi^2EI/0{\cdot}4816l^2$ . . . (Yassinski.)

20. A strut ACB is pin-jointed at both ends, and the point C is free to deflect in the horizontal plane, which contains A and B, but it cannot move at right angles to this plane. Thrusts P and Q are applied at A and B along the line AB, and a thrust $P - Q$ is applied at C in a direction parallel to AB. The cross section is uniform throughout.

Let $BC/CA = m$; $Q/P = n^2$; $P/EI = a^2$; $AC = l$. Show that the critical value of P is given by *

$$al\left(\cot al+\frac{1}{n}\cot mnal\right)=\frac{(n^2-1)^2}{n^2(m+n)^2}.$$

Hence show that, when $m = 1$ and $n = 0{\cdot}5$, the error made by supposing the whole strut to carry the mean thrust $\frac{1}{2}(P+Q)$ is about 2 per cent.

21. Two uniform struts AB and CD are pin-jointed at the ends for flexure in the plane $ABCD$. A cross member EF connects the two, being hinged to both at their middle points E, F.

Let P = the thrust in AB, m^2P = the thrust in CD;
I = the moment of inertia of the cross section of AB, and n^2I = that of CD.
$a^2 = P/EI$.

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* This result is due to Miss B. M. Cave-Browne-Cave.

Show that instability will occur when

$$\frac{\tan al}{al} + \frac{1}{m^2}\left(\frac{\tan\dfrac{mal}{n}}{\dfrac{mal}{n}} - 1\right) = 1.$$

If the load on CD be a tension, and we write $m^2 = -\mu^2$, write down the corresponding equation. If $m = 0$, show that the criterion for instability is

$$\frac{\tan al}{al} + \frac{(al)^2}{3n^2} = 1$$

Hence show that, for all practical purposes, when m^2 is not less than $-0{\cdot}5$, the error will not exceed about 5 per cent. if we use the formula $P_1 + P_2 = \pi^2 E(I_1 + I_2)L^2$, where P_1 and P_2 denote the two (positive or negative) end loads, and I_1 and I_2 denote the two moments of inertia.*

22. A uniform strut whose axis initially has the shape given by

$$y_0 = \frac{c}{2}\left(1 - \cos\frac{2\pi x}{l}\right),$$

with the notation of § 216, is encastré at both ends. Show that the bending moments at the ends and middle of the strut are equal, and are given by $2\pi^2 cP/(4\pi^2 - a^2l^2)$, provided $al < 2\pi$. (Salmon.)

23. If the axis of a uniform strut initially has the form $y_0 = c\sin\frac{\pi x}{l}$, and the ends are encastré so that each makes an angle $\pi c/l$ with the line of thrust, show that the bending moments at the ends are given by

$$M_0 = \pi calP.\cot\frac{al}{2}./(\pi^2 - a^2l^2),$$

whilst the bending moment at the centre is given by

$$M_c = \pi cP.\left(\pi - al\operatorname{cosec}\frac{al}{2}\right)/(\pi^2 - a^2l^2).$$

Show that M_0 and M_c do not become infinite when $al = \pi$, and that $M_c < M_0$, but $\rightarrow M_0$ as $al \rightarrow 2\pi$. Show also that the conditions of this problem give a worse case than the conditions of Example 20. (Salmon.)

24. The jib of a derrick makes an angle of 45° with its vertical supporting post, and the head of the jib, which carries the pulley for the purchase, is supported by a horizontal tie rope 20 ft. long, attached to the top of the post. The lead in of the purchase, which is a single hawser, lies above the jib in the same vertical plane, and makes an angle of 15° with it. The jib is of solid circular section 1 ft. diameter, wood; elastic limit 3·2 tons/in.^{2} "a" in Rankine-Gordon formula 1/750. Determine the purchase load that will cause failure of the jib. (R.N.E.C., Keyham, 1928.)

25. A strut is acted on by an end thrust P and terminal bending couples M_1 and M_2. Show that this is equivalent to a pin-jointed strut with the load eccentric at each end by amounts M_1/P and M_2/P, and that the bending moment is given by

$$M = M_1\frac{\sin a(l-x)}{\sin al} + M_2\frac{\sin ax}{\sin al}.$$

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* Case and Griffiths: *Aeronautical Research Committee*, R. & M., 403.

CHAPTER XIX

TAPERED STRUTS

243. Introductory.—In a pin-jointed strut failure always occurs on account of the stress exceeding a certain limit, at least in all the struts that are ever used in engineering or building, on account of a real or fictitious eccentricity, although it happens that if the ratio l/k exceed a certain value, and eccentricity of loading be small, the strength may be calculated from the criterion of stability without appreciable error. The stress in a strut is due partly to direct thrust and partly to bending moments arising from the deflection. The bending moment on any section is proportional to the deflection of the centroid of that section. Consequently, if the section of the strut be uniform, the stress will be greatest on the central cross section. In view of this it appears that we could obtain a more economical strut by designing it so that the maximum stress on all sections is the same, by tapering the strut towards the ends. Such a strut will be called a strut of uniform stress. A great variety of tapered struts are used in practice, frequently showing a considerable lack of understanding on the part of their designers. The two commonest are those which have an elliptic profile, and those which taper in straight lines with or without a parallel central portion, and it is this type which is most likely to be ill-designed. The problem of the lightest strut has been exhaustively treated * by H. A. Webb and W. H. Barling in connection with aeroplane design, and the reader is referred to their work. We shall consider here the strut of uniform stress, and then pass on to a few remarks about elliptically tapered and straight-tapered struts. Finally, we shall show how to find the strength of any given strut of non-uniform section.

STRUTS OF UNIFORM STRESS

244. General Equations.—We assume that the load has an eccentricity h, that the maximum stress on every cross section is to have the same value under a given end load, and consider the problem of finding the shape of the strut in side elevation.

In Fig. 290 AOB is the axis of the strut, and flexure takes place in the plane of the paper. The planes through AB and Oy, perpendicular to the paper, are planes of symmetry.

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* See *Aeronautics*, Dec. 4, 11, 1918; *Royal Aeronautical Society*, Reprint No. 9; *Aeronautic Research Committee's Reports*, R. & M., 343 and 363.

Let b = the thickness, in the plane of bending, at a distance x from the mid-section.
I = the moment of inertia of the same section.
S = the area of the same section.
f = the maximum permissible stress.
P = the end load.
E = Young's Modulus.

We shall use the suffix 0 to denote the central section, and the suffix 1 to denote the end sections.

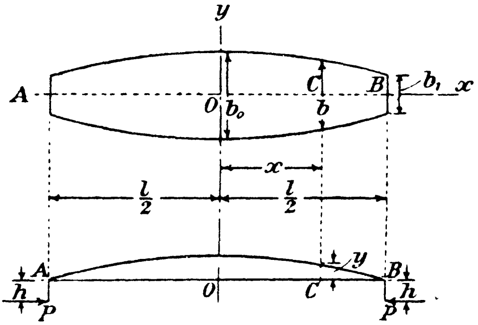


FIG. 290.

The bending moment on the section C is $P(y + h)$, hence at every section we require

$$\frac{P}{S} + \frac{P(y + h)b}{2I} = f. \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

The equation of the deflection curve is

$$-EI\frac{d^2y}{dx^2} = P(y + h) \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

From these two equations we have to find the shape of the strut, i.e. we must find an equation connecting b and x.

From (i) we have

$$\frac{P(y + h)}{2f} = \frac{I}{b}\left(1 - \frac{P}{fS}\right) \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

Differentiate this twice:

$$\frac{P}{2f}\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{I}{b}\left(1 - \frac{P}{fS}\right)\right\} = \frac{d}{db}\left\{\frac{I}{b}\left(1 - \frac{P}{fS}\right)\right\}\frac{db}{dx}.$$

$$\frac{P}{2f}\cdot\frac{d^2y}{dx^2} = \frac{d}{db}\left\{\frac{I}{b}\left(1 - \frac{P}{fS}\right)\right\}\frac{d^2b}{dx^2} + \frac{d^2}{db^2}\left\{\frac{I}{b}\left(1 - \frac{P}{fS}\right)\right\}\left(\frac{db}{dx}\right)^2 \quad \text{(iv)}$$

Substitute $\frac{d^2y}{dx^2}$ and $(y + h)$ from (iii) and (iv) in (ii):

$$\frac{d}{db}\left\{\frac{I}{b}\left(1 - \frac{P}{fS}\right)\right\}\frac{d^2b}{dx^2} + \frac{d^2}{db^2}\left\{\frac{I}{b}\left(1 - \frac{P}{fS}\right)\right\}\left(\frac{db}{dx}\right)^2 = -\frac{P}{EI}\cdot\frac{I}{b}\left(1 - \frac{P}{fS}\right) \quad \text{(v)}$$

Y

This is the differential equation of the meridian curve of the strut in the plane of bending. A first integral is obtainable on multiplying by

$$2\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}\cdot\frac{db}{dx}$$

We get

$$\left(\frac{db}{dx}\right)^2\left[\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}\right]^2=\int_b^{b_0}\frac{P}{EI}\cdot\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}^2 db.$$

This satisfies the condition $\frac{db}{dx}=0$, when $b=b_0$, which is necessary if the shape of the strut is to be a smooth curve.

From this we have

$$\sqrt{\frac{P}{E}}\cdot\frac{dx}{db}=\frac{\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}}{\sqrt{\int_b^{b_0}\frac{1}{I}\frac{d}{db}\left\{\left(1-\frac{P}{fS}\right)\frac{I}{b}\right\}^2 db}}$$

Integrating this we have

$$x\sqrt{\frac{P}{E}}=\int_b^{b_0}\frac{\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}}{\sqrt{\int_b^{b_0}\frac{1}{I}\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}^2 db}}\cdot db \quad . \quad (1)$$

which satisfies the condition $x=0$ when $b=b_0$.

At the end of the strut, when $x=\frac{l}{2}$ and $y=0$, we have from (iii)

$$\frac{I_1}{b_1}\left(1-\frac{P}{fS_1}\right)=\frac{Ph}{2f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation gives b_1 since I_1 and S_1 are known functions of b_1.

Then, since $b=b_1$ when $x=\frac{l}{2}$, we must have from (vi)

$$\frac{l}{2}\sqrt{\frac{P}{E}}=\int_{b_1}^{b_0}\frac{\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}}{\sqrt{\int_b^{b_0}\frac{1}{I}\frac{d}{db}\left\{\frac{I}{b}\left(1-\frac{P}{fS}\right)\right\}^2 db}}\cdot db \quad . \quad . \quad (3)$$

which is an equation for finding b_0, the thickness of the strut at the centre.

The thickness at any other point is then given by (1), whilst the deflection curve is given by (iii). We have thus found a complete general solution of the problem.

If the eccentricity is neglected (1) and (3) remain unaltered, but (2) reduces to

$$P=fS_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These equations appear formidable, but we shall see that they are quite manageable in certain useful cases.

245. Solid Strut of Uniform Stress.—Let all the cross sections be similar and similarly situated curves so that we can write

$$S = ab^2$$
$$I = \beta b^4$$

where a and β are constants. Also write

$$b = b_0 \cos \varphi$$

Then (1) and (3) become

$$x\sqrt{\frac{P}{3EI_0}} = \int_0^{\phi} \frac{\left(\cos^2 \varphi - \dfrac{P}{3fS_0}\right)d\varphi}{\sqrt{1 + \dfrac{8P \log_e \cos \varphi}{3fS_0 \sin^2 \varphi} + \dfrac{P^2 \sec^2 \varphi}{3f^2S_0^2}}} \qquad (5)$$

and

$$\frac{l}{2}\sqrt{\frac{P}{3EI_0}} = \int_0^{\phi_1} \frac{\left(\cos^2 \varphi - \dfrac{P}{3fS_0}\right)d\varphi}{\sqrt{1 + \dfrac{8P \log_e \cos \varphi}{3fS_0 \sin^2 \varphi} + \dfrac{P^2 \sec^2 \varphi}{3f^2S_0^2}}} \qquad (6)$$

where $b_1 = b_0 \cos \varphi_1$.

If the eccentricity be neglected we have

$$P = fS_1,$$

and therefore

$$\frac{P}{fS_0} = \frac{S_1}{S_0} = \frac{b_1^2}{b_0^2} = \cos^2 \varphi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the eccentricity cannot be neglected (2) gives

$$\frac{b_1^2}{b_0^2} = \frac{P}{fS_0}\left(1 + \frac{ah}{2\beta b_1}\right) = \cos^2 \varphi_1 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In the one case φ_1 is given in terms of $\dfrac{P}{fS_0}$ by (7), in the other by (8), so that in both cases (6) gives $\dfrac{l}{2}\sqrt{\dfrac{P}{3I_0}}$ as a function of $\dfrac{P}{fS_0}$ only. Hence, for a given value of $\dfrac{P}{fS_0}$, I_0, and so b_0, can be found by graphical integration, plotting the integrand of (6) on a base of φ.

Again, for a given value of $\dfrac{P}{fS_0}$, (5) gives $x\sqrt{\dfrac{P}{3EI_0}}$ as a function of $\dfrac{b}{b_0}$, so that by graphical integration we can find $2x/l$ for any values of b/b_0. Thus we can draw the shape of the strut. Curves obtained in this way by Webb and Barling are shown in Figs. 291 and 292.

To design a tapered strut of this type we must obtain a first approximation to the value of P/fS_0 to settle which curve is to be used.

When the eccentricity is negligible we obtain the required approximation thus. Assume that the strut is so long that only stability failure

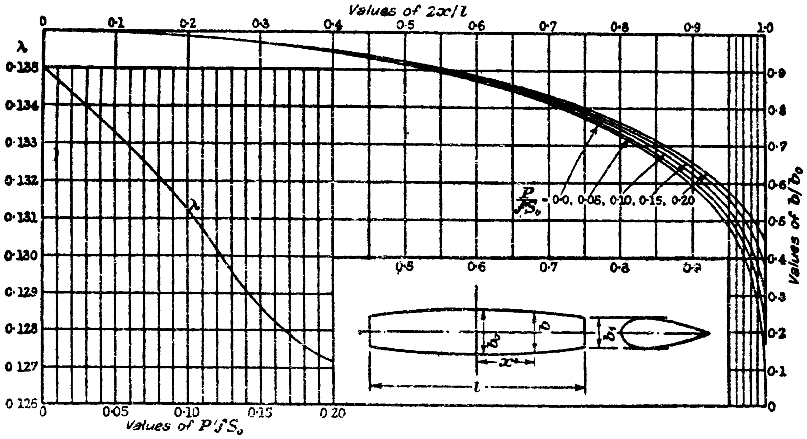


FIG. 291.

need be considered ; this corresponds with taking $f = \infty$. We then have from (6), since b_1 is then zero and so $\varphi_1 = \frac{\pi}{2}$.

$$\frac{l}{2}\sqrt{\frac{P}{3EI_0}} = \int_0^{\frac{\pi}{2}} \cos^2 \varphi \,.\, d\varphi = \frac{\pi}{4},$$

or

$$\mathbf{P} = \frac{3}{4}\frac{\pi^2 \mathbf{E} \mathbf{I}_0}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which gives

$$\mathbf{b}_0{}^4 = 0{\cdot}135\frac{l^2\mathbf{P}}{\beta\mathbf{E}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This gives an approximate value of b_0, and hence of S_0, so that a first approximation to the value of P/fS_0 is obtained.

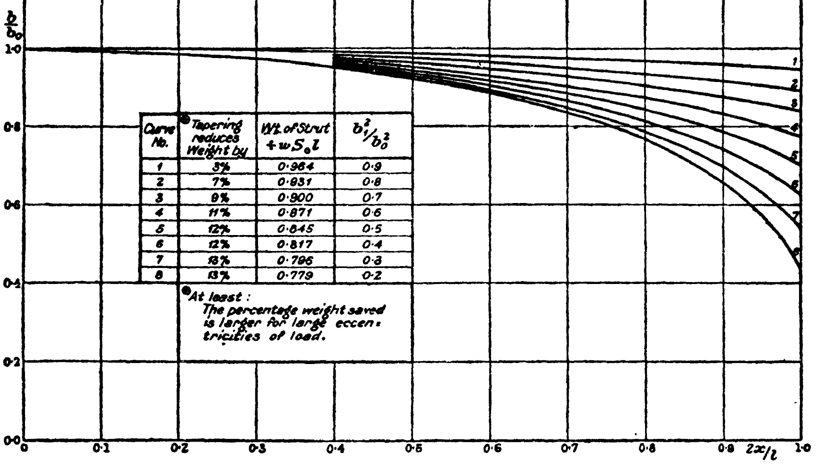
When stress failure is considered we obtain instead

$$\mathbf{b}_0{}^4 = \lambda\frac{l^2\mathbf{P}}{\beta\mathbf{E}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where λ is a constant given in the left-hand part of Fig. 291.

The question arises in connection with the above: When can the eccentricity be neglected ? Webb and Barling find that an eccentricity $h = 0{\cdot}01\, b_1$ will not affect the strength of a tapered strut by more than 1 per cent. provided $P/fS_0 \ngtr 0{\cdot}22$, and, accordingly, decide to neglect eccentricity if $P/fS_0 \ngtr 0{\cdot}2$.

With short tapered struts the eccentricity will not in general be negligible, but it is found that with given values of l, b_0 and b_1 the curves giving the shapes of the struts are nearly identical for all values of h, although, of course, the failing load of a given strut depends very much on the value of h. The curves given in Fig. 292 are drawn for $h = 0$, but if they are used for other values of h the error in b will not exceed



Curve No.	*Tapering reduces Weight by	Wt. of Strut $\div wS_0l$	b_1^2/b_0^2
1	3%	0·964	0·9
2	7%	0·931	0·8
3	9%	0·900	0·7
4	11%	0·871	0·6
5	12%	0·845	0·5
6	12%	0·817	0·4
7	13%	0·796	0·3
8	13%	0·779	0·2

*At least: The percentage weight saved is larger for large eccentricities of load.

FIG. 292.

0·2 per cent. In these cases b_1 is calculated from (8), which may be written

$$b_1{}^2 = \frac{P}{fa}\left(1 + \frac{ah}{2\beta b_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

When $\frac{h}{b_1} > 0{\cdot}03$ and $\frac{b_1}{b_0} > 0{\cdot}5$, Webb and Barling give the following formula for calculating b_0:

$$b_0 = \left(0{\cdot}933 + 0{\cdot}163\frac{ah}{\beta b_1}\right)\sqrt{\frac{P}{fa}} + \left(0{\cdot}191 - 0{\cdot}019\frac{ah}{\beta b_1}\right)\frac{l}{2}\sqrt{\frac{fa}{E\beta}} \quad (13)$$

When $\frac{h}{b_1} < 0{\cdot}03$, b_0 must be found by interpolation from the curves shown in Fig. 293.

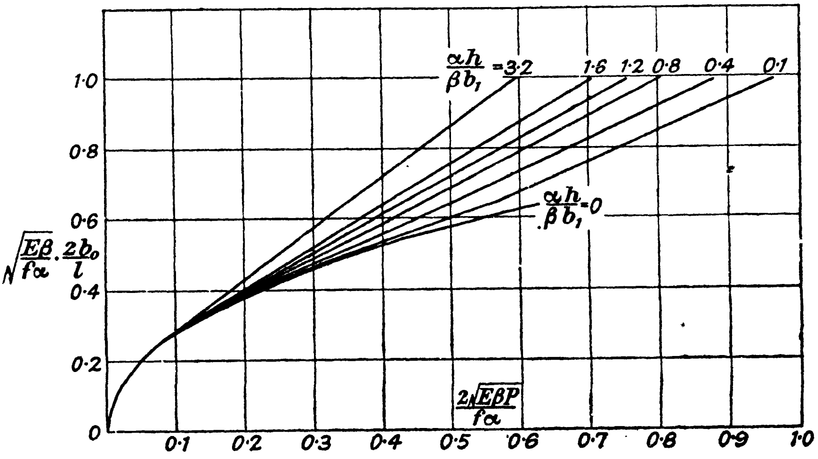


FIG. 293.

From the foregoing we can draw up the following rules for design :

(*a*) Find a first approximation to b_0 from (10) and the corresponding value of P/fS_0.

(*b*) i. If $P/fS_0 < 0{\cdot}2$ find λ from Fig. 291 for this value of P/fS_0. Hence find a corrected value of b_0 from (11) and the new value of P/fS_0.

With this value of P/fS_0 choose the appropriate curve for the profile from Fig. 291.

(*b*) ii. If the first value of $P/fS_0 > 0{\cdot}2$, we must estimate as well as possible the value of h/b_1. Then b_1 is found from (12) and b_0 from (13) or Fig. 293.

Next find the value of b_1^2/b_0^2 and select the profile from Fig. 292.

The weight of tapered struts designed on the principles given here is found from Fig. 294, where w is the weight per unit volume.

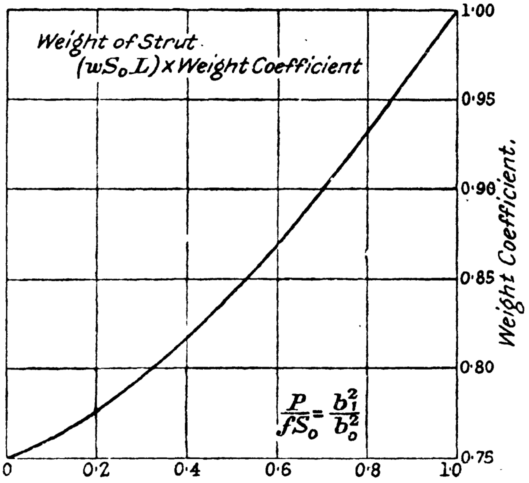


FIG. 294.

Example 1.—Design a solid pin-jointed strut of uniform strength to take an end load of 4,000 lbs., the distance between the pins being 70″. The strut is to be of timber for which $f = 5{,}500$ lbs./in.^{2}, and $E = 1{\cdot}6 \times 10^6$ lbs./in.^{2} The cross sections are such that $\alpha = 2{\cdot}8$ and $\beta = 0{\cdot}178$.

As a first approximation to b_0 we have from (9) :

$$4{,}000 = \frac{0{\cdot}75\pi^2 \times 1{\cdot}6 \times 10^6 \times 0{\cdot}178 b_0^4}{4{,}900}$$

which gives

$$b_0^4 = 9{\cdot}3 \text{ in.}^4$$
$$b_0^2 = 3{\cdot}05 \text{ in.}^2$$
$$b_0 = 1{\cdot}75 \text{ ins.}$$

Then $S_0 = 2{\cdot}8 \times 3{\cdot}05 = 8{\cdot}55$ in.^{2}

$$\therefore \frac{P}{fS_0} = \frac{4{,}000}{5{,}500 \times 8{\cdot}55} = 0{\cdot}0852.$$

Hence, from Fig. 291, $\lambda = 0{\cdot}132$, and the small value of P/fS_0 shows that we can neglect any fictitious eccentricity. We now get a second approximation to b_0 from (11)

$$b_0^4 = \frac{0{\cdot}132 \times 4{,}900 \times 4{,}000}{0{\cdot}178 \times 1{\cdot}6 \times 10^6} = 9{\cdot}09 \text{ ins.}^4$$

$$\therefore\ b_0^2 = 3{\cdot}015 \text{ in.}^2, \text{ and } b_0 = 1{\cdot}736 \text{ ins.}$$

$$\frac{P}{fS_0} = \frac{4{,}000}{5{,}500 \times 2{\cdot}8 \times 3{\cdot}015} = 0{\cdot}086.$$

The shape of the strut is now found from the curves of Fig. 291 by interpolation :—

x	0	7″	14″	21″	25″	30″	32″	34″	35″
$2x/l$	0	0·2	0·4	0·6	0·714	0·86	0·914	0·97	1·0
b/b_0	1	0·99	0·95	0·88	0·81	0·71	0·595	0·415	0·28
b	1·736″	1·72″	1·65″	1·53″	1·405″	1·23″	1·03″	0·72″	0·49″

The strut would probably be made with $b_0 = 1{\cdot}75''$, the other values of b being increased in proportion. This strut is about 13 per cent. lighter than a parallel strut of the same strength.

Example 2.—Design a solid pin-jointed strut of uniform strength for an end load of 11,000 lbs., the length between the pins being 48″. Assume that the eccentricity of loading is $0{\cdot}1b_1$ and take the same values of the constants as in Example 1.

We have

$$\frac{ah}{\beta b_1} = \frac{2{\cdot}8 \times 0{\cdot}1}{0{\cdot}178} = 1{\cdot}57$$

$$\frac{P}{fa} = \frac{11{,}000}{5{,}500 \times 2{\cdot}8} = 0{\cdot}715$$

$$\frac{fa}{E\beta} = \frac{15{,}400}{0{\cdot}285 \times 10^6} = 0{\cdot}054.$$

From equation (13) we get, then,

$$\begin{aligned} b_0 &= (0{\cdot}933 + 0{\cdot}256)0{\cdot}845 + (0{\cdot}191 - 0{\cdot}0298) \times 24 \times 0{\cdot}233 \\ &= 1{\cdot}91''. \end{aligned}$$

Next, from (12) :—

$$b_1^2 = 0{\cdot}715(1 + 0{\cdot}785) = 1{\cdot}275$$

$$\therefore\ b_1 = 1{\cdot}13''.$$

Hence $b_1^2/b_0^2 = 0{\cdot}35$.

The profile of the strut is a curve which is half-way between curves 6 and 7 in Fig. 292. By interpolation we have :—

$2x/l$	0·0	0·2	0·4	0·6	0·8	0·9	1·0
b/b_0	1·0	0·99	0·96	0·90	0·80	0·715	0·594
x ins.	0	9·6	19·2	28·8	38·4	43·2	48
b ins.	1·91	1·89	1·83	1·72	1·53	1·37	1·14

246. Tapered Hollow Struts of Uniform Thickness.

—We shall next consider hollow struts made of material of uniform small thickness.

Let t = the thickness of the material. Then if t is so small that t^2/b^2 is negligible we can write

$$S = abt$$

$$I = \beta b^3 t.$$

Also let $b = b_0 \cos^2 \varphi$.

Then the equations for the shape of the strut (1) and (3) become

$$x\sqrt{\frac{P}{4EI_0}} = \int_0^{\phi} \frac{\left(\cos^2\varphi - \frac{P}{2fS_0}\right)\cos\varphi\,.\,d\varphi}{\sqrt{1 + \frac{3P\log_e\cos\varphi}{fS_0\sin^2\varphi} + \frac{P^2\sec^2\varphi}{2f^2S_0^2}}} \quad . \quad . \quad (14)$$

$$\frac{l}{2}\sqrt{\frac{P}{4EI_0}} = \int_0^{\phi_1} \frac{\left(\cos^2\varphi - \frac{P}{2fS_0}\right)\cos\varphi\,.\,d\varphi}{\sqrt{1 + \frac{3P\log_e\cos\varphi}{fS_0\sin^2\varphi} + \frac{P^2\sec^2\varphi}{2f^2S_0^2}}} \quad . \quad . \quad (15)$$

where $b_1 = b_0\cos^2\varphi_1$.

Also, for a very long strut when the eccentricity can be neglected,

$$\text{and}\quad \left.\begin{aligned} P &= fS_1 \\ \frac{P}{fS_0} &= \frac{S_1}{S_0} = \frac{b_1}{b_0} = \cos^2\varphi_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

When the eccentricity is not negligible we find from (2)

$$\frac{b_1}{b_0} = \frac{P}{fS_0}\left(1 + \frac{ah}{2\beta b_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The shapes of the struts, when $h = 0$, can be drawn from these equations in the same way as before for given values of P/fS_0, using φ as an auxiliary variable for the graphical integrations involved. The curves obtained by Webb and Barling in this way are shown in Fig. 295. In this case they decide, from the same considerations as above (p. 332), to neglect any fictitious eccentricity if $P/fS_0 \ngtr 0{\cdot}4$.

A first approximation to the value of b_0 is found by considering only stability failure, i.e. putting $f = \infty$ and $\varphi_1 = \pi/2$ in (15); this gives

$$\frac{l}{2}\sqrt{\frac{P}{EI_0}} = \int_0^{\frac{\pi}{2}} \cos^3\varphi\,.\,d\varphi = \frac{4}{3}$$

Hence

$$P = \frac{64}{9}\frac{EI_0}{l^2} = \frac{64E\beta b_0^3 t}{9l^2}$$

Therefore a first approximation to b_0 is given by

$$\mathbf{b}_0^3 = \frac{9}{64}\frac{l^2\mathbf{P}}{\mathbf{E}\beta\mathbf{t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When stress failure is considered as well, we have

$$\mathbf{b}_0^3 = \lambda\frac{l^2\mathbf{P}}{\mathbf{E}\beta\mathbf{t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where λ is a constant given by the curve on the left of Fig. 295.

We have thus the following rules for designing struts of this type:

(*a*) Find a first approximation to b_0 from (18), and the corresponding value of P/fS_0.

(*b*) With this value of P/fS_0 find λ from Fig. 295.

(*c*) Find b_0 more accurately from (19), and the new value of P/fS_0.

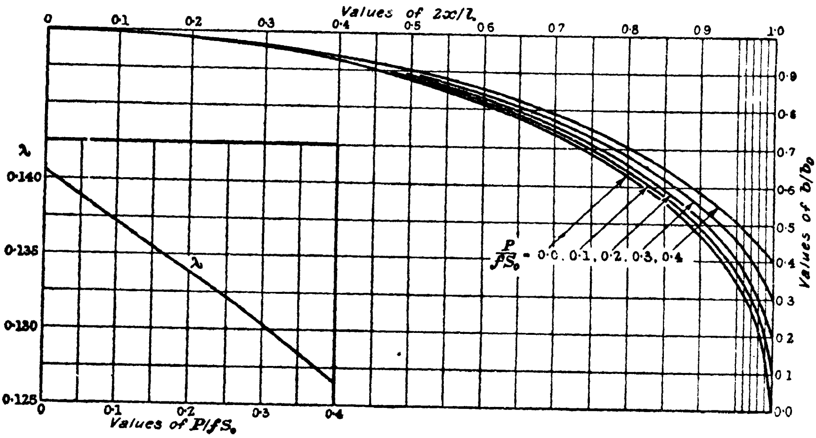


FIG. 295.

(*d*) The profile of the strut is then given by selecting a suitable curve from Fig. 295.

If required, the weight of the strut is $\mu w S_0 l$, where w is the weight per unit volume, and μ is a constant given by Fig. 296.

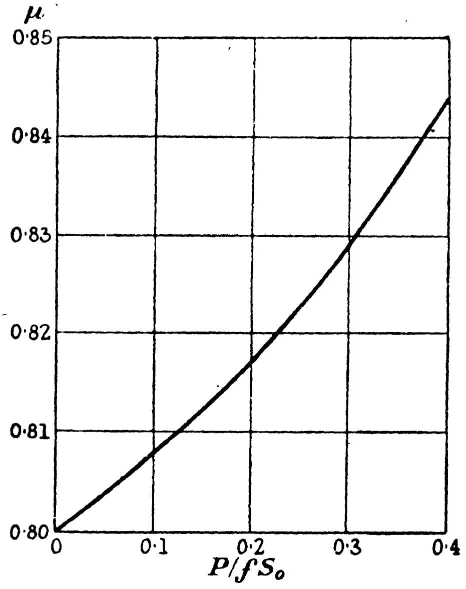


FIG. 296.

Example.—Design a tapered tubular pin-jointed steel strut of circular cross section to take an end load of 120 tons, the length between the pins

80 ft., and the thickness of the walls of the tube being 1 in. Take $f = 13{\cdot}5$ tons/in.^{2}, and $E = 13{,}500$ tons/in.^{2} Estimate the saving of weight effected by tapering.

For thin circular tubes we have approximately :

$$S = \pi bt \text{ and } I = \frac{\pi b^3 t}{8}$$

where b is the mean diameter. Hence

$$\alpha = \pi \text{ and } \beta = \frac{\pi}{8} = 0{\cdot}393.$$

From (18), since $l = 960''$, and $t = 1''$,

$$b_0^3 = \frac{9 \times 960^2 \times 120}{64 \times 13{,}500 \times 0{\cdot}393 \times 1} = 2{,}940 \text{ in.}^3$$

$$\therefore\ b_0 = 14{\cdot}33''$$

Then $$S_0 = 14{\cdot}33\pi, \text{ in.}^2$$

$$\frac{P}{fS_0} = \frac{120}{13{\cdot}5 \times 14{\cdot}33\pi} = 0{\cdot}198$$

From Fig. 295, $\lambda = 0{\cdot}134$.
Hence, from (19), $b_0^3 = 2{,}800$ in.^{3}

$$b_0 = 14{\cdot}1''$$

This gives $S_0 = 14{\cdot}1\pi$, in.^{2} and $\dfrac{P}{fS_0} = 0{\cdot}2$.

As this is so nearly equal to the previous value no further correction is necessary. The profile of the strut is then given by the curve marked $P/fS_0 = 0{\cdot}2$ in Fig. 295. The diameter 15″ from the ends will be 4·8″.

The area of the middle cross section is $S_0 = 14{\cdot}1 \times \pi = 44{\cdot}2$ in.^{2}, the length is 960″, and Fig. 296 gives $\mu = 0{\cdot}817$. Taking the weight of steel as 490 lbs./ft.^{3}, the weight of the tapered strut is

$$(0{\cdot}817) \times 490 \times \frac{44{\cdot}2 \times 960}{1{,}728} = 9{,}800 \text{ lb.} = 4{\cdot}40 \text{ tons.}$$

For a parallel strut we should have

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^3 E b^3 t}{8l^2}$$

giving $b = 12{\cdot}83''$, whence $S = 40{\cdot}4$ ins.^{2}

The weight would be

$$\frac{40{\cdot}4 \times 960 \times 490}{1{,}728 \times 2{,}240} \text{ tons} = 4{\cdot}94 \text{ tons.}$$

Hence the weight saved by tapering is 0·54 tons or 11 per cent. of the weight of the parallel strut.

247. Elliptically Tapered Struts.—Tapered struts are sometimes made with their longitudinal section in the form of an ellipse with its ends cut off. This is a bad type of strut as, even with "long" struts, stress failure will probably occur near the ends before the Euler crippling load is reached. As they are no easier to design or make than struts of uniform stress we shall not discuss elliptic struts in detail. We must point out, however, that when the failing load of a strut of arbitrary taper is required, it is extremely unreliable to draw an ellipse that more or less fits the profile of the given strut, and to calculate the Euler failing load of this elliptic strut. A full treatment of the subject has been given by the author in *Aeronautics.*

STRUTS WITH OTHER TAPERS

248. Rules for Design of Straight-Taper Struts.—The most simple kind of tapered strut is one in which the longitudinal section consists of a central parallel portion and two conical ends, as shown in Fig. 297, *a*. The length of the parallel portion may vary from l, the whole length of the strut, in which case we have a simple straight strut, to zero, in which case we have the double conical strut shown in Fig. 297, *b*.

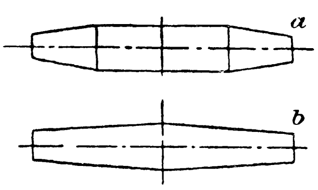


FIG. 297.

These struts have been considered by H. A. Webb and E. D. Lang,* who give rules for the optimum design; struts of the type *b*, Fig. 297, have been approximately treated by Pippard and Pritchard,† and accurately by A. Berry. Webb and Miss Lang show that the lightest possible straight-tapered strut is obtained when:

(i) The length of the parallel portion is one-half the total length.

(ii) The end thickness b is one-half the thickness b_0 of the parallel portion.

(iii) The thickness of the parallel portion is given by

$$b_0{}^4 = \frac{l^2P}{8E\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

with our previous notation.

Such a strut will have a weight only about 1 per cent. in excess of the weight of the ideal tapered strut. The saving of weight compared with a straight strut is slightly more than 12 per cent., and this is the best that can be obtained with a straight taper. A strut of the type *b* (Fig. 297) will be heavier than this; the maximum possible saving of weight in this case is about 10·5 per cent., when b_1/b_0 equals about 0·58. These remarks will apply only to "long" solid struts, when effects of eccentricity are comparatively unimportant. The reader must be reminded that the wrong taper may be worse than no taper, and this is of great importance.

The rules just given enable us to design a strut which will not become unstable before a certain end load is reached. In dealing with tapered struts, however, we should consider whether failure may not occur on account of the maximum stress at some section in the tapered part exceeding the maximum stress on the central section. The present author has investigated this matter and has found that the greatest stress in a strut designed according to the above rules occurs somewhere in the last tenth of the length from either end. This suggests that, in the absence of experimental evidence, it will hardly be safe to rely on the above rules.

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* *Royal Aeronautical Society,* Reprint No. 12.

† *Aeroplane Structures,* p. 227.

Safe rules are the following: (i) as above; (ii) make the thickness at the ends $= 0{\cdot}6 \times$ the thickness at the centre; (iii) as above; always provided $P/fS_0 < 0{\cdot}2$.*

249. To Find the Failing Load of a Solid Strut of Given Shape.—

We have remarked that, except in the case of the strut of uniform stress, the Euler crippling load is not necessarily the least load which will cause stress failure in long tapered struts, and that the actual failing load may be considerably less than the Euler load. The surest practical method of finding the failing load of a given tapered strut, which is not designed as in § 245, consists of three stages:

(i) Find the Euler crippling load in the manner indicated below; let this be P_e.

(ii) Draw a strut of uniform stress to take this load P_e as shown in § 245.

(iii) Compare the stresses in the two struts at all points in the length.

We require, then, to be able to find the Euler failing load of a strut of any given arbitrary shape. One method will be described below.

250. To Find the Euler Crippling Load of a Strut Symmetrical about the Central Section.†

—Let ACB (Fig. 298) be axis of the deflected strut under the action of an end load P acting along the line AOB. Take the origin O at the centre of the line AB.

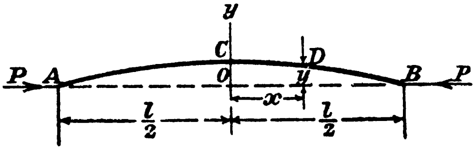

FIG. 298.

The ends A and B are supposed pin-joints. Then, at any point D on the axis we have

$$-EI\frac{d^2y}{dx^2} = Py \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where I is the moment of inertia of the cross section at D and is a function of x. Then, integrating both sides of (i) we have

$$\frac{dy}{dx} = -\int_0^x \frac{Py}{EI}\,dx,$$

---

* For further treatment of the problem of tapered struts the reader is referred to the following: *Aeronautics in Theory and Experiment*, Cowley and Levy, Ch. ix. Articles by A. Morley in *Engineering*, April 24, 1914, and Sept. 21, 1917; also an article by Bairstow and Stedman, *Engineering*, Oct. 2, 1914.

† See also *Aeronautical Research Committee*, Report No. 543, by A. A. Griffith.

the constant of integration being zero, since $\frac{dy}{dx} = 0$ when $x = 0$. Integrating both sides of this we have

$$y = \int_x^{\frac{l}{2}} \int_0^x \frac{Py}{EI} dxdx \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

since $y = 0$ when $x = \frac{l}{2}$.

Let y_0 be the deflection at the centre, and assume

$$y = y_0 \cos \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

This gives the correct value of y at the centre and ends of the strut, and the difference between the actual and assumed deflection curves will nowhere be large. Consequently the difference between their second integrals will be very small indeed. From (ii) and (iii) we have, then

$$y_0 = \int_0^{\frac{l}{2}} \int_0^x \frac{Py_0}{EI} \cos \frac{\pi x}{l} \, . \, dx \, . \, dx.$$

Hence

$$\frac{E}{P} = \int_2^{\frac{l}{2}} \int_0^x \frac{\cos \frac{\pi x}{l}}{I} dx \, . \, dx$$

or *

$$\frac{E}{P} = \int_0^{\frac{l}{2}} \frac{\left(\frac{l}{2} - x\right) \cos \frac{\pi x}{l}}{I} dx \quad . \quad . \quad . \quad . \quad . \quad (21)$$

To obtain the crippling load we have only to plot a curve of the integrand in this equation, find its area, and deduce the value of P from (21). The same method may readily be extended to the case of unsymmetrical struts.†

EXAMPLES XIX

1. Design a solid tapered strut of uniform strength for a load of 2,000 lbs., the length between pin-centres being 70″. Take $E = 1{\cdot}6 \times 10^6$ lbs./in.^{2}, $f = 5{,}500$ lbs./in.^{2}, $a = 2{\cdot}18$, $\beta = 0{\cdot}130$.

2. Find the thickness at the middle and ends of a solid tapered strut 50″ long for an end load of 12,000 lbs., using the same values of the constants as in question 1, and estimate the percentage weight saved by tapering.

3. A circular solid wooden strut is 1″ diam. at the middle, the eccentricity of loading is 0·2″, and the strut is designed with the correct taper according

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* Applying the theorem $\int_0^c \int_0^x f(x).dx\,dx = \int_0^c (c - x)f(x).dx$. See Edwards' *Integral Calculus*, Vol. I, p. 381.

† See an article by the author, *Engineering*, Dec. 20, 1918, for various applications of this method to problems of flexure.

to Fig. 292. Taking the above values of E and f calculate the failing load of the strut. The length of the strut is 18″ and the diameter at the ends is ¾″.

4. In a straight-tapered strut of the type (*a*) Fig. 297, the length of the parallel portion is λl, l being the total length; I_0 is the moment of inertia of the cross section of the parallel portion. If the Euler crippling load is $k^2\frac{EI_0}{l^2}$, show that k is given by the equation

$$k \tan \lambda k = \frac{1-\mu}{1-\lambda} + k \cot \frac{k(1-\lambda)}{\mu},$$

where μ = the ratio of the diameter at the ends to the diameter of the central section.* (Webb and Lang.)

5. Show that a tapered strut, having a parabolic profile, has a crippling load $P = (E/l^2) \,.\, (\sqrt{I_1} - \sqrt{I_0})^2 \,.\, \{(\pi/\log\sqrt{I_1/I_0})^2 + \frac{1}{4}\}$ where I_1 and I_2 denote the values of I at the large and small ends respectively. (J. A. Wilcken, *Proc. Univ. of Durham Phil. Soc.*).

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* The solution of the equation $\frac{d^2y}{dx^2} + \frac{c^2}{m^2}.\frac{y}{x^4} = 0$ is

$$y = x\left(A \sin \frac{c}{mx} + B \cos \frac{c}{mx}\right)$$

where A and B are constants of integration.

CHAPTER XX

BEAMS UNDER LATERAL AND LONGITUDINAL LOADS COMBINED

251. Deflection Due to Lateral Loads Influenced by End Loads.—In Chapter XII, § 139, we pointed out the general nature of the problem of flexure due to eccentric thrusts or tensions, and showed how to estimate the stresses when the deflection of the beam is negligible; in the last chapter we have dealt with the case when the deflection is important and due entirely to the end thrust. We must now consider the case when the flexure is due partly to lateral loads, and partly to end loads (see § 139). In such cases the stress on any section is that arising from the resultant bending moment and the direct axial force, and is calculated from (1), § 139.

252. Beam Supported at Each End, Carrying a Uniformly Distributed Transverse Load, and End Thrust.—Let OA (Fig. 299)

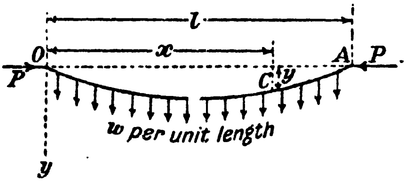

FIG. 299.

be a beam of uniform cross section, pin-jointed at the ends, with a load w per unit length and a thrust P in the line OA. Let I be the moment of inertia of the section of the beam for bending in the plane of w. Then at any point C, where $OC = x$, the bending moment is

$$M = Py, \text{ due to the thrust } P$$

$$+\frac{wx}{2}(l - x), \text{ due to } w \text{ (§ 102)} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The deflection equation is then

$$EI\frac{d^2y}{dx^2} = -Py - \frac{w}{2}(lx - x^2)$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{w}{2EI}(lx - x^2)$$

Now write

$$\frac{P}{EI} = a^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(ii)}$$

The equation for y becomes

$$\frac{d^2y}{dx^2} + a^2y = -\frac{wa^2}{2P}(lx - x^2).$$

We have for the solution

$$y = A\cos ax + B\sin ax - \frac{1}{P}\left(\frac{wlx}{2} - \frac{wx^2}{2} + \frac{w}{a^2}\right) \qquad \text{(iii)}$$

When

$$x = 0,\ y = 0$$

$$\therefore A = \frac{w}{a^2P}.$$

Also, when $x = l$, $y = 0$,

$$\therefore A\cos al + B\sin al = \frac{w}{a^2P}.$$

Hence

$$B = \frac{w}{a^2P}(\operatorname{cosec} al - \cot al)$$

$$= \frac{w}{a^2P}\tan\frac{al}{2};$$

Substituting in (iii) for A and B gives

$$y = \frac{w}{a^2P}\left(\cos ax + \tan\frac{al}{2}\sin ax\right) - \frac{1}{P}\left(\frac{wlx}{2} - \frac{wx^2}{2} + \frac{w}{a^2}\right)$$

$$= \frac{w}{a^2P}\left(\frac{\cos ax\cos\frac{al}{2} + \sin ax \,.\, \sin\frac{al}{2}}{\cos\frac{al}{2}} - 1\right) - \frac{wx}{2P}(l - x)$$

or

$$y = \frac{w}{a^2P}\left\{\cos a\left(\frac{l}{2} - x\right) . \sec\frac{al}{2} - 1\right\} - \frac{wx}{2P}(l - x). \quad . \quad (1)$$

The maximum value of y occurs when $x = \frac{l}{2}$, and is:

$$y_{max} = \frac{w}{a^2P}\left(\sec\frac{al}{2} - 1\right) - \frac{wl^2}{8P} \quad . \; . \; . \; . \; . \; . \quad (2)$$

Substituting in (i) the value of y given by (iv) we get:

$$M = \frac{w}{a^2}\left\{\cos a\left(\frac{l}{2} - x\right) . \sec\frac{al}{2} - 1\right\} \quad . \; . \; . \quad (3)$$

The bending moment is a maximum when $x = \frac{l}{2}$, and we have

$$\mathbf{M_{max}} = \frac{\mathbf{w}}{\mathbf{a^2}}\left(\mathbf{sec}\,\frac{\mathbf{al}}{\mathbf{2}} - \mathbf{1}\right) \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

If we write $al = 2\theta$, and adopt the approximation for sec θ given in § 215, we obtain the following approximate expression for the maximum bending moment:

$$M^*_{\max} = \frac{1{\cdot}02P_e}{P_e - P} \cdot \frac{wl^2}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When P is zero, the error arising from the use of (5) instead of (4) is 2 per cent.; when $P/P_e = 0{\cdot}985$ the error is $-0{\cdot}6$ per cent., which shows that the more convenient approximate expression (5) has all the accuracy necessary in practice.

Example.—The coupling rod of a locomotive is of rectangular section 4″ deep × 1·25″ wide. The maximum thrust in the rod is 12 tons, applied at the centre of the section at each end. Inertia and gravity combined produce a maximum transverse load of 24 lbs. per inch length. The length of the rod between centres is 100″.

Find the value of the central bending moment and deduce the maximum intensity of stress in the central section. (Mech. Sc. Trip. B., 1909.)

We have $$I = \frac{1{\cdot}25 \times 64}{12} = 6{\cdot}66 \text{ ins.}^4$$

$$P = 12 \text{ tons.}$$

Taking $E = 13{,}500$ tons/in.^{2} we get

$$a^2 = \frac{12 \text{ tons}}{13{,}500 \text{ tons/in.}^2 \times 6{\cdot}66 \text{ ins.}^4} = \frac{1}{7{,}490 \text{ in.}^2}$$

$$a = \frac{1}{86{\cdot}5 \text{ ins.}}.$$

$$l = 100 \text{ ins.}$$

$$\therefore\ al = 1{\cdot}155 \text{ radian.}$$

$$\frac{al}{2} = 0{\cdot}5775 \text{ radian} = 33{\cdot}1^\circ.$$

$$\sec\frac{al}{2} = 1{\cdot}194.$$

Then, from (4) of § 252,

$$M_{\max} = 24 \frac{\text{lbs.}}{\text{ins.}} \times 7{,}490 \text{ ins.}^2 \times 0{\cdot}194$$

$$= 34{,}900 \text{ lbs. ins.}$$

The stress due to bending $= \dfrac{34{,}900 \times 2}{6{\cdot}66} = 10{,}500$ lbs./in.^{2}

The area of the cross section is 5 in.^{2}, hence the direct compressive stress due to thrust is

$$\frac{12 \times 2{,}240}{5} = 5{,}380 \text{ lbs./in.}^2$$

Hence the maximum total stress is 10,500 + 5,380 = 15,880 lbs./in.^{2}

It will be interesting to compare the value of the bending moment calculated above with that given by the approximate formula (5), § 152.

We have

$$P_e = \frac{\pi^2 \times 13{,}500 \times 6{\cdot}66}{10{,}000} = 88{\cdot}6 \text{ tons.}$$

---

* By writing the B.M. due to w as $\frac{wl^2}{8}\cos\frac{\pi x}{l}$, the origin being at the centre of the beam, instead of the algebraic expression, Perry obtained the expression (5) without the factor 1·02 (*Phil. Mag.*, March, 1892).

Then

$$M_{max} = \frac{1 \cdot 02 \times 88 \cdot 6}{76 \cdot 6} \times \frac{24 \times 10{,}000}{8} = 35{,}400 \text{ lbs. ins.}$$

This is about 1·3 per cent. greater than the value found by the more exact formula, and is an error on the safe side.

253. Beam Supported at Each End, Loaded with a Uniformly Distributed Lateral Load, Terminal Couples and End Thrust.—

In Fig. 300 let *OCA* represent the strained axis of the beam, the supports

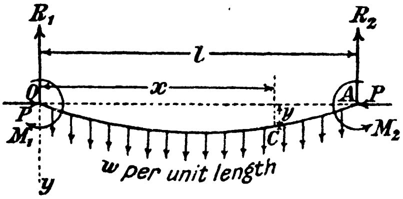

FIG. 300.

being at O and A. Let the lateral load be w per unit length, and let the couples applied to the ends be M_1 and M_2. Let R_1 and R_2 be the reactions at the supports, which are such that they exercise no constraint on the direction of the axis of the beam. Let P denote the end thrust, which is supposed to act along the direction of the unstrained axis of the beam.

Then the bending moment at any point C, distant x from y, is

$M = Py$ due to the end load P,

$+ \frac{l-x}{l}M_1 + \frac{x}{l}M_2$ due to the terminal couples (§ 106)

$+ \frac{wx}{2}(l-x)$ due to the lateral load (§ 102). . . . (i)

Hence the equation of the strained axis of the beam is

$$EI\frac{d^2y}{dx^2} = -Py - \frac{l-x}{l}M_1 - \frac{x}{l}M_2 - \frac{w}{2}(lx - x^2).$$

Let us write

$$a^2 = \frac{P}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

Then the equation becomes

$$\frac{d^2y}{dx^2} + a^2y = -\frac{a^2}{P}\left\{\frac{l-x}{l}M_1 + \frac{x}{l}M_2 + \frac{w}{2}(lx - x^2)\right\} \quad (\text{iii})$$

The solution of this is

$$y = A\cos ax + B\sin ax - \frac{1}{P}\left(M_1\frac{l-x}{l} + M_2\frac{x}{l} + \frac{wlx}{2} - \frac{wx^2}{2} + \frac{w}{a^2}\right) \quad . \quad . \quad (\text{iv})$$

The conditions to be satisfied at the ends are

$y = 0$ when $x = 0$
$y = 0$ when $x = l$.

The first gives:

$$0 = A - \frac{1}{P}\left(M_1 + \frac{w}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (\text{v})$$

and the second gives:

$$0 = A \cos al + B \sin al - \frac{1}{P}\left(M_2 + \frac{w}{a^2}\right) \quad . \quad . \quad . \quad (\text{vi})$$

From (v) we have:

$$A = \frac{1}{P}\left(M_1 + \frac{w}{a^2}\right),$$

and substituting this in (vi) leads to

$$B = -\frac{1}{P}\left(M_1 + \frac{w}{a^2}\right)\cot al + \frac{1}{P}\left(M_2 + \frac{w}{a^2}\right)\operatorname{cosec} al.$$

If we now substitute these values of A and B in (i) and (iv) we obtain the equations of the bending moment curve and deflection curve. After some straightforward simplification the results are:

$$y = \frac{M_1}{P}\left[\frac{\sin a(l-x)}{\sin al} - \frac{l-x}{l}\right] + \frac{M_2}{P}\left[\frac{\sin ax}{\sin al} - \frac{x}{l}\right]$$
$$+ \frac{w}{a^2P}\left[\frac{\sin a(l-x)}{\sin al} + \frac{\sin ax}{\sin al} - 1\right] - \frac{wx}{2P}(l-x). \quad (6)$$

and

$$M = \left(M_1 + \frac{w}{a^2}\right)\cos ax + \left[\left(M_2 + \frac{w}{a^2}\right)\operatorname{cosec} al\right.$$
$$\left. - \left(M_1 + \frac{w}{a^2}\right)\cot al\right]\sin ax - \frac{w}{a^2} \quad . \quad . \quad . \quad . \quad (7)$$

We shall find it convenient later to have expressions for the slopes of the beam at the ends. From (6) we have, by differentiation,

$$\frac{dy}{dx} = \frac{M_1}{lP}\left[1 - \frac{al \cos a(l-x)}{\sin al}\right] + \frac{M_2}{lP}\left[\frac{al \cos ax}{\sin al} - 1\right]$$
$$+ \frac{w}{aP}\left[\frac{\cos ax}{\sin al} - \frac{\cos a(l-x)}{\sin al}\right] + \frac{w}{2P}(2x - l).$$

When $x = 0$ this gives

$$\left[\frac{dy}{dx}\right]_{x=0} = \frac{1 - al\cot al}{lP}M_1 + \frac{al\operatorname{cosec} al - 1}{lP}M_2 + \frac{wl}{P}\left(\frac{\tan\frac{al}{2}}{al} - \tfrac{1}{2}\right) \quad (8)$$

and, when $x = l$,

$$\left[\frac{dy}{dx}\right]_{x=l} = -\frac{al\operatorname{cosec} al - 1}{lP}M_1 - \frac{1 - al\cot al}{lP}M_2 - \frac{wl}{P}\left(\frac{\tan\frac{al}{2}}{al} - \tfrac{1}{2}\right) \quad (9)$$

It will frequently be found in practice that it is convenient to adopt the following notation, suggested by A. Berry:

Put $al = 2\theta$, and let

$$\left.\begin{aligned} \frac{6(2\theta \operatorname{cosec} 2\theta - 1)}{(2\theta)^2} &= f(\theta) \quad . \quad . \quad . \\ \frac{3(1 - 2\theta \cot 2\theta)}{(2\theta)^2} &= \varphi(\theta) \quad . \quad . \quad . \\ \frac{3(\tan \theta - \theta)}{\theta^3} &= \psi(\theta) \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Then (8) and (9) can be written:

$$\left[\frac{dy}{dx}\right]_{x=o} = \frac{l \,.\, \varphi(\theta)}{3EI} M_1 + \frac{l \,.\, f(\theta)}{6EI} M_2 + \frac{wl^3}{24EI} \psi(\theta) \quad . \quad . \quad (11)$$

$$\left[\frac{dy}{dx}\right]_{x=l} = - \frac{l \,.\, f(\theta)}{6EI} M_1 - \frac{l \,.\, \varphi(\theta)}{3EI} M_2 - \frac{wl^3}{24EI} \psi(\theta) \quad . \quad . \quad (12)$$

The functions f, φ, ψ have been tabulated by Berry and are given at the end of Pippard and Pritchard's *Aeroplane Structures*.

To find the "Maximum" bending moment we proceed as follows. Let us write:

$$\left.\begin{aligned} C &= M_1 + \frac{w}{a^2} \\ D &= \left(M_2 + \frac{w}{a^2}\right) \operatorname{cosec} al - \left(M_1 + \frac{w}{a^2}\right) \cot al \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

$$= \quad M_2 \operatorname{cosec} 2\theta - M_1 \cot 2\theta + \frac{w}{a^2} \tan \theta$$

Then (12) becomes

$$M = C \cos ax + D \sin ax - \frac{w}{a^2} \quad . \quad . \quad . \quad . \quad . \quad (\text{vii})$$

When M is a maximum we must have

$$\frac{dM}{dx} = 0$$

which gives

$$\tan ax = \frac{D}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Solving this equation for x, we find the position of the "maximum" bending moment. Its value is then obtained from (7).

From (14) we have

$$\cos ax = \frac{C}{\sqrt{C^2 + D^2}} \text{ and } \sin ax = \frac{D}{\sqrt{C^2 + D^2}},$$

hence from (7)

$$M_{max} = \sqrt{C^2 + D^2} - \frac{w}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The positions of the points of inflexion are found by solving the equation $M = 0$, i.e.

$$C \cos ax + D \sin ax \doteq \frac{w}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

254. Approximate Formulæ.—The following approximations * will be found extremely useful for arithmetical work and may be relied upon for accuracy within 1 per cent. for all values of θ between 0 and $\frac{\pi}{2}$.

$$\left.\begin{aligned} f(\theta) &= \frac{P_e + 0{\cdot}2P}{P_e - P} \qquad \varphi(\theta) = \frac{P_e - 0{\cdot}38P}{P_e - P} \\ \psi(\theta) &= \frac{P_e - \frac{1}{70}P}{P_e - P} \end{aligned}\right\} \quad . \quad . \quad . \qquad (17)$$

where P_e is the Euler crippling load of the beam $= \pi^2 EI/l^2$.

The maximum bending moment occurs at a distance from O (Fig. 300) given by

$$\frac{l}{2} + \frac{\mathrm{M}_2 - \mathrm{M}_1}{\mathrm{w}l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

and its value is

$$\mathrm{M}_{\max} = \frac{\mathrm{P_e}}{\mathrm{P_e} - \mathrm{P}}\left\{\tfrac{1}{2}(\mathrm{M}_1 + \mathrm{M}_2)\left(1 + 0{\cdot}26\,\frac{\mathrm{P}}{\mathrm{P_e}}\right) + 1{\cdot}02\,\frac{\mathrm{w}l^2}{8}\right\} + \frac{(\mathrm{M}_1 - \mathrm{M}_2)^2}{2\mathrm{w}l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

whilst the distances of the points of inflexion, if any, from O are

$$\frac{l}{2} + 0{\cdot}9\,\frac{\mathrm{M}_2 - \mathrm{M}_1}{\mathrm{w}l} \pm 1{\cdot}2\sqrt{\frac{\mathrm{M}_{\max}}{\mathrm{w}}} \quad . \quad . \quad . \quad . \qquad (20)$$

When the beam is in tension instead of compression the corresponding exact formulæ are easily worked out. The following approximate expressions will, however, be sufficiently accurate in all practical cases : †

$$\left.\begin{aligned} f(\theta) &= \frac{P_e - 0{\cdot}12P}{P_e + P} \quad . \quad . \quad . \\ \varphi(\theta) &= \frac{P_e + 0{\cdot}32P}{P_e + P} \quad . \quad . \quad . \\ \psi(\theta) &= \frac{P + \frac{1}{80}P}{P_e + P} \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

$$\mathrm{M}_{\max} = \frac{\mathrm{P_e}}{\mathrm{P_e} + \mathrm{P}}\left\{\tfrac{1}{2}(\mathrm{M}_1 + \mathrm{M}_2)\left(1 - 0{\cdot}21\,\frac{\mathrm{P}}{\mathrm{P_e}}\right) + 0{\cdot}98\,\frac{\mathrm{w}l^2}{8}\right\} + \frac{(\mathrm{M}_1 - \mathrm{M}_2)^2}{2\mathrm{w}l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

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* Due to H. A. Webb, *Aeronautics*, Jan. 1, 1919.

† Exact expressions will be found on p. 351. For the case of a beam freely supported over a span of fixed length see Ex. 13, p. 355.

The positions of the maximum bending moment and points of inflexion are given by the formulæ as above.

255. Continuous Beams with Longitudinal Forces and Lateral Loads.*—Fig. 301 shows two consecutive bays of a continuous beam, in which each bay is subjected to a uniformly distributed load, an end thrust, and at each point of the beam over a support there are applied forces and couples in the plane of the distributed loads, so that the end loads in consecutive bays will in general be different, and the bending moments on either side of a support will be different. The equation connecting the bending moments at *A*, *B* and *C* is formed exactly as in § 194; taking the origin at *A* for *AB* and at *B* for *BC* we equate the values of the slopes at *B* for the two bays, using the equations (11) and (12) of § 253.

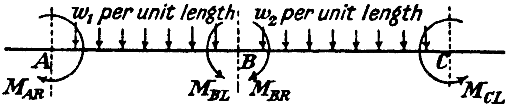

FIG. 301.

If we employ the notation of § 253, and use the suffixes 1 and 2 to denote the bays *AB* and *BC* respectively, we arrive at the following equation:

$$\frac{l_1 . f(\theta_1)}{I_1}M_{AR} + \frac{2l_1 . \varphi(\theta_1)}{I_1}M_{BL} + \frac{2l_2 . \varphi(\theta_2)}{I_2}M_{BR} + \frac{l_2 f(\theta_2)}{I_2}M_{CL}$$
$$+ \frac{w_1 l_1^3}{4} . \frac{\psi(\theta_1)}{I_1} + \frac{w_2 l_2^3}{4} . \frac{\psi(\theta_2)}{I_2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

When concentrated loads as well act on any bay, or when the supports are not at the same level, the corresponding modifications can easily be deduced.

Approximate expressions:—

Corresponding with the algebraic approximations of § 254 Webb † gives the following equations:

$$\frac{1}{(P_{e1}-P_1)l_1}\left[\left(1+0{\cdot}2\frac{P_1}{P_{e1}}\right)M_{AR}+2\left(1-0{\cdot}38\frac{P_1}{P_e}\right)M_{BL}\right.$$
$$\left.+\frac{w_1 l_1^2}{4}\left(1-\frac{1}{70}\cdot\frac{P_1}{P_{e1}}\right)\right]+\frac{1}{(P_{e2}-P_2)l_2}\left[\left(1+0{\cdot}2\frac{P_2}{P_{e2}}\right)M_{CL}\right.$$
$$\left.+2\left(1-0{\cdot}38\frac{P_2}{P_{e2}}\right)M_{BR}+\frac{w_2 l_2^2}{4}\left(1-\frac{1}{70}\frac{P_2}{P_{e2}}\right)\right]=0 \quad . \quad (24)$$

when P_1 and P_2 are both thrusts. This form of the equation is much more convenient for arithmetical work than the trigonometric form, particularly when the section of the beam may not be known at first and several trial calculations have to be made.

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* For graphical treatment see *Phil. Mag.*, Jan., 1914; *Engineering*, Dec. 20, 1918.

† *Loc. cit.*, p. 349.

If any bay be in tension instead of compression the coefficients of the corresponding M's are modified as in § 254, and $-\frac{1}{70}$ is replaced by $+\frac{1}{80}$ in the coefficients of $wl^2/4$.

To adapt the formulæ of § 253 and § 255 to the case of tension $f(\theta)$, $\phi(\theta)$, and $\psi(\theta)$ must be given the following values:

$$f(\theta) = \frac{6(1 - 2\theta \operatorname{cosech} 2\theta)}{(2\theta)^2}$$

$$\phi(\theta) = \frac{3(2\theta \coth 2\theta - 1)}{(2\theta)^2}$$

$$\psi(\theta) = \frac{3(\theta - \tanh \theta)}{\theta^3}$$

Also
$$M = C' \cosh ax + D' \sinh ax + \frac{w}{a^2}$$

where
$$C' = -\left(M_1 - \frac{w}{a^2}\right)$$

$$D' = \left(M_1 - \frac{w}{a^2}\right) \coth 2\theta - \left(M_2 - \frac{w}{a^2}\right) \operatorname{cosech} 2\theta,$$

M being a maximum when $\tanh ax = D'/C'$, and is then
$$\sqrt{C'^2 + D'^2} + w/a^2.$$

We may remark here that a continuous beam is not necessarily unstable when the end load in one bay is equal to the Euler load of that bay (considered as a pin-jointed strut by itself), on account of the fixing moments at the supports; but this is of little practical interest. We are concerned here, as in most practical cases, with stresses and not stability, so that the stability of a continuous beam is not of much importance. Readers interested in the mathematical theory may refer to the work of Messrs. Cowley and Levy.*

Example.—Fig. 302 represents a wing spar for a certain aeroplane, the moment of inertia and area of the cross section being 3·52 ins.^{4} and 2·48 in.^{2}, and $E = 1{\cdot}4 \times 10^6$ lbs./in.^{2} There are no external couples applied at the points A, B, C, etc., and the supports are all collinear. The loads in the various bays are given in the diagram. It is required to draw the bending moment diagram as accurately as possible.

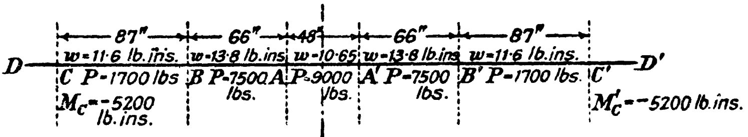

FIG. 302.

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* *Loc. cit.*, p. 340, or *Aeronautical Research Committee Reports*, R. & M., 364 and 373.

The calculations are most conveniently done in tabular form, and we shall illustrate the use of the approximate formulæ only as these suffice for all practical work.

$$E = 1{\cdot}4 \times 10^6 \text{ lb./in.}^2$$

——	CB	BA	AA'
P lb.	1,700	7,500	9,000
l in.	87	66	48
w lb. per in.	11·6	13·8	10·65
EI	$4{\cdot}92 \times 10^6$	$4{\cdot}92 \times 10^6$	$4{\cdot}92 \times 10^6$
$P_e = \pi^2 EI/l^2$	6,420	11,130	21,000
P/P_e	0·265	0·674	0·429
$1 + 0{\cdot}2\dfrac{P}{P_e}$	1·053	1·135	1·086
$0{\cdot}38\dfrac{P}{P_e}$	0·101	0·256	0·163
$1 - 0{\cdot}38\dfrac{P}{P_e}$	0·899	0·744	0·837
$\dfrac{1}{70}\cdot\dfrac{P}{P_e}$	0·0038	0·0086	0·0061
$1 - \dfrac{1}{70}\cdot\dfrac{P}{P_e}$	0·9962	0·9914	0·9939
$\dfrac{wl^2}{4}$	22,000	15,000	6,110
$\dfrac{wl^2}{4}\left(1 - \dfrac{1}{70}\cdot\dfrac{P}{P_e}\right)$	21,950	14,900	6,050
$P_e - P$	4,720	3,630	12,000
$(P_e - P)l$	411,000	239,500	576,000

The equations for finding the fixing moments can now be written down from (24):—

$$\frac{1}{411{,}000}(1{\cdot}053M_C + 1{\cdot}798M_B + 21{,}950)$$

$$+ \frac{1}{239{,}000}(1{\cdot}488M_B + 1{\cdot}135M_A + 14{,}900) = 0$$

and

$$\frac{1}{239{,}000}(1{\cdot}135M_B + 1{\cdot}488M_A + 14{,}900)$$

$$+ \frac{1}{576{,}000}(1{\cdot}674M_A + 1{\cdot}086M_A' + 6{,}050) = 0$$

Also, $M_C = -5{,}200$, and $M_A' = M_A$.

These equations reduce to :—

$$4{\cdot}358M_B + 1{\cdot}950M_A = -42{,}350$$
$$1{\cdot}135M_B + 2{\cdot}633M_A = -17{,}410$$

We find

$$M_A = -3{,}000 = M_A'$$
$$M_B = -8{,}370$$

We now proceed to find the position and magnitude of the maximum bending moment, and the positions of the points of inflexion :—

——	CB	BA	AA^1
M_1	— 5,200	— 8,370	— 3,000
M_2	— 8,370	— 3,000	— 3,000
$M_2 + M_1$	— 13,570	— 11,370	— 6,000
$M_2 - M_1$	— 3,170	5,370	0
wl	1,010	810	511
$\dfrac{M_2 - M_1}{wl}$	— 3·13	6·64	0
$\dfrac{l}{2}$	43·5	33	24
x for M_{max} [equation (18)]	40·4	39·6	24
$1 + 0{\cdot}26\dfrac{P}{P_e}$	1·069	1·176	1·112
$\left(1 + 0{\cdot}26\dfrac{P}{P_e}\right)\dfrac{M_1 + M_2}{2}$	— 7,250	— 6,670	— 3,336
$1{\cdot}02 \times \dfrac{wl^2}{8}$	11,200	7,650	3,110
$\left(1 + 0{\cdot}26\dfrac{P}{P_e}\right)\dfrac{M_1 + M_2}{2} + 1{\cdot}08\dfrac{wl^2}{8}$	3,950	980	— 226
$\left[\left(1 + 0{\cdot}26\dfrac{P}{P_e}\right)\dfrac{M_1 + M_2}{2} + 1{\cdot}08\dfrac{wl^2}{8}\right] \times \dfrac{P_e}{P_e - P}$	5,370	3,000	— 396
$\dfrac{(M_2 - M_1)^2}{2wl^2}$	57	250	0
M_{max} [equation (19)]	5,427	3,250	— 396
$\dfrac{M_{max}}{w}$	468	236	— 37
$0{\cdot}9\dfrac{M_2 - M_1}{wl}$	2·82	5·96	0
$\dfrac{l}{2} + 0{\cdot}9\dfrac{M_2 - M_1}{wl}$	46·32	38·96	24
$1{\cdot}2\sqrt{M_{max}/w}$	25·9	16·1	imaginary
Distances of points of inflexion from left hand end of bay (equation 20)	20·42 72·22	22·86 55·06	— —

EXAMPLES XX

1. An eccentric rod AB of uniform section and length l is subjected to an end thrust P. Owing to friction on the eccentric sheave the end B is subjected to a bending moment λP; the other end A is free from bending moment. Prove that at a distance x from A the deflection y of the rod is given by $y = \lambda(\sin ax/\sin al - x/l)$, where $a^2 = P/EI$.

If $l = 100''$ and the diameter of the rod is $2''$, calculate the greatest thrust the rod can withstand without buckling; $E = 14{,}000$ tons/in.^{2} (Mech. Sc. Trip., 1922.)

2. If the yield point of the steel in question 1 is 20 tons/in.^{2}, will the rod fail at a lower load than there indicated, taking $\lambda = 0{\cdot}9$ ins.?

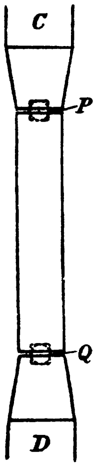

Fig. 303.

3. Fig. 303 illustrates an arrangement sometimes employed for testing struts. The load is applied to the strut by C and D, acting through steel balls P and Q fitting into cylindrical holes bored in the ends of the specimen and in the pieces C, D.

In a particular case the strut is of circular section 0·5″ diameter and 18″ long. It is estimated that when the load is 1 ton a friction couple of 0·01 ton-inch has to be overcome before angular movement can take place at the ends of the strut. If the ends were free from friction show that a load of 2,800 lbs. would buckle the strut. In the case described above show that if the axial load is increased to 4,000 lbs. the strut will not buckle, and that it would then take a transverse force of 12 lbs. applied at the centre of the strut to produce breakdown. $E = 30 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip. B., 1910.)

4. A strut initially straight is continuous through a number of supports. A, B, C are three consecutive supports, AB and BC being of lengths a and b. If on applying an axial load P the strut begins to buckle, show that M_1, M_2, M_3, the bending moments at A, B, C, are connected by the relation

$$\frac{M_1 \sin b\alpha + M_3 \sin a\alpha - M_2 \sin (a+b)\alpha}{M_1 b + M_3 a - M_2(a+b)} = \frac{\sin a\alpha . \sin b\alpha}{ab\alpha}$$

where $\alpha^2 = P/EI$. (Mech. Sc. Trip. B., 1912.)

5. A beam of varying cross section, with pin-joints at its ends, carries a lateral load w per unit length and an axial thrust P. The beam is symmetrical about the middle cross section. By the method of § 250, show that the deflection δ at the centre can be found approximately from the equation

$$\delta\left[1 - \frac{P}{E}\int_0^{\frac{l}{2}} \frac{\frac{l}{2} - x}{I} \cos\frac{\pi x}{l} dx\right] = \frac{l^2}{8}\int_0^{\frac{l}{2}} \frac{w\left(\frac{l}{2} - x\right)}{EI} \cos\frac{\pi x}{l}.dx.$$

x being measured from the centre of the beam.

6. A mild steel connecting rod, 5 ft. in length, is required to transmit a thrust of 25 tons between pins on two levers that turn in opposite directions in the same plane. Assuming that the pins are 2″ diameter, and act with $\mu = 0{\cdot}15$, find a suitable diameter for the connecting rod, allowing a maximum stress not greater than 6 tons/in.^{2} (R.N.C., Greenwich, 1922.)

7. The coupling rod of a locomotive is of length 8 ft., the cross section being such that the area is 6·5 in.^{2} and the moment of inertia 9·8 ins.^{4} The maximum thrust in the rod is 16 tons, and the lateral inertia load is 20 lbs. per inch run. Taking the density of the metal as 490 lbs./ft.^{3}, find the maximum stress when gravity and inertia act together.

8. The following particulars refer to a locomotive coupling rod: l = length between centres, w = weight per unit length, r = crank length, ω = angular velocity of wheels, P = thrust. When the rod is in its lowest position, allowing for the variation of inertia force along the rod on account

of the deflection of the latter, show that the maximum bending moment is

$$\frac{w}{2}EI\left(1+\frac{rw^2}{g}\right)\left(\frac{P^2}{4E^2I^2}+\frac{w\omega^2}{gEI}\right)^{-\frac{1}{2}}(\operatorname{sech}\theta l-\sec\varphi l)$$

where

$$\varphi^2=\frac{P}{2EI}+\sqrt{\frac{P^2}{4E^2I^2}+\frac{w\omega^2}{gEI}}$$

$$\theta^2=-\frac{P}{2EI}+\sqrt{\frac{P^2}{4E^2I^2}+\frac{w\omega^2}{gEI}}$$

(H. Mawson in *Phil. Mag.*, Oct., 1915.)

9. Referring to Fig. 302, the following are particulars of a certain aeroplane spar: $AB = A'B' = 81''$; $BC = B'C' = 84''$; $CD = C'D' = 20''$. Distributed load 5·5 lbs./inch. End thrusts in AB and $A'B'$ 2,730 lbs.; in BC and $B'C'$ 640 lbs.; $M_A = M_A' = 0$. Area and moment of inertia of spar section 2·17 in.^{2} and 1·8 ins.^{4}, depth of section 2·65″; $E = 1{\cdot}38 \times 10^6$ lbs./in.^{2} Calculate the bending moment at B, the value of the maximum B.M. in the bay AB, and the maximum compressive stress in the spar.

10. If in § 252 the beam be encastré at each end show that the "maximum" bending moment is $\frac{wl^2}{24}f\left(\frac{\theta}{2}\right)$ in the notation of § 252.

11. A freely supported beam AB, of length l, carries a load W at a distance a from A, and is acted on by an axial thrust P. Show that the deflection and bending moment at a distance x from A are given by

$$y=\frac{W}{aP}\left[\frac{\sin a(l-x)\sin aa}{\sin al}-\sin a\{a-x\}\right]-\frac{aW}{lP}(l-x)+\frac{W}{P}\{a-x\}$$

$$M=\frac{W}{a}\left[\frac{\sin a(l-x)\sin aa}{\sin al}-\sin a\{\imath-x\}\right],$$

where the { } terms are omitted if $x > a$.

12. A freely supported uniform beam of length l carries a uniform load w per unit length, and a longitudinal pull P; show that the central deflection y_0 is given by

$$y_0=\frac{w}{a^2P}\left(\operatorname{sech}\frac{al}{2}-1+\frac{a^2l^2}{8}\right),\text{ where } a^2=P/EI.$$

13. A uniform beam of length l is supported by fixed pin joints at the ends, and carries a uniformly distributed load w. Assuming that the deflection curve is given approximately by $y = y_0 \sin (\pi x/l)$, show that the extension is $\pi^2y_0^2/4l$. Hence, using the approximation $\operatorname{sech}\frac{al}{2}=\frac{P_e-{\cdot}26P}{P_e+P}$, where $P_e = \pi^2EI/l^2$, show that P, the tension in the beam, is given by the equation

$$\frac{4nI}{A}=\frac{w^2l^8}{\pi^8E^2I^2}\left(\frac{1{\cdot}23n-0{\cdot}03}{n(n+1)}\right)^2$$

where A is the cross sectional area, and $n = P/P_e$.

CHAPTER XXI

FRAMEWORKS WITH STIFF JOINTS

256. Nature of the Problem.—In practice the several members of structural frameworks are frequently attached to each other rigidly, so that any flexure of one member involves the flexure of all the others. We may divide such frameworks into two general classes: those which depend on the rigidity of the joints for maintaining their shape and those which do not. In the former case there will always be a smaller number of bars than is required to constitute a simply stiff frame (p. 38), in the latter the number may be such that the frame would, if pin-jointed, be simply stiff or redundant.

In any case the first problem to attack is the estimation of the bending moments at the ends of each member, for, when that is done, we can proceed according to the principles of Chapters IX and XII to draw the bending moment diagram for each member and to calculate the stresses. To solve this problem we make use of the results obtained in Chapter XIV. There we found expressions for the angular deflections of the ends of a beam under various conditions of loading; here we apply these results to express the physical condition that the ends of all the members rigidly connected at one joint must rotate by the same amount. By this means we obtain sufficient equations to enable us to determine the unknown bending moments. It will be seen from these remarks that, when we have only two bars meeting at one point, the problem is analogous to that of a continuous beam, and the same equations may be used. The following examples should make the method clear to the reader.

257. Rectangular Portal.*—In Fig. 304, *ABCD* is a framework of rods or beams of the same material rigidly joined together at right angles at *B* and *C*, whilst the ends *A* and *D* are fixed rigidly vertical in the ground. *AB* carries a uniformly distributed horizontal load *w* per unit length. It is required to investigate the bending moments in *AB* and the thrust in *BC*; the effects of the thrusts on the flexure of the members may be neglected.

Let M_1 = the bending moment at *A* on *AB*.
M_2 = the couple acting on *AB* on account of *BC*, and vice versa.
M_3 = the similar couple at *C*.
M_4 = the bending moment at *D* on *CD*.
P = the thrust in *BC*.
I_1 = the moment of inertia of the cross sections of *AB* or *CD*.
I_2 = ditto, for *BC*.

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* Problems of this nature may often, with advantage, be solved by the use of strain energy, see Appendix, p. 555.

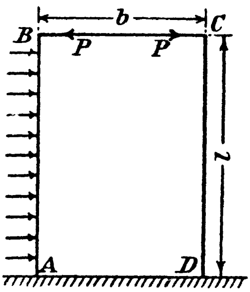

FIG. 304.

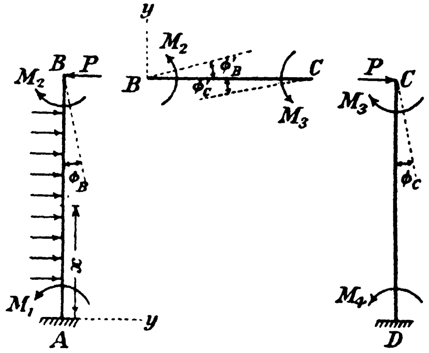

FIG. 305.

The directions of the couples are shown in Fig. 305, where the beams have been drawn separately for the sake of clearness.

For AB take the origin at A and measure y to the right. The bending moment at a point in AB at a height x above A is

$$-M_2 - \frac{w}{2}(l-x)^2 + P(l-x).$$

Hence for AB we have

$$EI_1\frac{d^2y}{dx^2} = M_2 + \frac{w}{2}(l-x)^2 - P(l-x).$$

Integrating this twice we find

$$EI_1\frac{dy}{dx} = M_2x - \frac{w}{6}(l-x)^3 + \frac{P}{2}(l-x)^2 + A$$

$$EI_1y = \frac{M_2x^2}{2} + \frac{w}{24}(l-x)^4 - \frac{P}{6}(l-x)^3 + Ax + B.$$

The conditions to be satisfied are $\frac{dy}{dx} = 0$ and $y = 0$, when $x = 0$, which give

$$A = \frac{wl^3}{6} - \frac{Pl^2}{2}$$

$$B = -\frac{wl^4}{24} + \frac{Pl^3}{6}$$

Let $\varphi_B =$ the slope of AB at B, in the direction shown in Fig. 305, then, from the above equations,

$$-\varphi_B = \left[\frac{dy}{dx}\right]_{x=l} = \frac{1}{EI_1}\left(M_2l + \frac{wl^3}{6} - \frac{Pl^2}{2}\right)$$

or

$$\varphi_B = -\frac{l}{EI_1}\left(M_2 + \frac{wl^2}{6} - \frac{Pl}{2}\right) \quad . \quad . \quad . \quad . \quad (\text{i})$$

Let δ_B = the movement of B to the right, then

$$EI_1\delta_B = \frac{M_2 l^2}{2} + Al + B$$

$$= \frac{M_2 l^2}{2} + \frac{wl^4}{8} - \frac{Pl^3}{3}$$

$$\therefore \delta_B = \frac{l^2}{EI_1}\left(\frac{M_2}{2} + \frac{wl^2}{8} - \frac{Pl}{3}\right) \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The slope, φ_C, and deflection, δ_C, of DC at C can be deduced by writing M_3 for M_2, $-P$ for P, and omitting w in (i) and (ii) above; thus

$$\varphi_C = -\frac{l}{EI_1}\left(M_3 + \frac{Pl}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$\delta_C = \frac{l^2}{EI_1}\left(\frac{M_3}{2} + \frac{Pl}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Let φ_B' and φ_C' be the slopes of BC at B and C as shown. Then from equations (17) and (18), § 170, writing M_2 for M_1, $-M_3$ for M_2, and $w = 0$, we have

$$\varphi_B' = \frac{b}{6EI_2}(2M_2 - M_3) \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

$$\varphi_C' = -\frac{b}{6EI_2}(M_2 - 2M_3) \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

Since the corners are rigid, we must have

$$\varphi_B = \varphi_B' \text{ and } \varphi_C' = \varphi_C.$$

From (i) and (v) we have, then,

$$-\frac{l}{EI_1}\left(M_2 + \frac{wl^2}{6} - \frac{Pl}{2}\right) = \frac{b}{6EI_2}(2M_2 - M_3)$$

whilst (iii) and (vi) give

$$-\frac{l}{EI_1}\left(M_3 + \frac{Pl}{2}\right) = -\frac{b}{6EI_2}(M_2 - 2M_3).$$

If we write

$$k = \frac{bI_1}{lI_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

these two last equations reduce to

$$-(2k + 6)M_2 + kM_3 + 3lP = wl^2 \quad . \quad . \quad . \quad \text{(viii)}$$

and

$$kM_2 - (2k + 6)M_3 - 3lP = 0 \quad . \quad . \quad . \quad . \quad \text{(ix)}$$

Again, we must have $\delta_B = \delta_C$; hence, from (ii) and (iv),

$$\frac{M_2}{2} + \frac{wl^2}{8} - \frac{lP}{3} = \frac{M_3}{2} + \frac{lP}{3}$$

or

$$M_2 - M_3 - \frac{4}{3}lP = -\frac{wl^2}{4} \quad . \quad . \quad . \quad . \quad \text{(x)}$$

Equations (viii), (ix) and (x) enable us to find M_2, M_3 and P. Solving them we obtain the values

$$\left.\begin{aligned} M_2 &= -\frac{23k+6}{(k+6)(2k+1)} \cdot \frac{wl^2}{24} \\ M_3 &= -\frac{25k+18}{(k+6)(2k+1)} \cdot \frac{wl^2}{24} \\ P &= \frac{3k+2}{2k+1} \cdot \frac{wl}{8} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\text{xi})$$

Now, considering the equilibrium of DC, we have

$$M_4 = lP + M_3$$

Substituting the above values of P of M_3, this gives

$$M_4 = \frac{9k^2+35k+18}{(2k+1)(k+6)} \cdot \frac{wl^2}{24} \quad . \quad . \quad . \quad (\text{xii})$$

For the equilibrium of AB we have

$$M_1 = M_2 - lP + \frac{wl^2}{2}$$

which gives

$$M_1 = \frac{15k^2+73k+30}{(k+6)(2k+1)} \cdot \frac{wl^2}{24} \quad . \quad . \quad . \quad . \quad (\text{xiii})$$

Equations (xi), (xii), (xiii) provide a complete solution of the problem, since the bending moment diagrams for each member can now be drawn in the ordinary way.

For variations of this problem see Examples XXI, p. 367.

258. Secondary Stresses in Triangulated Frameworks.*—In the ordinary process of estimating the loads in the members of frameworks, whether they be simply stiff or redundant, we assume that all the joints are pin-joints. The stresses which arise from these loads are called direct stresses, or *primary stresses*. In practice the perfect, frictionless pin-joint does not exist. If the joint is really a pin-joint there is always friction present which resists the free rotation of the members round the pin. Frequently the joint does not even pretend to be pin-jointed, and the members are riveted or bolted together to form a rigid joint.

Consider the framework $ABCD$ shown in Fig. 306, acted on by the opposing forces P. The members AB, BC, CD, DA will all be in tension, whilst BD will be in compression. If the bars were all freely pinned together at the corners the framework would distort in the manner shown by the dotted lines, the angles A and C being decreased, whilst B and D are increased. But if the bars are rigidly fixed together this change of angle cannot take place, so that the frame must distort as shown in Fig. 307, where the full lines show the undistorted frame, and the dotted lines the distorted frame. We see, then, that the bars are bent, so that there must be corresponding bending moments and stresses, and these are sometimes of considerable importance. In the case of pin-jointed frameworks, friction will have a similar effect, but to a much less extent of course. The stresses arising from these causes are called ***secondary stresses.***

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* For a detailed treatment of this subject see *Modern Framed Structures*, by Johnson, Bryan, and Turneaure, Pt. II (Wiley).

Secondary stresses may arise from other causes: in a pin-jointed frame the joint may be eccentric, i.e. the axis joining the pin-centres may not coincide with the line of centroids of the members, in which case bending moments will be applied to the ends of the bars. The weights of the members themselves also introduce bending stresses which must sometimes be considered. It will usually be found that the most important secondary stresses are those which are due to the rigidity of the joints.

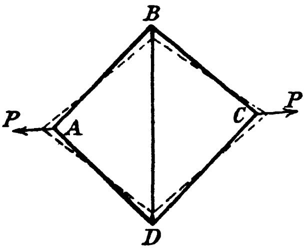

FIG. 306.

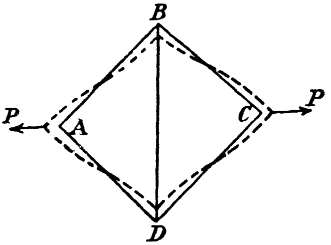

FIG. 307.

259. Secondary Stresses Due to Rigid Joints.

(i) The first step in the calculations is to estimate the direct stresses, and hence the elongations, for all the members of the frame on the assumption that all the joints are pin-joints.

(ii) The second step is to find the changes in the angles of all the triangles of the frame, using the values of the stresses found previously. This may be done by drawing a displacement diagram or by calculation.

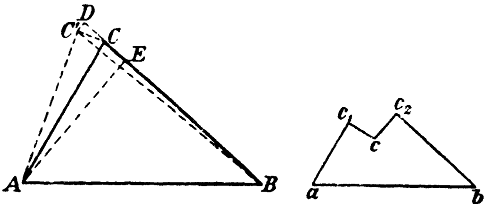

FIG. 308.

Thus, in Fig. 308, abc_2cc_1 is the displacement diagram for the triangle ABC, drawn so that ab represents the elongation of AB, ac_1 that of AC and bc_2 that of BC, as in Chapter II. Then c_1c, at right angles to AC, represents the movement of C due to the rotation of AC round A, so that c_1c/AC gives the change in the angle A. Similarly cc_2/BC gives the change in B, and the change in C is found from the condition

$$\delta A + \delta B + \delta C = 0.$$

Again, suppose, on account of an elongation of BC only, C moves to C', so that $AC' = AC$, and CC' can be regarded as a straight line perpendicular to AC. Draw $C'D$ and AE perpendicular to BC. Then, since δA is small, we have

$$\delta A = \frac{CC'}{AC} = \frac{CD \operatorname{cosec} CC'D}{AC} = \frac{CD}{AC \sin CC'D} = \frac{CD}{AC \sin C} = \frac{CD}{AE}.$$

Now $BC = CE + EB = AE(\cot C + \cot B)$, hence

$$\delta A = \frac{CD}{BC}(\cot C + \cot B),$$

where CD represents the elongation of BC.

If p_A denote the stress in BC, $\frac{CD}{BC} = \frac{p_A}{E}$, and we have

$$\delta A = \frac{p_A}{E}(\cot B + \cot C).$$

Similar expressions can be derived for the changes in the angle A arising from changes in the lengths of the other sides, and the other angles can be treated in the same way. Thus, if A, B, C are the angles of a triangle, whilst p_A, p_B, p_C are the tensile stresses in the opposite sides, the increases in the angles are given by

$$\left.\begin{aligned}
\delta A &= \frac{p_A - p_C}{E}\cot B + \frac{p_A - p_B}{E}\cot C \quad . \quad . \\
\delta B &= \frac{p_B - p_A}{E}\cot C + \frac{p_B - p_C}{E}\cot A \quad . \quad . \\
\delta C &= \frac{p_C - p_B}{E}\cot A + \frac{p_C - p_A}{E}\cot B \quad . \quad .
\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

These formulæ enable us to calculate the alterations in all the angles of the framework.

(iii) The third step is to find the bending moments which must be applied to the ends of the members to bring about the changes of angle found above.

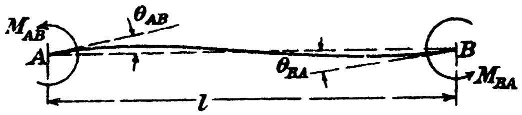

FIG. 309.

In Fig. 309, let AB be the unstrained axis of a beam acted on by couples M_{AB} and M_{BA}, as shown, so that the angular deflections are θ_{AB}

and θ_{BA} at the ends.* Then, from (17) and (18), p. 232, putting $w = 0$, we see that

$$\theta_{AB} = \frac{l}{6EI}(2M_{AB} - M_{BA}) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\theta_{BA} = \frac{l}{6EI}(2M_{BA} - M_{AB}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Solving these equations for M_{AB} and M_{BA} we get

$$\left.\begin{aligned} M_{AB} &= \frac{2EI}{l}(2\theta_{AB} + \theta_{BA}) \quad . \quad . \\ M_{BA} &= \frac{2EI}{l}(2\theta_{BA} + \theta_{AB}) \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which give the bending moments in terms of the deflections, at the ends of any bar.

Now consider a joint A in a framework $ABCDE$ shown in Fig. 310,

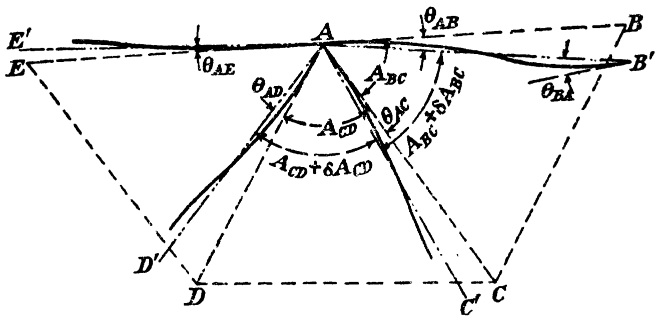

FIG. 310.

where the triangles ABC, ACD, ADE are made by the unstrained axes of the members. Let AB', AC', etc., be the directions which AB, AC, etc., would take up if the frame were pin-jointed, so that $C'AB' - CAB$ is the increase of the angle at A subtended by BC, and so on.

Let A_{BC} be the angle at A opposite BC, A_{CD} the angle at A opposite CD, and so on.

Let θ_{AB} be the angle through which AB rotates, θ_{AC} the angle through which AC rotates, and so on.

Now, suppose the A-ends of the bars AB, AC, etc., were brought back to their original directions by the application of couples. These couples would be given by (4) above. But then the joint would probably not be in equilibrium, and in order to get into a position of equilibrium it would rotate through some angle. Let this angle be θ_A, positive in a clockwise direction. Then the final angular deflection of the end A

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* The suffix notation used is this: the first letter of the suffix denotes that end of the bar to which the slope or bending moment refers, and the second letter denotes the other end. The positive direction is such that the deflection at one end appears to be towards the right when viewed from the other end, i.e. the bending moments are clockwise.

of the bar AB will be $\theta_A - \theta_{AB}$. Similarly if θ_B is the angle through which the joint B rotates bodily, the final deflection of the end B of AB will be $\theta_B - \theta_{BA}$.

Then, if M_{AB} denotes the bending moment at A on the bar AB, we have from (4)

$$M_{AB} = 2E\frac{I_{AB}}{l_{AB}}\{2(\theta_A - \theta_{AB}) + (\theta_B - \theta_{BA})\}$$

where I_{AB} and l_{AB} denote the moment of inertia and length of AB. Now it is clear that $\theta_{BA} = \theta_{AB}$, hence

$$\mathbf{M}_{\mathbf{AB}} = 2\mathrm{E}\frac{\mathbf{I}_{\mathbf{AB}}}{\boldsymbol{l}_{\mathbf{AB}}}(2\boldsymbol{\theta}_{\mathbf{A}} + \boldsymbol{\theta}_{\mathbf{B}} - 3\boldsymbol{\theta}_{\mathbf{AB}}) \quad . \quad . \quad . \quad . \quad (5)$$

There will be a similar expression for the bending moments on all the other bars at A.

For the equilibrium of the join A we must have

$$M_{AB} + M_{AC} + M_{AD} + M_{AE} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, from (5), dividing by $2E$,

$$\boldsymbol{\Sigma}\frac{\mathbf{I}_{\mathbf{AB}}}{\boldsymbol{l}_{\mathbf{AB}}}(2\boldsymbol{\theta}_{\mathbf{A}} + \boldsymbol{\theta}_{\mathbf{B}} - 3\boldsymbol{\theta}_{\mathbf{AB}}) = \mathbf{0} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and there will be a similar equation for each joint. In these equations, the angles θ_{AB} are known, and the angles θ_A, θ_B ... are unknown. There is one of these angles for each joint, so that there are as many equations of the type (7) as there are unknown quantities. When θ_A, θ_B, ... have been found, the bending moments are given by (5), and so the stresses are found.

In general we are faced with the problem of solving a large number of simultaneous equations. The work may be simplified by the method of successive approximation in the case of complicated frameworks. The process consists of assuming values for θ_A, etc., e.g. for the joint A, assume $\theta_A = \frac{1}{2}(\theta_{AB} + \theta_{AE})$ and so on. Test these assumed values by substituting in the equations (7) : if the summation on the left-hand side vanishes, the assumed values are correct. If, however, the summation does not vanish but has the value σ_A, as a second approximation reduce the previously assumed value of θ_A by

$$\frac{\sigma_A}{\frac{I_{AB}}{l_{AB}} + \frac{I_{AC}}{l_{AC}} + \frac{I_{AD}}{l_{AD}} + \frac{I_{AE}}{l_{AE}} + \ldots} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the previous value of θ_B by a quantity given by a similar expression. Now test these values : usually they will be found to satisfy the equations sufficiently well, but if not the process must be repeated.

An alternative method of approximation is to consider the structure in parts, neglecting the effects of joints remote from that under consideration. Thus, in Fig. 311, if we wish to find the bending moments at the joint C, we could approximate by neglecting the bending moments at

A, *K*, *E*, *F*, and deal with the frame *CDGHB*, which would only involve solving five simultaneous equations.

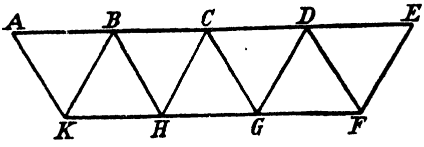

FIG. 311.

260. Effects of Lateral Loads.—So far we have considered only the flexure which arises from the distortion of the triangles composing the framework; if we wish to consider also the effects of lateral loads in producing flexure, instead of equations (2) and (3) we must use suitable expressions such as those given on pp. 232 and 241 for the slopes of the ends of the members. The procedure then follows exactly the same lines.

261. Effects of End Loads.*—In some cases the direct thrust or tension in a member will exert a large influence on the flexure; we must then use the expressions given on pp. 348-9 for the slopes of the ends of the members. The work then becomes extremely laborious when the frame has a large number of members.

Matters are complicated by the fact that the end loads themselves are functions of the bending moments, the number of simultaneous equations to solve is large, and they are not all simple algebraic equations as the expressions for the slopes involve trigonometric functions. This latter difficulty may be avoided by using the approximate expressions given on p. 349, but even then the work will be terrible. Usually, however, it is sufficiently accurate to calculate the end loads as if the structure were pin-jointed, and, using these values, proceed to the calculation of the bending moments. The method will be made clear by the example below.

Example.—Consider the structure shown in Fig. 312. The spruce beams *AB* and *DC* are encastré at *A* and *D*, and connected by the member *BC*, to which they are rigidly connected at *B* and *C*. The steel tie rod *AC* is pin-jointed at its ends. The loads are shown in the figure, and the other necessary data are

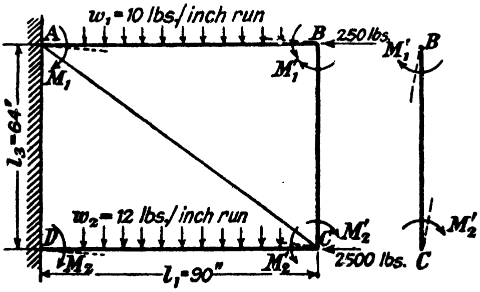

FIG. 312.

AB:—$I = 6{\cdot}0$ ins.^{4}; CD:—$I = 8{\cdot}0$ ins.^{4}; BC:—$I = 0{\cdot}7$ ins.^{4};
$S = 5{\cdot}9$ in.^{2}; thickness in plane of bending $1{\cdot}5''$.

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* See a paper *Equivalent Eccentricities due to Secondary Stresses*, C. N. Ross, Sydney, 1926.

For the three members $E = 1{\cdot}4 \times 10^6$ lbs./in.^{2}
We shall neglect the stretching of the tie AC.
Let the suffix 1 refer to AB, 2 to DC, and 3 to BC.
Then the end loads, if the structure were pin-jointed would be:

$$\begin{aligned} P_1 &= 250 \text{ lbs.} = \text{thrust in } AB \\ P_2 &= 3{,}890 \text{ lbs.} = \text{thrust in } DC \\ P_3 &= 450 \text{ lbs.} = \text{thrust in } BC \end{aligned}$$

$$\begin{aligned} \text{Let } M_1 &= \text{the bending moment at } \left.\begin{matrix} A \\ B \end{matrix}\right\} \text{on } AB \\ M_1' &= \;,, \quad ,, \quad ,, \quad ,, \\ M_2 &= \;,, \quad ,, \quad ,, \quad ,, \quad \left.\begin{matrix} D \\ C \end{matrix}\right\} \text{on } BC. \\ M_2' &= \;,, \quad ,, \quad ,, \quad ,, \end{aligned}$$

As usual these will be considered positive when they have a sagging action on AB and DC; the bending moments on BC will then be as shown by the arrows. The positive directions of the slopes are indicated by the dotted lines in the figure.

Then, from (12), p. 348, the slope of AB at B is

$$-\frac{l_1M_1}{6EI_1}f(\theta_1) - \frac{l_1M_1'}{3EI_1}\varphi(\theta_1) - \frac{w_1l_1^3}{24EI_1}\psi(\theta_1),$$

and the slope of BC at B is

$$\frac{l_3M_1'}{3EI_3}\varphi(\theta_3) - \frac{l_3M_2'}{6EI_3}f(\theta_3).$$

Equating these two we get

$$\frac{l_1M_1}{6I_1}f(\theta_1) + \frac{l_1M_1'}{3I_1}\varphi(\theta_1) + \frac{l_3M_1'}{3I_3}\varphi(\theta_3) - \frac{l_3M_2'}{6I_3}f(\theta_3) = -\frac{w_1l_1^3}{24I_1}\psi(\theta_1) \quad . \quad \text{(i)}$$

Similarly, considering the joint C, we get

$$\frac{l_1M_2}{6I_2}f(\theta_2) + \frac{l_1M_2'}{3I_2}\varphi(\theta_2) - \frac{l_3M_1'}{6I_3}f(\theta_3) + \frac{l_3M_2'}{3I_3}\varphi(\theta_3) = -\frac{w_2l_1^3}{24I_1}\psi(\theta_2) \quad . \quad \text{(ii)}$$

The conditions that the slopes at A and D are zero give

$$M_1\varphi(\theta_1) + \frac{M_1'}{2}f(\theta_1) + \frac{w_1l_1^2}{8}\psi(\theta_1) = 0 \quad . \quad . \quad . \quad \text{(iii)}$$

$$M_2\varphi(\theta_2) + \frac{M_2'}{2}f(\theta_2) + \frac{w_2l_1^2}{8}\psi(\theta_2) = 0 \quad . \quad . \quad . \quad \text{(iv)}$$

Taking the values of P_1, P_2, P_3, etc., given above we have, in the notation of § 253,

$$\begin{array}{lll} f(\theta_1) = 1{\cdot}0290 & f(\theta_2) = 1{\cdot}4635 & f(\theta_3) = 1{\cdot}2753 \\ \varphi(\theta_1) = 1{\cdot}0166 & \varphi(\theta_2) = 1{\cdot}2568 & \varphi(\theta_3) = 1{\cdot}1541 \\ \psi(\theta_1) = 1{\cdot}0249 & \psi(\theta_3) = 1{\cdot}3922 & \psi(\theta_3) = 1{\cdot}2340 \end{array}$$

Equations (i) to (iv) become, then,

$$\begin{aligned} 2{\cdot}58M_1 + 40{\cdot}3M_1' - 19{\cdot}45M_2' &= -\,51{,}700 \\ 19{\cdot}45M_1' - 2{\cdot}75M_2 - 39{\cdot}90M_2' &= 63{,}400 \\ 1{\cdot}017M_1 + 0{\cdot}515M_1' &= -\,10{,}380 \\ 1{\cdot}257M_2 + 0{\cdot}732M_2' &= -\,16{,}900 \end{aligned}$$

Solving these we get

$$\begin{array}{ll} M_1 = -\,9{,}700 \text{ lb. ins.} & M_2 = -\,12{,}700 \text{ lb. ins.} \\ M_1' = -\,1{,}310 \text{ lb. ins.} & M_2' = -\,1{,}350 \text{ lb. ins.} \end{array}$$

We can now, if we wish, proceed to a second approximation thus: taking the above values of the bending moments and calculating the corrected values of the reactions at the ends of the beams, we find new values for the end loads:

$$P_1 = 291 \text{ lbs.} \qquad P_2 = 3{,}540 \text{ lbs.} \qquad P_3 = 357 \text{ lbs.}$$

Then, proceeding as above, we obtain new values for the bending moments :

$$M_1 = -9{,}540 \text{ lb. ins.} \qquad M_2 = -12{,}570 \text{ lb. ins.}$$
$$M_1' = -1{,}340 \text{ lb. ins.} \qquad M_2' = -1{,}360 \text{ lb. ins.}$$

We see that the labour of the second approximation is hardly worth while. These last values of the bending moments give

$$P_1 = 292 \text{ lbs.}, \; P_2 = 3{,}545 \text{ lbs.}, \; P_3 = 359 \text{ lbs.}$$

The bending-moment diagrams for the two members AB and BC are shown in Fig. 313 ; it will be seen that making the joint B stiff has very little effect on the beam AB. The most serious effect is the introduction of bending moments at the ends of BC, the greater one being 1,360 lb. ins. This, combined with the direct thrust, gives a stress

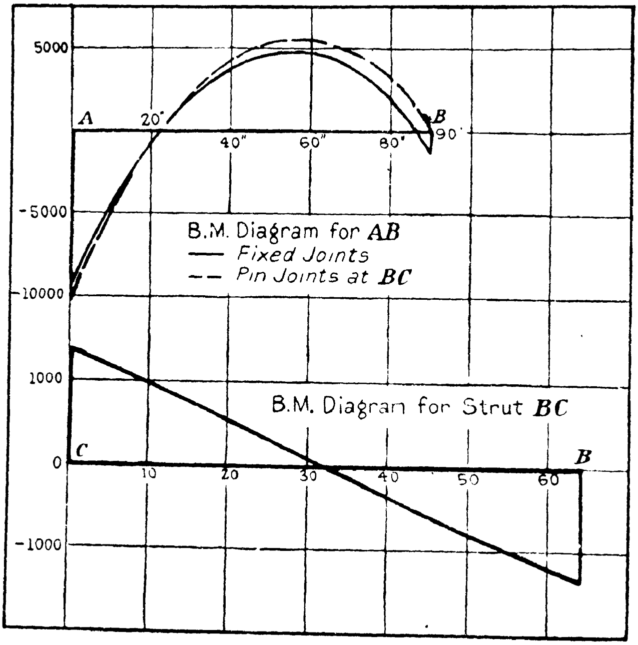

FIG. 313.

$$\frac{1{,}360 \times 0{\cdot}75}{0{\cdot}7} + \frac{359}{5{\cdot}9} = 1{,}520 \text{ lbs./in.}^2$$

Taking the compressive strength of the material (spruce) as 5,000 lbs./in.^{2} we see that, even if the stress were proportional to the load, the factor of safety is only 3·7. As a pin-jointed strut, the Euler failing load of BC is 2,400 lbs., or about seven times the load the member has to carry. Hence in this case **the structure as a whole is considerably weakened by making B and C stiff joints.** This is a most important point to notice as it is sometimes supposed that a frame with stiff joints must necessarily be stronger than the same frame with pin joints.

EXAMPLES XXI

1. In the example on p. 357, if the lower ends of AB and CD are hinged at A and D, show that

$$M_2 = -\frac{3wl^2}{8} \cdot \frac{1+2k}{2+3k}; \quad M_3 = -\frac{wl^2}{8} \cdot \frac{5+6k}{2+3k};$$

and
$$P = \frac{wl}{8} \cdot \frac{5+6k}{2+3k}.$$

2. In the example on p. 357, if there is a concentrated load W at B instead of the distributed load, and the ends A and D are built-in as in Fig. 304, show that

$$M_1 = M_4 = \frac{Wl}{2} \cdot \frac{3+k}{6+k}; \quad M_2 = -\frac{Wl}{2} \cdot \frac{3}{6+k} = M_3.$$

3. In Fig. 314, BC carries a vertical load w per unit length, and the corners B and C are stiff; show that the bending moments at B and C are $\frac{k}{k+2} \cdot \frac{wl^2}{6}$, k having the same meaning as above.

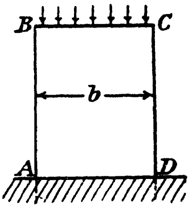

Fig. 314.

4. ABC is a triangular framework with stiff rigid corners: see Fig. 315.

The framework is loaded at A and supported by vertical reactions at B and C.

Find the greatest B.M.'s set up in AB and BC and the magnitude of the thrust and pull in these two members. The bars are of rectangular section 2″ × ½″. (Mech. Sc. Trip., 1911, B.)

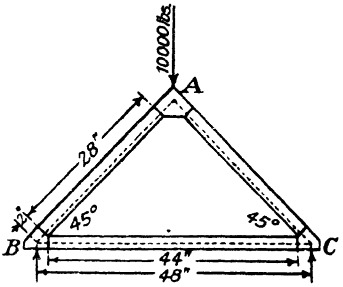

Fig. 315.

5. The radius of gyration of the cross section of all the members of the framework shown in Fig. 316 is k. All the joints are rigid connections. Prove that the bending moment set up in AB at A is $3W(l + 2b)k^2/a(l + a)$. (Mech. Sc. Trip., 1923, B.)

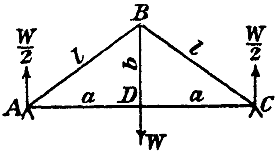

Fig. 316.

6. In Fig. 317 the horizontal joist and the supporting stanchions have the same section, and they are riveted together so that the joints at A and B are rigid. The stanchions are encastré vertically at their lower extremities. Prove that the joints A and B are subjected to bending moments $Wa^2/(4a+h)$ and that the joist AB is subjected to a horizontal thrust $3Wa^2/2h(4a + h)$. The deflections due to axial loads can be neglected. (Mech. Sc. Trip., 1922.)

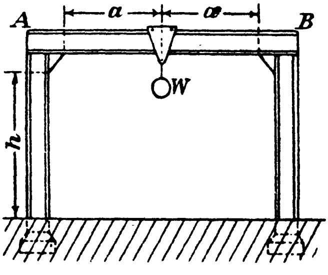

Fig. 317.

CHAPTER XXII

BENDING COMBINED WITH TORSION AND THRUST

262. Introductory.—Innumerable cases arise in practice where a shaft is subjected to bending as well as twist, the bending being due either to the weight of the shaft or to transverse loads from belts or cranks; in some cases, such as the propeller shafts of ships, there is an axial thrust in addition to torsion and bending. In such cases we have to consider (i) torsion stresses, (ii) tensile and compressive stresses due to bending, (iii) shear stresses due to bending, and then find the principal stresses. Frequently the shear stresses due to bending are unimportant, but they may have a large influence when the shaft is very short.

263. Torsion Combined with Pure Bending.—Suppose a round shaft is subjected to a uniform bending moment M and a torque T, and that bending takes place in vertical planes, the shaft becoming concave upwards.

Let d = the outside diameter of the shaft.
I = the flexural moment of inertia of the cross section, and
J = the torsional moment of inertia (pp. 106 and 180).
Then $J = 2I$.

At the bottom and top of the shaft there will be tensile stresses

$$p = \pm \frac{Md}{2I} = \pm \frac{Md}{J} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

At all points on the surface there will be shear stresses due to torsion given by (§ 89)

$$q = \frac{Td}{2J} = \frac{Td}{4I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

At the bottom of the shaft the principal stresses will be

$$\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q^2},$$

the upper sign giving a tensile stress and the lower a compressive stress giving p the + sign in (i). At the top of the shaft the principal stresses will be given by the same expression, the maximum compressive stress occurring when both p and the radical are given the negative sign. Thus the maximum tensile and compressive stresses are both given numerically by

$$\mathrm{p_{max}} = \frac{\mathrm{Md}}{4\mathrm{I}} + \sqrt{\frac{\mathrm{M^2d^2}}{16\mathrm{I}^2} + \frac{\mathrm{T^2d^2}}{16\mathrm{I}^2}} \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\mathrm{p_{max}} = \frac{\mathrm{Md}}{2\mathrm{J}} + \sqrt{\frac{\mathrm{M^2d^2}}{4\mathrm{J}^2} + \frac{\mathrm{T^2d^2}}{4\mathrm{J}^2}}. \quad . \quad . \quad . \quad (2)$$

according to which form we take for p and q from (i) and (ii).

The first of these expressions can be written in the form

$$\frac{d}{2I} \cdot \left[\frac{M}{2} + \tfrac{1}{2}\sqrt{M^2 + T^2}\right]$$

which is the tensile or compressive stress which would be produced by a bending moment

$$M' = \tfrac{1}{2}(M + \sqrt{M^2 + T^2}) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

acting alone ; M' is therefore sometimes called the *equivalent bending moment*.

Similarly (2) can be written

$$\frac{d}{2J}[M + \sqrt{M^2 + T^2}]$$

which is the maximum principal stress which would be produced by a torque

$$T' = M + \sqrt{M^2 + T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

acting alone ; T' is often called the *equivalent torque.*

Since $\sqrt{p^2/4 + q^2}$ is greater than $p/2$, the two principal stresses must be of opposite sign, and we have seen in Chapter VI that in this case the maximum principal stress cannot be regarded as a criterion of failure, at least when dealing with ductile materials. It follows that *the above expressions for equivalent bending moment and equivalent torque must not be used when static yield is under consideration,** but they are important in relation to fatigue. It was shown in Chapter VI that when dealing with unlike principal stresses we should use the maximum shear theory, or the maximum strain-energy theory, as a criterion of failure, the latter for preference. The maximum shearing stress is

$$\sqrt{\frac{p^2}{4} + q^2} = \frac{d}{2J}\sqrt{M^2 + T^2},$$

which is the same as would be produced by a torque

$$\mathbf{T} = \sqrt{\mathbf{M}^2 + \mathbf{T}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If we adopt the strain-energy theory we must have

$$p_1^2 + p_2^2 - \frac{2}{m}p_1p_2 = f^2,$$

---

* *Cf.* Ex. 1 below, which shows that (4) errs on the side of danger, which in fact is obvious. See footnote, p. 372.

where p_1 and p_2 are the principal stresses and f is the yield point of the material. We have

$$p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}$$

$$p_2 = \frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2}$$

Substituting these values of p_1 and p_2 we get

$$p^2 + \frac{2(m+1)}{m}q^2 = f^2.$$

If we now put $p = Md/2I$ and $q = Td/4I$, this equation becomes

$$\frac{d^2}{4I^2}\left(M^2 + \frac{m+1}{2m}T^2\right) = f^2.$$

Hence, if $\mathbf{T}$ be the pure torque required to produce failure we must have

$$\frac{d^2}{4I^2}\left(M^2 + \frac{m+1}{2m}T^2\right) = \frac{d^2}{4I^2}\cdot\frac{m+1}{2m}\mathbf{T}^2$$

which gives

$$\mathbf{T} = \sqrt{\frac{2\mathrm{m}}{\mathrm{m}+1}\mathrm{M}^2 + \mathrm{T}^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This will give a slightly larger value of $\mathbf{T}$ than the maximum shear-stress theory; the Misés-Hencky formula will be obtained by putting $m = 2$ in (6).

264. Torsion Combined with Bending and Shear.—If, in addition to the bending moment M and torque T, there be a shearing force F acting on any transverse section in a vertical direction, at the ends of a horizontal diameter there will be shearing stresses of amount (p. 203)

$$\frac{16F}{3\pi d^2}\text{for a solid shaft.}$$

These shearing stresses will occur on planes at right angles to the axis of the shaft, and on the horizontal plane through the axis of the shaft. In the same planes there will be the torsional shear stresses

$$\frac{Td}{2J} = \frac{16T}{\pi d^3}\text{ for a solid shaft.}$$

Hence the total shear stress will be

$$\mathrm{q} = \frac{16F}{3\pi d^2} + \frac{16T}{\pi d^3} = \frac{\mathbf{16}}{\pi \mathrm{d}^3}\left(\frac{\mathrm{Fd}}{3} + \mathrm{T}\right) \quad . \quad . \quad . \quad . \quad (7)$$

At these points (on the horizontal diameter) there will be no longitudinal stresses due to bending, so that (7) will also give the numerical value of the equal and opposite principal stresses. At the top and bottom of the cross section the shear stresses due to bending are zero, so that there the principal stresses are the same as in § 263. The maximum shear stress at the top or bottom, for a solid shaft is

$$\frac{d}{2J}\sqrt{M^2+T^2}=\frac{16}{\pi d^3}\sqrt{M^2+T^2} \quad . \quad . \quad . \quad . \quad (8)$$

Comparing (7) and (8), according to the maximum shear-stress theory the shaft will fail on a diameter in the plane of bending, or perpendicular to that plane, according as

$$\frac{Fd}{3}+T < \text{or} > \sqrt{M^2+T^2}$$

i.e. according as

$$(Fd)^2+6(Fd)T < \text{or} > 9M^2 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

An analogous criterion can be worked out for a hollow shaft, or the strain-energy theory may be used in either case.

265. Torsion Combined with Bending, Shear and Thrust.—In many cases a shaft is subjected to thrust as well as the forces dealt with in the preceding paragraph ; when this is so we must add to the longitudinal stresses due to bending, given by (i) of § 263, the longitudinal stress due to the thrust.

Let P = the thrust, and S = the area of the cross section of the shaft, and suppose as before that the shaft is subjected to a sagging bending moment M. Then, at the top of a cross section the compressive stress is

$$p=\frac{P}{S}+\frac{Md}{2I},$$

whilst at the bottom it is

$$p=\frac{P}{S}-\frac{Md}{2I}.$$

The principal stresses can then be worked out as before.

Example 1.—A mild steel shaft 4″ diameter is subjected to a torque of 6 tons. ins. and a uniform bending moment of 4 tons. ins. Calculate the principal stresses and the maximum shear stress. Also calculate the equivalent torque (*a*) according to the maximum principal stress theory, (*b*) according to Haigh's strain-energy theory, taking $m = 10/3$.

We have

$$I=\frac{\pi\times 4^4 \text{ ins.}^4}{64}=4\pi \text{ ins.}^4$$

$$J=8\pi \text{ ins.}^4$$

The maximum direct stress due to bending is

$$p=\frac{4 \text{ tons. ins.} \times 2 \text{ ins.}}{4\pi \text{ ins.}^4}=0{\cdot}636 \text{ tons/in.}^2$$

The maximum shear stress due to twisting is

$$q=\frac{6 \text{ tons. ins.} \times 2 \text{ ins.}}{8\pi \text{ ins.}^4}=0{\cdot}478 \text{ tons/in.}^2$$

Hence the principal stresses are

$$0{\cdot}318 \pm \sqrt{0{\cdot}101+0{\cdot}228} \text{ tons/in.}^2$$
$$=0{\cdot}318 \pm 0{\cdot}574 \text{ tons/in.}^2$$

The maximum principal stress = 0·892 tons/in.^{2}
„ minimum „ „ = — 0·256 „
„ maximum shear „ = 0·574 „

The maximum principal stress will be compressive on one side of the shaft and tensile on the other.

The angles between the principal planes and the plane of a normal section of the shaft are given by

$$\tan 2\theta = \frac{2 \times 0{\cdot}478}{0{\cdot}636} = 1{\cdot}5$$

$$2\theta = 56^\circ\ 18' \text{ or } 236^\circ\ 18'$$
$$\theta = 28^\circ\ 9' \text{ or } 118^\circ\ 9'$$

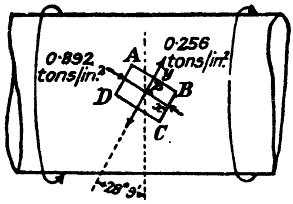

FIG. 318.

In Fig. 318 the shaft is supposed bent concave upwards perpendicular to the plane of the figure, the directions of the principal planes at the point P are Px and Py; the sides AB and CD of the element $ABCD$ suffer a tensile stress 0·256 tons/in.^{2} across them, and the sides AD and BC a compressive stress of 0·892 tons/in.^{2} On the opposite side of the shaft the directions of the principal planes will be the same but compression and tension are interchanged.

According to the maximum shear theory the equivalent torque is, from (5),

$$\sqrt{4^2 + 6^2} = 7{\cdot}2 \text{ tons. in.}$$

The strain-energy theory, by (6), gives

$$\sqrt{\frac{20}{13} \times 16 + 36} = 7{\cdot}78 \text{ tons. in};$$

which is 8 per cent. greater.*

Example 2.—A pulley weighing 600 lbs. is mounted on a shaft 2·5″ diameter midway between bearings which are 24″ apart, and the shaft is transmitting a torque of 5,000 lb. ins. Calculate the principal stresses at the extremities of (*a*) a vertical diameter, (*b*) a horizontal diameter, of a cross section of the shaft close to the pulley, also the maximum shear stress at the same points.

For the shaft we have

$$I = \frac{\pi \times 2{\cdot}5^4}{64} = 1{\cdot}92 \text{ ins.}^4$$

$$J = 3{\cdot}84 \text{ ins.}^4;\quad S = 4{\cdot}9 \text{ in.}^3$$

The maximum bending moment is

$$M = 300 \times 12 = 3{,}600 \text{ lbs. ins.},$$

and we shall take this value as obtaining on the section under consideration, although it will actually be slightly less by an amount depending on the width of the pulley.

The maximum shearing force is $F = 300$ lbs.

(*a*) At the top and bottom of the section the direct stress due to bending is

$$p = \frac{3{,}600 \text{ lbs. ins.} \times 1{\cdot}25 \text{ ins.}}{1{\cdot}92 \text{ ins.}^4} = 2{,}340 \text{ lbs./in.}^2$$

---

* The erroneous formula $M + \sqrt{M^2 + T^2}$ would give 11·07 tons. ins.

The shear stress due to torsion is

$$q = \frac{5{,}000 \text{ lbs. ins.} \times 1{\cdot}25 \text{ ins.}}{3{\cdot}84 \text{ ins.}^4} = 1{,}630 \text{ lbs./in.}^2$$

The shear stress due to F is zero.

Hence the principal stresses are

$$1{,}170 \pm \sqrt{1{,}170^2 + 1{,}630^2} \text{ lbs./in.}^2$$
$$= 1{,}170 \pm 2{,}000 \text{ lbs./in.}^2$$
$$= 3{,}170 \text{ and } -830 \text{ lbs./in.}^2$$

The maximum shear stress is 2,000 lbs./in.^{2}

(5) At the ends of a horizontal diameter p is zero, and the shear stress due to F is

$$\frac{4}{3} \times \frac{300}{4{\cdot}9} = 82 \text{ lbs./in.}^2$$

The shear stress due to torsion is 1,630 lbs./in.^{2} as before.

Hence $q = 1{,}630 + 82 = 1{,}712$ lbs./in.^{2}

This is the value of the maximum shear stress and of the principal stresses at these points, since $p = 0$. Hence the maximum shear stress is greatest at the ends of a vertical diameter.

EXAMPLES XXII

1. A uniform hollow steel shaft, 18″ external diameter and 12·5″ internal diameter, is supported between bearings 30 ft. apart and runs in salt water at 150 r.p.m. transmitting 13,500 H.P. Find the maximum principal stress due to combined bending and torsion. Density of salt water = 64 lbs./ft.^{3}; density of steel = 490 lbs./ft.^{3} (R.N.E.C., Keyham, 1923.)

2. A shaft 5″ diameter is subjected to a thrust of 15 tons along its axis. There is a bending moment on the shaft equal to half the twisting moment. If the maximum stress is 13,000 lbs./in.^{2}, find the H.P. which can be transmitted at 120 r.p.m. (R.N.E.C., Keyham, 1920.)

3. A solid steel marine engine weigh shaft is required to transmit a maximum torque of 7 tons. ft. Given that the shear stress should not exceed 9,000 lbs./in.^{2}, and that the angle of twist should be limited to 1° in a length of 20 ft., find a suitable diameter for the shaft. If the shaft were also subjected to a bending moment of 1 ton. ft., would this necessitate any modification of the original size ? if so, what would be the new diameter ? Take $C = 6{,}000$ tons/in.^{2} (R.N.E.C., Keyham, 1921.)

4. A solid steel shaft 6″ diameter transmits 1,000 H.P. at 200 r.p.m. Assuming that there is no constraint at the bearings, i.e. that each span of the shaft is simply supported and free-ended, find the maximum allowable span between bearings if the shear stress is limited to 8,000 lbs./in.^{2}, and then find the maximum direct stress. Take the weight of steel = 490 lbs./ft.^{3} (R.N.E.C., Keyham, 1922.)

5. The outside and inside diameters of a shaft are 16″ and 8″; the twisting moment is 200 tons. ft.; there is at the same time a bending moment of 50 tons. ft., and an end thrust of 35 tons. Find the maximum intensity of the compressive stress. (H.M. Dockyard Schools, 1921.)

6. In Fig. 319, C and D are two wheels keyed to a shaft 1″ diameter. The wheels are acted on by tangential forces as shown, so that the torque transmitted is 250 lb. ins. The shaft is carried in bearings at A and B. Determine the shear stress due to torque, the longitudinal stresses due to bending, and the principal stresses, for the top point of the cross section at P. (Intercoll. Exam., Cambridge, 1912.)

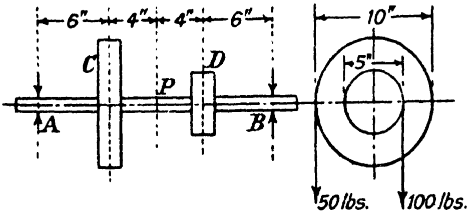

FIG. 319.

7. Fig. 320 below indicates the general arrangement and dimensions of the back axle of a motor car, weighing 2,400 lbs. The casing of the back axle has to transmit the braking torque, as well as support the part of the weight shown on the sketch. The car, when running at 30 miles per hour, is stopped in 40 ft. by the back axle brakes. Find the principal stresses set up at the section AB, which is $\frac{1}{4}$ in. thick and of external diameter $2\frac{1}{2}$ in. (Intercoll. Exam., Cambridge, 1919.)

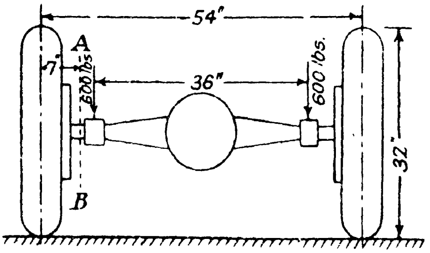

FIG. 320.

8. A shaft, 6 in. diameter and 10 ft. long from centre to centre of the bearings, carries a flywheel 12 ft. diameter, weighing 4 tons, midway between the bearings. The flywheel is in gear with a pinion vertically below it, and transmits 180 H.P., the flywheel shaft making 100 revolutions per minute. Find the pressure between the teeth and the maximum bending moment on the shaft. The power is conveyed to the shaft by a pure torque on one side of the flywheel. Estimate the maximum tensile, compressive and shear stresses in the material of the shaft due to the combined bending and twisting action. (Birmingham, 1910.)

9. An engine has an overhung crankshaft and the stroke is 12″. The centre line of the crankpin and connecting rod is 5·5″ outside the centre of the bearing. The thrust on the crank-pin is 2,000 lbs. at right angles to the crank. If the maximum shear stress due to combined bending and twisting is not to exceed 10,000 lbs./in.^{2}, find a suitable diameter for the crankshaft journal.

CHAPTER XXIII

STABILITY OF BENT AND TWISTED RODS

266. Introductory.—In dealing with the combination of bending and twisting above we have neglected the effects of the deformation of the axis of the rod, but in certain instances this is not permissible. In general, when a rod is both bent and twisted, the applied torque will not have its axis at right angles to the principal axes of inertia of the cross section, so that it will have components about these axes and thus will influence the bending. In the cases we have considered above this interaction between the bending moments and torque has been ignored. We shall now examine certain problems in which this mutual action has to be considered.

In §§ 267-272 we are considering the same *type* of phenomenon as the buckling of a straight strut, i.e. we find critical values of the external forces which bring about a change of phase in the elastic equilibrium.

267. Stability of a Straight Circular Rod Subjected to Thrust and Torsion.[*]—In Fig. 321 let OA be the unstrained axis of the rod, which is perfectly straight; let there be a truly axial thrust P and torque T applied to the ends of the rod. Suppose the axis can be held by this system in the position OCA, and let $OC'A$ and $OC''A$ be the projections of OCA on the planes xy and zx, so that $CC'' = y$, and $CC' = z$.

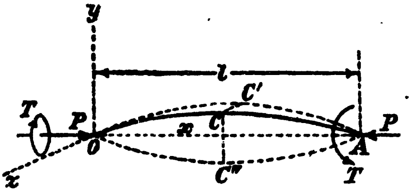

FIG. 321.

The axes of the section at C, which were originally parallel to the axes of y and z, now make with the axis of x angles whose cosines are $\frac{dy}{dx}$ and $\frac{dz}{dx}$. Hence the torque T has components about these axes, in their deflected position, which are given by $T\frac{dy}{dx}$ and $T\frac{dz}{dx}$; the former tends to increase the curvature produced by the end thrust in the plane

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[*] The method of solution given here is due to H. A. Webb.

xz, and the latter to decrease the curvature in the plane xy. For bending in the two planes, then, we have

$$-EI\frac{d^2y}{dx^2} = Py - T\frac{dz}{dx} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$-EI\frac{d^2z}{dx^2} = Pz + T\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

To solve these equations multiply (ii) by $i(=\sqrt{-1})$ and add the result to (i), writing

$$y + iz = u.$$

Then we get

$$EI\frac{d^2u}{dx^2} + iT\frac{du}{dx} + Pu = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

The auxiliary equation is

$$EI \, . \, D^2 + iT \, . \, D + P = 0,$$

the roots of which are

$$D = \frac{1}{2EI}[-iT \pm \sqrt{-T^2 - 4EI \, . \, P}]$$

$$= \frac{i}{2EI}[-T \pm \sqrt{T^2 + 4EIP}]$$

Hence the solution of (iii) is

$$u = Ae^{-i(\alpha x - \beta x)} + Be^{-i(\alpha x + \beta x)}$$

$$= e^{-i\alpha x}(A' \sin \beta x + B' \cos \beta x) \quad . \quad . \quad . \quad \text{(iv)}$$

where

$$\alpha = \frac{T}{2EI}$$

$$\beta = \frac{\sqrt{T^2 + 4EIP}}{2EI}$$

and A' and B' are constants of integration. When $x = 0$ or l we have $y = z = 0$, and therefore $u = 0$. Hence we must have $B' = 0$ and

$$A' \sin \beta l = 0.$$

Thus the criterion for instability is $\beta l = \pi$, which gives

$$\frac{\pi^2}{l^2} = \frac{T^2}{4E^2I^2} + \frac{P}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This may be written

$$\mathbf{P} = \mathbf{P_e} - \frac{\mathbf{T^2}}{\mathbf{4EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where P_e is the Euler crippling load of the rod.

268. Non-Circular Rods.—Let I_1 and I_2 be the principal moments of inertia of the cross section, the corresponding principal axes being originally parallel to Oz and Oy (Fig. 318). Then in (i) and (ii) we must write I_1 and I_2 respectively instead of I. From these equations we can show that the end load required to cause instability is the smaller of the two values given by

$$\left.\begin{aligned} P &= \frac{\pi^2 EI_1}{l^2} - \frac{T^2}{4EI_2} \\ P &= \frac{\pi^2 EI_2}{l^2} - \frac{T^2}{4EI_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly, if P is zero, the torque required to produce instability is given by

$$T = \frac{2\pi E\sqrt{I_1 I_2}}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

269. Stability of Thin Deep Cantilever with Concentrated Load.*—Suppose we have a uniform beam OA (Fig. 322) with the end O fixed in the direction Ox, whilst the principal axes of the cross sections are parallel to Oy and Oz. If the principal moments of inertia of the

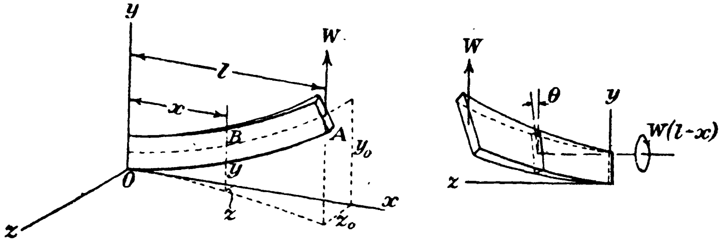

FIG. 322.

cross section are comparable with one another, and the beam be bent by a load W at A, parallel to Oy, flexure will take place entirely in planes parallel to xOy. But if the stiffness of the beam for bending in the plane xOz be much smaller than for bending in the plane xOy, it is possible for the load W to hold the beam deflected so that its axis is not in the plane xy, and in this case the beam is also twisted.

Let I_1 = the moment of inertia of the cross section about the principal axis which is parallel to Oz, and I_2 that about the principal axis parallel to Oy, and suppose that I_2 is small compared with I_1.

Suppose that the load W can deflect the end A to the position $y_0 z_0$ as shown, and let θ be the total twist of the beam between O and some section B distant x from O.

The section at B is then acted on by a couple $W(l - x)$ about an axis parallel to Oz, which makes with the principal axes of the section angles whose cosines are approximately unity and $-\theta$. Thus the component bending moments about the two principal axes are $W(l - x)$ and

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* A. G. M. Michell, *Phil. Mag.*, 1899; and Pradtl, "*Kipperscheinungen*," Nürnberg, 1899.

$- W(l - x)\theta$, so that the equations for the deflected axis are

$$EI_1\frac{d^2y}{dx^2} = W(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$EI_2\frac{d^2z}{dx^2} = - W(l - x)\theta \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The couple $W(l - x)$ also has a torque component about the tangent to the axis of the beam at B equal to $W(l - x)\frac{dz}{dx}$, whilst the vertical force W exerts a torque $W(z_0 - z)$; the former tends to increase θ whilst the latter tends to decrease it. Hence we have

$$CK\frac{d\theta}{dx} = W(l - x)\frac{dz}{dx} - W(z_0 - z) \quad . \quad . \quad . \quad \text{(iii)}$$

where K is the torsion constant for the section, and C is the modulus of rigidity. The calculation of K is dealt with in Chapter XXX.

Differentiating (iii) with respect to x we have

$$CK\frac{d^2\theta}{dx^2} = W(l - x)\frac{d^2z}{dx^2}$$

$$= - \frac{W^2(l - x)^2}{EI_2}\theta, \text{ from (ii).}$$

This may be written

$$\frac{d^2\theta}{dx^2} + \beta^2(l - x)^2\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

where

$$\beta^2 = \frac{W^2}{CKEI_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

The solution of (iv), in terms of Bessel Functions,* is

$$\theta = \sqrt{l - x}\left[A \,.\, J_{\frac{1}{4}}\left\{\frac{\beta}{2}(l - x)^2\right\} + B.J_{-\frac{1}{4}}\left\{\frac{\beta}{2}(l - x)^2\right\}\right] \text{(vi)}$$

where A and B are constants of integration.

When $x = l$, $z = z_0$, therefore $\frac{d\theta}{dx} = 0$, from (iii), so that A must be zero.

When $x = 0$, $\theta = 0$, and the criterion for instability is

$$J_{-\frac{1}{4}}\left(\frac{\beta l^2}{2}\right) = 0$$

which gives $\frac{\beta l^2}{2} = 2$, very closely, or

$$W = \frac{4\sqrt{CKEI_2}}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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* *Funktionentafeln mit Formeln und Kurven*, by Jahnke and Emde (Teubner), contains tables of Bessel Functions and a most useful collection of differential equations with their solutions in terms of Bessel Functions.

270. Thin Deep Cantilever with Distributed Load.—We shall next consider the case when the load on the cantilever is uniformly distributed.

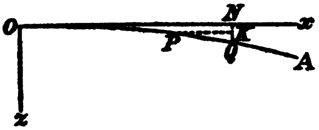

FIG. 323.

Let w be the vertical load per unit length in the direction Oy in Fig. 323 which shows the projection of the distorted axis of the beam on the xz plane. Let P be the point (x, y, z), and let Q be a point (x', y', z') between P and the free end A.

The force parallel to Oy, due to an element dx' at Q, is $w.dx'$. This produces a torque round $PK = w.dx'.QK$, PK and QK being parallel to Ox and Oz respectively. Hence in the notation of § 269,

$$T = -\int_x^l w.QK.dx' \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

l being the length of the beam.

Then, as in the previous case, we form the equations

$$-EI_2\frac{d^2z}{dx^2} = \frac{w}{2}(l-x)^2\theta + T\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad (\text{ii})$$

$$CK\frac{d\theta}{dx} = \frac{w}{2}(l-x)^2\frac{dz}{dx} + T \quad . \quad . \quad . \quad . \quad (\text{iii})$$

We shall suppose that flexure in the plane xy can be neglected, so that we omit the last term of (ii). Then (iii) gives

$$CK\frac{d^2\theta}{dx^2} = -w(l-x)\frac{dz}{dx} + \frac{w}{2}(l-x)^2\frac{d^2z}{dx^2} + \frac{dT}{dx} \quad . \quad (\text{iv})$$

From (i) we have

$$\frac{dT}{dx} = +[w.QK]_{x'=x} - w\int_x^l \frac{d(QK)}{dx}\,.\,dx' \quad . \quad . \quad . \quad (\text{v})$$

Now, when $x' = x$, QK is zero, so that the term [ ] vanishes. Also

$$QK = z' - z$$

$$\therefore \frac{d(QK)}{dx} = -\frac{dz}{dx}$$

$$\therefore \int_x^l \frac{d(QK)}{dx}dx' = -\int_x^l \frac{dz}{dx}\,.\,dx' = -(l-x)\frac{dz}{dx}.$$

Hence, from (v),

$$\frac{dT}{dx} = w(l-x)\frac{dz}{dx}.$$

Substituting in (iv) we have

$$CK\frac{d^2\theta}{dx^2} = \frac{w}{2}(l-x)^2\frac{d^2z}{dx^2}$$

$$= -\frac{w^2}{4EI_2}(l-x)^4\theta,$$

from (ii), remembering that we are neglecting $T\frac{dy}{dx}$. Hence

$$\frac{d^2\theta}{dx^2} + \frac{w^2(l-x)^4}{4CKEI_2} \,.\, \theta = 0$$

or

$$\frac{d^2\theta}{dx^2} + \lambda^2(l-x)^4\theta = 0,$$

where

$$\lambda^2 = \frac{w^2}{4CKEI_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

The solution of this equation is

$$\theta = (l-x)^{\frac{1}{2}}\left[AJ_{\frac{1}{6}}\left\{\frac{\lambda(l-x)^3}{3}\right\} + BJ_{-\frac{1}{6}}\left\{\frac{\lambda(l-x)^3}{3}\right\}\right]$$

In the same manner as § 269, we find that the criterion for instability is

$$J_{-\frac{1}{6}}\left(\frac{\lambda l^3}{3}\right) = 0$$

which gives

$$\frac{\lambda l^3}{3} = 2{\cdot}15$$

i.e.

$$\frac{wl^3}{6\sqrt{CKEI_2}} = 2{\cdot}15$$

or

$$wl^3 = 12{\cdot}9\sqrt{CKEI_2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

271. Thin Deep Beam under Constant Bending Moment.—The next problem of this nature which we shall consider is that shown in Fig. 324, where $ABCD$ is a thin deep beam acted on by terminal couples M in the plane of its depth, when unstrained, which is the plane xy. We suppose that dimensions are such that the flexure in the plane xy is negligible, and seek to find what value of M, if any, will cause the beam to bend parallel to the xz plane. If this happens there will be twist about the axis Ox.

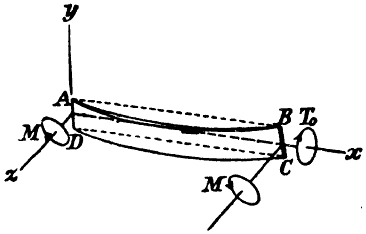

FIG. 324.

In the first place we assume that the only constraint applied to the ends is such that the edges AD and BC remain in the plane xy. Then, if the beam twist in the manner shown, it follows that there must be a "fixing" torque T_0 applied to the ends by the fixings.

Hence

$$EI_2\frac{d^2z}{dx^2} = M\theta + T_0\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$CK\frac{d\theta}{dx} = -M\frac{dz}{dx} - T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The flexure in the plane xy being negligible, we can omit the last term in (i).

Eliminating z from (i) and (ii) we get

$$\frac{d^2\theta}{dx^2} + \beta^2\theta = 0$$

where
$$\beta^2 = \frac{M^2}{CKEI_2}.$$

Hence

$$\theta = A \cos \beta x + B \sin \beta x.$$

where A and B are constants.

Since $\theta = 0$ when $x = 0$, we must have $A = 0$. Then, as θ also vanishes when $x = l$, we must have

$$B \sin \beta l = 0$$

Hence, either $B = 0$, in which case there is no twist, or $\beta l = \pi$,

i.e.
$$\mathrm{M} = \frac{\pi}{l}\sqrt{\mathrm{CKEI}_2}\,. \qquad (7)$$

This is the smallest value of M which will hold the beam deflected in the supposed manner ; θ then becomes indeterminate and the beam is unstable.

If the ends of the beam are encastré in the xz plane, " fixing " moments must be introduced at the ends, acting in this plane. We find that the critical value of M is given by

$$M = \frac{2\pi}{l}\sqrt{CKEI_2}\,. \qquad (8)$$

These problems have been investigated experimentally by Carrington,* whose results, so far as they go, confirm the formulæ obtained by analysis.

272. The Case of I-Beams.—The above results require modification† when the beam carries stiffening flanges, such as those on **I**, **T** beams etc., since the lateral flexure of the flanges now becomes important.

Referring to Fig. 325, let ζ be lateral deflection of the top of the web relative to the axis, and let I_3 denote the moment of inertia of each flange about the axis BB. Then the extra shearing force on the flange is

$$EI_3\frac{d^3\zeta}{dx^3}$$

The moment of this about the axis of the beam is

$$\frac{h}{2}EI_3\frac{d^3\zeta}{dx^3}.$$

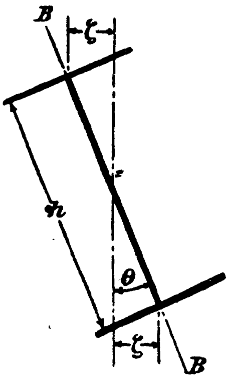

FIG. 325.

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* *Phil. Mag.*, 1922.

† M. S. Timochenko, *Sur la Stabilité des Systèmes Élastiques.* Pub. A. Dumas, Paris, or see *Annales des Ponts et Chaussées*, Fasc. III–V, 1913.

There will be a corresponding force on the bottom flange, so that on account of the shear in the flanges there is a torque

$$hEI_3\frac{d^3\zeta}{dx^3} = \frac{h^2}{2}EI_3\frac{d^3\theta}{dx^3},$$

since $\zeta = \frac{h}{2}\theta$ approximately. Hence the left-hand side of equation (iii), § 269, becomes

$$CK\frac{d\theta}{dx} - \frac{EI_3h^2}{2}\frac{d^3\theta}{dx^3}.$$

273. Uniform Bending Moment.—The equations of § 271 now become

$$EI_2\frac{d^2z}{dx^2} = M\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$CK\frac{d\theta}{dx} - \tfrac{1}{2}EI_3h^2\frac{d^3\theta}{dx^3} = -M\frac{dz}{dx} - T_0 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

neglecting flexure in the plane xy. Differentiating the second we have

$$\begin{aligned} CK\frac{d^2\theta}{dx^2} - \tfrac{1}{2}EI_3h^2\frac{d^4\theta}{dx^4} &= -M\frac{d^2z}{dx^2} \\ &= -\frac{M^2}{EI_2}\theta \end{aligned}$$

or

$$\frac{d^4\theta}{dx^4} - \frac{2CK}{EI_3h^2}\frac{d^2\theta}{dx^2} - \frac{2M^2\theta}{E^2I_2I_3h^2} = 0 \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

It will be found that

$$\theta = A\sin\frac{n\pi x}{l}$$

satisfies (iii), and all the terminal conditions, provided that

$$\frac{n^4\pi^4}{l^4} + \frac{n^2\pi^2}{l^2}\cdot\frac{2CK}{EI_3h^2} = \frac{2M^2}{E^2I_2I_3h^2}$$

or

$$M^2 = \frac{n^2\pi^2}{l^2}(EI_2CK) + \frac{n^4\pi^4}{l^4}(EI_2)\,(EI_3)\frac{h^2}{2}$$

and then A is indeterminate. Hence, if we give n the value unity we shall obtain the least value of M consistent with the assumed deformation, and this will be the critical value. Thus

$$M = \frac{\pi}{l}\sqrt{EI_2CK}\,\sqrt{1 + \frac{\pi^2EI_3}{2CK}\frac{h^2}{l^2}} \quad . \quad . \quad . \quad . \quad (9)$$

274. Other Cases.—The following are some of the results which have been obtained by M. Timochenko (*loc. cit.*), and are given for reference. They all refer to **I** section beams.

(i) Constant bending moment M in xy plane, axis fixed in direction at the ends in the xz plane.

$$M = \frac{2\pi\sqrt{CKEI_2}}{l}\sqrt{1 + 2\pi^2\frac{EI_3}{CK}\frac{h^2}{l^2}}$$

(ii) Same as (i), but ends free in xz plane, with the addition of an axial thrust P.

$$M = \frac{\pi\sqrt{CKEI_2}}{l}\sqrt{1 + \frac{\pi^2}{2}\cdot\frac{EI_3}{CK}\frac{h^2}{l^2}}\sqrt{1 - \frac{Pl^2}{\pi^2 EI_2}}$$

(iii) Same as (ii), but ends fixed in xz plane.

$$M = \frac{2\pi\sqrt{CKEI_2}}{l}\sqrt{1 + 2\pi^2\frac{EI_3}{CK}\cdot\frac{h^2}{l^2}}\sqrt{1 - \frac{Pl^2}{4\pi^2 EI_2}}$$

(iv) Cantilever with load W at the end.

$$W = \frac{k\sqrt{CKEI_2}}{l}, \text{ where } k \text{ is given by}$$

$\frac{2CKl^2}{EI_3h^2}$ . .	0·1	1·0	2·0	4·0	8·0	12·0	16·0	24·0	32·0	40
k . . .	44·3	15·7	12·2	9·76	8·03	7·2	6·73	6·19	5·87	5·64

(v) Beam of length l carrying a distributed load w per unit length, the ends being free to rotate in the xz plane; $w = k\sqrt{CKEI_2}/l^3$, where k is given below, and $a^2 = \frac{EI_3h^2}{2CKl^2}$.

$1/a^2$.	0·40	4·0	8·0	16	32	48	64	96	160	320	∞
1. $k/8$.	17·9	6·63	5·32	4·54	4·08	3·94	3·81	3·73	3·65	3·58	3·54
2. $k/8$.	11·6	4·54	3·8	3·43	3·28	3·27	3·22	3·25	3·27	3·31	3·54
3. $k/8$.	27·7	9·77	7·43	6·01	5·09	4·76	4·5	4·3	4·08	3·87	3·54

1. Load applied along axis; 2, load applied on top; 3, load applied at bottom.

(vi) The same as (v), but the ends encastré in the xz plane, a^2 having the same meaning as in (v), 1.

$1/a^2$ . . .	0·4	4	8	16	32	96	128	200	400
k . . .	488	160·8	119·2	91·2	73·04	58·00	55·84	53·44	51·20

(vii) Beam supported at each end, with a concentrated load W at the centre. The critical load W is given by $W = k\sqrt{CKEI_2}/l^2$, k being obtained from the following table:

$1/a^2$ . . .	0·4	4·0	8·0	16	32	64	96	160	320	∞
k . . .	86·4	31·9	25·6	21·8	19·4	18·3	17·9	17·5	17·2	17·0

The load is here supposed to be applied at a point on the axis of the beam.

(viii) The same as (vii), but with ends prevented from rotating in the xz plane.

$1/a^2$ . . .	0·4	4·0	8·0	16	32	64	96	160	320	400
k . . .	268	88·8	65·5	50·2	40·2	34·2	31·8	30·0	28·5	28·2

If there be an axial thrust as well as the bending moments in cases (v) and (vi), the above values must be multiplied by the following factors :

$$\text{(v) and (vii)} \quad \sqrt{1-\frac{Pl^2}{\pi^2 EI_2}}$$

$$\text{(vi) and (viii)} \quad \sqrt{1-\frac{Pl^2}{4\pi^2 EI}}$$

EXAMPLES XXIII*

1. If a narrow deep beam, such as shown in Fig. 324, carries a vertical load W at the middle point, prove that torsional instability will occur when

$$Wl^2 = 16{\cdot}94\sqrt{ECI_2K}.$$

2. In question 1, if the ends are encastré in the horizontal plane, prove that the critical value of W is given by

$$Wl^2 = 25{\cdot}9\sqrt{ECI_2K}.$$

3. In the case of § 271, if the section of the beam varies in such a manner that

$$I_2 = I_0\left(1+\frac{ax}{l}\right) \text{ and } K = K_0\left(1+\frac{ax}{l}\right),$$

and the ends of the beam are free in direction in the xz plane, show that the twist is given by

$$\left(1+\frac{ax}{l}\right)^2\frac{d^2\theta}{dx^2}+\frac{a}{l}\left(1+\frac{ax}{l}\right)\frac{d\theta}{dx}+\frac{M^2}{CEI_0K_0}\theta = 0.$$

Hence, by the substitution $1+\dfrac{ax}{l} = e^{\xi}$, solve the equation and show that instability will occur when

$$M > \frac{\pi a\sqrt{CEI_0K_0}}{l.\log_e(1+a)},$$

a being a constant which gives the rate of change of I_2 and K along the beam.

4. In § 271, if there be an axial thrust P in addition to the couples M, and if the section is not uniform, prove that the twist is given by

$$\frac{d}{dx}\left[\frac{EI_2}{M}\cdot\frac{d}{dx}\left(CK\frac{d\theta}{dx}\right)+M\theta\right]+\frac{PCK}{M}\cdot\frac{d\theta}{dx} = 0.$$

5. A narrow deep beam, simply supported at both ends, carries a uniformly distributed load w per unit length. Show that the twist is given by

$$\frac{d^2\theta}{dx^2}+\frac{w^2x^2(l-x)^2}{4ECI_2K}\theta = 0,$$

and hence show that the beam will be unstable if

$$wl^3 > 28{\cdot}3\sqrt{ECI_2K}$$

6. A thin deep cantilever of uniform section carries a total load W which decreases uniformly from a maximum at the fixed end to zero at the free end. Show that the condition for instability is

$$J_{-\frac{1}{8}}\left(\frac{Wl^2}{12\sqrt{CKEI_2}}\right) = 0$$

that is when $$Wl^2 = 26{\cdot}51\sqrt{CKEI_2}$$

7. If the section of a cantilever carrying a load W at the free end tapers from the fixed end, considered as origin, in such a way that $I_2 = I_0\left(1 - \frac{x}{l}\right)$ and $K = K_0\left(1 - \frac{x}{l}\right)$ show that instability occurs when

$$J_0\left(\frac{Wl^2}{\sqrt{CK_0EI_0}}\right) = 0,$$

i.e. when $$Wl^2 = 2{\cdot}405\sqrt{CK_0EI_0}.$$

8. In question 7 if the total load W be uniformly distributed show that the criterion for torsional instability is $Wl^2 \gtrless 9{\cdot}62\sqrt{CK_0EI_0}$, whilst, if it decrease uniformly from a maximum at the fixed end to zero at the free end, the criterion is $Wl^2 \gtrless 21{\cdot}65\sqrt{CK_0EI_0}$.

9. A strut of uniform wide thin section is acted upon by an eccentric end load, the load being eccentric by an amount h along the major principal axis. Prove that the load P which will cause torsional instability is given by

$$\frac{P}{EI_2}\left(1 + \frac{Ph^2}{CK}\right) = \frac{\pi^2}{l^2},$$

where I_2 is the smaller moment of inertia of the cross section, if the ends of the axis are free to take up any angular position in the xz plane.

10. If the beam in § 271 be subjected to an axial thrust P, prove that the criterion of instability is

$$\frac{M^2}{CKEI_2} + \frac{P}{EI} = \frac{\pi^2}{l^2},$$

the ends being free to bend in the xz plane.

* The results of examples 6, 7, 8 are due to H. A. Webb, those of 1, 2, 5, 9 are due to Michell.

CHAPTER XXIV

SPRINGS

275. General Properties of Springs.—The common purpose of all kinds of springs is to absorb energy and restore it slowly or rapidly according to the function of the particular spring under consideration. Thus, in the case of clockwork a certain amount of work is done by an external agency in winding up, i.e. deforming, the spring; this work is stored in the form of strain energy and is regained when the spring is allowed to return to its original shape. In clockwork the resumption of the spring's original shape takes place slowly. The other most common use of springs is for absorbing shocks, such as the springs of buffers of railway rolling-stock and the springs of wheels on all manner of vehicles. In such cases some of the kinetic energy of the moving body, the truck, or that due to the vertical motion of the wheels and axles, is converted into strain energy in the spring, the effects of the blow on the truck as a whole being thereby reduced. The springs, in returning to their original shape, give back this energy tending to reverse the relative motion of the colliding bodies. Springs are also used to provide a means of restoring various mechanisms to their original configuration against the action of some external force, or when an external force is removed.

The properties of a spring which are usually of most interest to the engineer are (i) its capacity for absorbing energy, (ii) the deformation produced by a given load, or vice versa, provided of course that the safe working stress of the material is not exceeded, and sometimes (iii) its natural frequency of vibration.

Springs in practice belong usually to one of two definite families: springs in which a length of rod or wire is made into a coil of some kind, and springs consisting of one or more approximately flat plates.

The " stiffness " of a spring is the load required to produce unit deflection.

The " resilience " of a spring is its capacity for storing energy without exceeding a certain stress limit.

276. Coiled Springs.—Coiled springs may be divided into (i) ordinary helical springs, when the axis of the wire has the form of a helix described on a right circular cylinder; (ii) helical springs in which the axis of the wire is a helix described on a right circular cone; (iii) spiral springs, when the axis forms a plane spiral curve.

277. Geometry of Helical Springs.—In Fig. 326, *CQB* is a helix described on a cylinder whose axis is *OA*, this helix being the central line of the wire forming the spring. *FQD* is a generator of the cylinder; *Dx* is the tangent at *D* to the cross section *CDH* of the cylinder, and *Qx* is the tangent at *Q* to the helix. *Qy* is in the tangent plane *FDx* and perpendicular to *Qx*. *QK* is parallel to *Dx*.

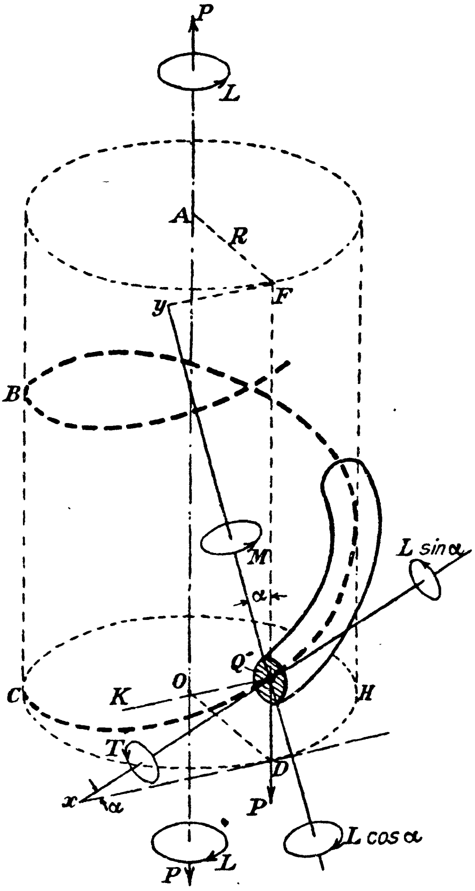

FIG. 326.

Let R be the radius of the cylinder, and let α be the pitch angle of the helix, so that $FQy = QxD = \alpha$.

Let l = the length of wire.

n = the number of complete convolutions.

Then

$$l = 2\pi nR \sec \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Let the spring be acted on by axial forces P and axial couples L. Now forces P acting along the axis of the cylinder are equivalent to forces P along DF and couples $P.R$ about QK.

The force P along QD * can be resolved into components $P\cos\alpha$ along yQ, i.e. a shearing force in the plane of the normal cross section of the wire at Q, and $P\sin\alpha$ along Qx, i.e. perpendicular to the same cross section. Usually the effects of these forces can be neglected.

The couple $P.R$ about QK can be resolved into components $T = PR\cos\alpha$ about Qx, i.e. a torsional couple acting on the wire, and $M = PR\sin\alpha$ about Qy, i.e. a flexural couple tending to decrease the curvature of the wire.

Similarly a couple L, whose axis is OA, will have a torsion component $L\sin\alpha$ about Qx in the same direction as T, as shown in Fig. 326, and a flexural component $L\cos\alpha$ about Qy tending to increase the curvature of the wire.

278. Close-Coiled Helical Spring : Axial Pull.—When the coils of a helical spring are so close together that they can be regarded as practically lying in planes at right angles to the axis of the helix, the angle α is very small, and we speak of the spring as "close-coiled."

With such springs under an axial tension P only, the bending couple $PR\sin\alpha$ becomes negligible in comparison with the torsion couple; we can also take $l = 2\pi nR$, with sufficient accuracy.

Let δ be the axial extension of the spring due to the load P applied gradually. Then

$$\text{the work done by } P = \tfrac{1}{2}P\delta.$$

The torsion couple at any point on the central line is $T = PR$, approximately. From equation (7), § 92, the strain energy of the deformed spring is

$$U = \frac{lT^2}{2CJ},$$

where l is the total length of wire, i.e. $2\pi nR$. Hence the energy stored in the spring is

$$U = \frac{\pi nR^3P^2}{CJ} = \frac{32nR^3P^2}{d^4C}, \quad \dots\dots \quad (1)$$

if the spring be made of round wire of diameter d.

Equating this to the work done by P we obtain an expression for the deflection :

$$\tfrac{1}{2}P\delta = \frac{\pi nR^3P^2}{CJ}$$

$$\therefore \delta = \frac{2\pi nR^3P}{CJ} = \frac{64nR^3P}{d^4C} \quad \dots\dots \quad (2)$$

Putting $\delta = 1$, we have

$$\text{stiffness} = \frac{CJ}{2\pi nR^3} = \frac{d^4C}{64nR^3} \quad \dots\dots \quad (3)$$

If q denote the maximum permissible shear stress, the resilience and maximum load can be expressed in terms of q thus :

---

* We are considering now the forces and couples transmitted *by* the part CQ of the wire *to* the part QB, and Fig. 326 is drawn accordingly.

From § 89, we have

$$q = \frac{Td}{2J} = \frac{PRd}{2J}$$

$$\therefore P = \frac{2Jq}{Rd} = \frac{\pi d^3 q}{16R}\text{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in (1) we obtain a formula for the stored energy in terms of the maximum stress :

$$\textbf{resilience} = \frac{4\pi nRJq^2}{Cd^2} = \frac{\pi^2 nRd^2q^2}{8C} = \frac{\mathbf{q}^2}{\mathbf{4C}} \times \textbf{volume of metal} \quad . \quad (5)$$

It will thus be seen that all such springs have equal weight for equal stress and equal resilience.

279. Close-Coiled Helical Spring : Axial Couple.—If a close-coiled spring is acted on by a couple L whose axis is the axis of the helix, we can neglect the torsional couple $L \sin \alpha$, and consider the wire as acted on everywhere by a flexural couple which is approximately equal to L.

Let θ = the total angle through which one end of the spring is turned relative to the other. Then the work done by L is $\frac{1}{2}L\theta$, θ being measured in radians.

From § 128 the strain energy of the spring is

$$\frac{L^2 l}{2EI} = \frac{\pi nRL^2}{EI}$$

Equating this to the work done by L we have

$$\boldsymbol{\theta} = \frac{\mathbf{2\pi nRL}}{\mathbf{EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For a circular section wire of diameter d we have

$$\theta = \frac{128nRL}{Ed^4}.$$

The maximum stress in the wire is given by

$$p = \frac{Ld}{2I} = \frac{32L}{\pi d^3}$$

Hence the maximum couple for a given stress is

$$\mathbf{L} = \frac{\boldsymbol{\pi}\mathbf{d^3p}}{\mathbf{32}}.$$

Putting θ equal to unity in (6) we have

$$\textbf{stiffness for torsion} = \frac{\mathbf{EI}}{\mathbf{2\pi nR}} \textbf{ per radian} \quad . \quad . \quad . \quad (7)$$

---

* Useful alignment charts expressing these formulæ were published by Mr. F. Fitchett in *Machinery*, Jan. 14, 1915.

The following practical rules are worth noting: A new close-coiled helical spring should be closed up solid *twice* in compression ; after this treatment it will take no more permanent set. If P be the load required to close the spring solid, the spring should never, even in tension, carry more than this load P.

Example.—Calculate the number of turns required for ¼″ deflection in a spring made of steel ¾″ diameter and forming a cylindrical coil 5″ mean diameter, if the load be 100 lbs. and $C = 12 \times 10^6$ lbs./in.^{2} (Mech. Sc. Trip., 1910.)

Assuming the coil to be close-coiled, we have from (2)

$$n = \frac{d^4 C\delta}{64 R^3 P}$$

$$d = 0{\cdot}75''. \quad d^4 = 0{\cdot}315 \text{ in}^4. \quad R = 2{\cdot}5'', \ R^3 = 15{\cdot}6 \text{ ins.}^3$$
$$P = 100 \text{ lbs.} \quad \delta = 0{\cdot}25''. \quad C = 12 \times 10^6 \text{ lbs./in.}^2$$

Hence
$$n = \frac{0{\cdot}315 \text{ in.}^4 \times 12 \times 10^6 \text{ lbs./in.}^2 \times 0{\cdot}25 \text{ in.}}{64 \times 15{\cdot}6 \text{ in.}^3 \times 100 \text{ lbs.}}$$
$$= 9{\cdot}45.$$

280. Open-Coiled Helical Spring : Axial Force.—We shall now consider a helical spring where the coils are not so close that the angle α can be treated as small.

Let P = the axial load, and let the rest of the notation be as in § 276. The torsion couple is $PR\cos\alpha$, and the flexural couple is $PR\sin\alpha$; hence the strain energy of the spring is

$$\frac{l(PR\cos\alpha)^2}{2CJ} + \frac{l(PR\sin\alpha)^2}{2EI}$$
$$= \frac{lP^2R^2}{2}\left(\frac{\cos^2\alpha}{CJ} + \frac{\sin^2\alpha}{EI}\right)$$

The work done by P is $\frac{1}{2}P\delta$; equating the two we have

$$\delta = lPR^2\left(\frac{\cos^2\alpha}{CJ} + \frac{\sin^2\alpha}{EI}\right)$$
$$= 2\pi nPR^3 \sec\alpha\left(\frac{\cos^2\alpha}{CJ} + \frac{\sin^2\alpha}{EI}\right) \quad . \quad . \quad . \quad . \quad (8)$$

The stiffness is the value of P obtained from this when δ = unity.

For a spring of circular wire of diameter d, (8) becomes

$$\delta = \frac{64nPR^3 \sec\alpha}{d^4 C}\left(\cos^2\alpha + \frac{2C}{E}\sin^2\alpha\right) \quad . \quad . \quad . \quad (9)$$

281. Open-Coiled Helical Spring : Axial Couple.—Let the spring be acted on by a couple L whose axis coincides with the axis of the helix. Then the torsion couple at any point is $L\sin\alpha$ and the flexural couple is $L\cos\alpha$. Hence, in this case, the strain energy is

$$\frac{l(L\sin\alpha)^2}{2CJ} + \frac{l(L\cos\alpha)^2}{2EI},$$

whilst the work done by the couple L in twisting one end of the spring through an angle θ relative to the other is $\frac{1}{2}L\theta$, the angle being measured in radians. Equating this to the strain energy we have

$$\theta = lL\left(\frac{\sin^2\alpha}{CJ} + \frac{\cos^2\alpha}{EI}\right)$$
$$= 2\pi nRL \sec\alpha\left(\frac{\sin^2\alpha}{CJ} + \frac{\cos^2\alpha}{EI}\right) \quad . \quad . \quad . \quad . \quad (10)$$

For a wire of circular cross section this gives

$$\theta = \frac{128nRL \sec a}{d^4E}\left(\frac{E}{2C}\sin^2 a + \cos^2 a\right) \quad . \quad . \quad . \quad (11)$$

In the case of open-coiled springs the stresses due to flexure and torsion must be found separately and the principal stresses calculated.

It should be noted that the formulæ of §§ 280 and 281 are only approximations as we have treated R and a as constants, whereas really they vary continuously as the load is applied, but the formulæ derived here are sufficiently accurate for all practical purposes.* The connections between R, a, δ are $2\pi Rn = l \cos a$, and

$$\delta = l \cos a.da.$$

282. Plane Spiral Springs.—Fig. 327 represents a spring whose central line is a plane spiral curve. One end of the spring is anchored to a pin at C, and the inner end is attached to the winding spindle. A couple M is applied to this spindle.

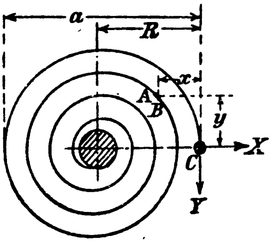

FIG. 327.

Let X and Y be the components of the reaction at C along and perpendicular to the line joining the axis of the spindle to the centre of the pin C. Let (x, y) be the coordinates of an element AB ($= ds$) referred to the same directions as shown in Fig. 327.

Then the bending moment on AB is $Yx - Xy$. Let $d\theta =$ the change in the angle between the tangents at the ends of the element AB, then

$$d\theta = ds\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)$$

where ρ_0 and ρ denote the radii of curvature at (x, y) before and after strain. But

$$\frac{1}{\rho} - \frac{1}{\rho_0} = \frac{Yx - Xy,}{EI}$$

where I is the moment of inertia of a cross section of the spring about its neutral axis perpendicular to the plane of the spring. Hence

$$d\theta = \frac{Yx - Xy}{EI}ds.$$

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* For a more exact treatment the reader is referred to Love's *Theory of Elasticity*, 3rd Ed., p. 421.

If I be constant this gives, for the total change of angle between the tangents at the extremities of the spring,

$$\theta = \frac{Y}{EI}\int xds - \frac{X}{EI}\int yds.$$

Now $\int xds$ and $\int yds$ represent the moment of the whole length of the spring about the directions Y and X respectively. The centroid of the spring will be approximately at the centre of the winding spindle, and if we assume this to be true we shall have

$$\int xds = lR \text{ and } \int yds = 0$$

where l is the total length of the spiral. Also $Y.R = M$, the couple applied to the spindle. Hence

$$\theta = \frac{Ml}{EI} \quad \text{. . . . .} \quad (12)$$

The work done in winding, i.e. the energy stored, is

$$\tfrac{1}{2}M\theta = \frac{M^2l}{2EI}.$$

The stress at any section will be given by

$$p = \frac{Yx}{Z} = \frac{Mx}{RZ}$$

where Z is the modulus of the cross section.

The maximum value of x is a (Fig. 327), so that the maximum stress is

$$p = \frac{Ma}{RZ} \quad \text{. . . . .} \quad (12\text{A})$$

If f stand for the maximum permissible stress, the maximum value of M is

$$M = \frac{RZf}{a} \quad \text{. . . . .} \quad (13)$$

and the resilience is

$$\frac{R^2Z^2lf^2}{2EIa^2} \quad \text{. . . . .} \quad (14)$$

When the spring is made of a very thin flat band of metal, anticlastic curvature arises (see p. 175), and the full treatment of the problem becomes more complicated, but is not of great interest to the engineer.*

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* A solution of the problem of the flexure of a broad thin band into a circle of radius comparable with the mean proportional between the width and thickness of the band will be found in the *Phil. Mag.*, Vol. 31, 1891 (H. Lamb).

Other papers of interest dealing with this class of spring are referred to Todhunter and Pearson's *History*, Vol. II, pt. i, pp. 466–470. The most important is that by Phillips. *Annales des Mines*, tome xx, 1861, "Mémoire sur le spiral réglant de chronometres et des montres."

283. Close-Coiled Conical Spiral Spring.—Fig. 328 represents a spring whose central line is a spiral curve drawn on a circular cone.

Let r_1 be the smallest, and r_2 the largest radius of the spiral.

Let n be the number of complete convolutions of the wire.

Let the spiral be such that the polar equation of its projection on a plane at right angles to the axis of the cone is of the form

$$r = a + b\theta.$$

Then, if we measure θ from the radius r_1 we have $a = r_1$ and

$$b = (r_2 - r_1)/2\pi n,$$

so that

$$r = r_1 + \frac{\theta}{2\pi n}(r_2 - r_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

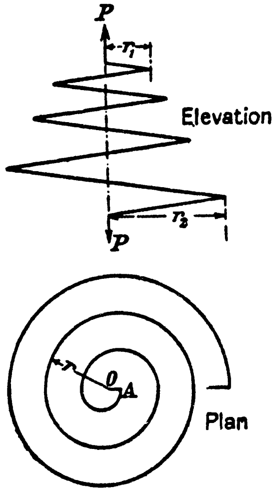

FIG. 328.

We shall suppose the spring to be "close-coiled," i.e. that the convolutions very nearly lie in planes perpendicular to the axis of the cone. Then the element of the spring at radius r is subjected to a torque Pr.

The strain energy per unit length is then $\frac{P^2r^2}{2CJ}$. Hence the energy stored in length ds, $= rd\theta$, is

$$dU = \frac{P^2r^3d\theta}{2CJ}$$

From (i) we have

$$d\theta = \frac{2\pi n.dr}{r_2 - r_1}$$

$$\therefore\ dU = \frac{\pi nP^2r^3dr}{CJ(r_2 - r_1)}.$$

Hence the total strain energy of the spring is

$$U = \frac{\pi nP^2}{CJ(r_2 - r_1)}\int_{r_1}^{r_2} r^3dr = \frac{\pi nP^2}{4CJ}(r_1 + r_2)(r_1^2 + r_2^2) \quad . \quad (15)$$

The work done by P is $\frac{1}{2}P\delta$, if δ denote the total axial extension of the spring. Equating this to U we have

$$\delta = \frac{\pi nP}{2CJ}(r_1 + r_2)(r_1^2 + r_2^2) = \frac{Pl(r_1^2 + r_2^2)}{2CJ}, \quad . \quad . \quad . \quad . \quad (16)$$

where l denotes the total length of the wire, approximately.

If the spring be acted on by an axial couple L instead of an axial pull, every element of the wire is subjected to the same bending moment, and the change of radius has no direct effect. The formula which we found for a cylindrical helical spring (§ 279) subjected to an axial torque is therefore true also for a close-coiled conical spring, l denoting the total length of wire. It is only in its effect on the value of l that the changing radius alters the formula.

284. Approximate Theory of Leaf Springs.—We shall now consider the type of spring shown in Fig. 329, consisting of n parallel strips

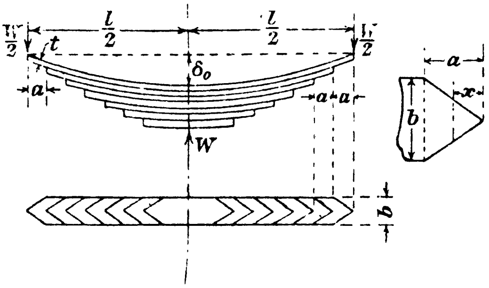

FIG. 329.

of metal of width b. The spring carries a central vertical load W which is balanced by equal end reactions $\frac{W}{2}$ as shown.

We shall assume now that the centre lines of all the plates are initially circular arcs of the same radius r, that each plate has a uniform thickness t, and overlaps the one below it by an amount $a = l/2n$ at each end, and that these overlaps are tapered in width to the triangular shape shown.

Now, since the plates are initially circular arcs of the same radius, each will, when unloaded, touch the one above it at its ends only. If, when the load is applied, the change of curvature of each plate is uniform and the same for all the plates, contact will continue to be at the ends only.

In these circumstances each plate will bear a downward load $\frac{W}{2}$ at its ends, and an upward load $\frac{W}{2}$ at the ends of the plate next below it. Thus the triangular overhanging ends are loaded as cantilevers, whilst the parallel portions are loaded with a uniform bending moment $\frac{1}{2}Wa$.

The moment of inertia of the cross section of the parallel portion of each plate is $\frac{1}{12}bt^3$, hence for this portion of the plate the new radius of curvature, ρ, is given by

$$\frac{1}{r} - \frac{1}{\rho} = \frac{\text{bending moment}}{EI} = \frac{6Wa}{Ebt^3}.$$

Next consider the triangular ends: at a distance x from the point the bending moment is $\frac{1}{2}Wx$, and the moment of inertia is $\frac{1}{12}\frac{x}{a}bt^3$; hence for this portion we have

$$\frac{1}{r}-\frac{1}{\rho}=\frac{Wx}{2}\times\frac{12a}{xbt^3E}=\frac{6Wa}{Ebt^3}.$$

Thus for the whole length of each plate we have

$$\frac{1}{\rho}=\frac{1}{r}-\frac{6Wa}{Ebt^3}=\text{constant}. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Thus all the plates are bent into circular arcs of radius ρ, and contact continues to be at the ends only.

The load W_0 which will straighten out all the plates is obtained by putting $\rho=\infty$ in (i) ; we get

$$W_0=\frac{Ebt^3}{6ar}=\frac{nEbt^3}{3lr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If δ_0 be the initial dip of the top plate, assumed to be small, we have $\delta_0=\frac{l^2}{8r}$ or $r=\frac{l^2}{8\delta_0}$ from the properties of a circle.

Substituting for r in (17) gives

$$W_c=\frac{8nEbt^3\delta_0}{3l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Since for each plate the bending moment $\div$ the moment of inertia is constant and the same for all the plates, the maximum fibre stress will also be constant for each plate and the same for all.

Let p = the maximum fibre stress for a given load W, then

$$p=\frac{\frac{1}{2}Wa\times\frac{t}{2}}{I}=\frac{3Wa}{bt^2}=\frac{3}{2}\frac{Wl}{nbt^2} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Hence, if f denote the stress corresponding with the load* W_0 required to straighten the plates,

$$W_0=\frac{2}{3}\frac{nbt^2f}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

From (18) and (20) we have

$$f=\frac{4t\delta_0}{l^2}E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

For a given material f and E will be prescribed, so that this equation fixes the proper relationship between the thickness and initial radius of the plates.

Since M/I is constant, the dip δ, for a load W, is given by (§ 164),

$$\delta_0-\delta=\frac{Ml^2}{8EI}=\frac{1}{8}\cdot\frac{Wa}{2}\cdot\frac{l^2}{E}\cdot\frac{12}{bt^3}=\frac{3Wal^2}{4Ebt^3}=\frac{3Wl^3}{8nEbt^3}. \quad . \quad (22)$$

which is the deflection of the ends of the spring relative to the centre.

---

* The load W_0 is generally known as the "proof" load.

Example.—Find the load required to straighten a carriage spring which has 6 strips, of breadth 3″ and thickness ⅜″, the top strip having a length of 3 ft., if the deflection of the top strip when unloaded is 2½″. The overlaps are each equal to half the total length of the bottom strip, and their breadth is uniformly tapered to a point. (Mech. Sc. Trip., 1912.)

From (22) of § 284, we have, since $\delta = 0$,

$$W = \frac{8nEbt^3\delta_0}{3l^3}.$$

$n = 6$, $b = 3$ in., $t = 0{\cdot}375$ ins., $\delta_0 = 2{\cdot}5''$, $l = 36''$.

Hence, taking $E = 30 \times 10^6$ lbs./in.^{2}

$$W = \frac{48 \times 30 \times 10^6 \text{ lbs./in.}^2 \times 3 \text{ in.} \times 0{\cdot}0527 \text{ in.}^3 \times 2{\cdot}5 \text{ in.}}{3 \times 46{,}656 \text{ in.}^3}$$

$$= 4{,}070 \text{ lbs.}$$

EXAMPLES XXIV

Unless otherwise stated take $E = 30 \times 10^6$ lbs./in.^{2}, $C = 12{\cdot}5 \times 10^6$ lbs./in.^{2}

1. A helical spring is made of steel wire ¼″ diameter. The coils, 60 in number, are in close formation, and the centre line of the wire lies on a cylinder 3″ diameter. The two ends of the spring are pulled apart by an axial pull of 10 lbs. What shear stress is set up in the wire and what elongation is produced? (Mech. Sc. Trip., 1915.)

2. A safety valve of 3″ diameter is to blow off at a pressure of 150 lbs./in.^{2} by gauge. It is held by a close-coiled compression spring of circular steel bar. The mean diameter is 6″ and the initial compression of the spring is 1″. Find the diameter of the steel and the number of convolutions necessary if the shearing stress allowed is 8 tons/in.^{2}, and $C = 5{,}000$ tons/in.^{2} (Mech. Sc. Trip., 1916.)

3. Two shafts in line, which are prevented from moving axially, are connected by a helical spring, the spring fitting loosely on the shafts and having its ends fixed to the shafts. Show that, if the coils of the spring are of circular cross section, and are inclined at 45° to the axis, the couple per unit angle of twist is given by $\dfrac{r^4}{8\sqrt{2}.nR}\left(\dfrac{E}{2} + C\right)$, where r is the radius of the cross section and the rest of the notation is as in § 280. (Mech. Sc. Trip., B, 1913.)

4. Obtain the deflection and angular twist of the free end of a helical spring of 10 coils, 10″ diameter, made of ½″ round steel, due to an axial load of 40 lbs. if the helix makes an angle of 60° with the axis. (R.N.C., Greenwich, 1922.)

5. A load of 50 lbs. weight is suspended from a close-coiled helical spring made of ¼″ diameter steel wire. If there be 10 coils of 2″ mean diameter, find the extension of the spring and the shear stress in the material. (R.N.E.C., Keyham, 1923.)

6. A close-coiled helical spring made of ¼″ round steel wire has 20 coils, mean diameter 3″. Find (i) its deflection under a load of 40 lbs., (ii) the shear stress in the wire due to this load, (iii) the work done in producing the extension. Take $C = 11 \times 10^6$ lbs./in.^{2} (R.N.E.C., Keyham, 1920.)

7. A steel carriage spring is to be 30″ long and to carry a central load of ½ ton. If the plates are 3″ wide and ¼″ thick, how many plates will be required if the stress is to be limited to 12 tons/in.^{2}? What will be the deflection of the spring at the centre? To what radius should each piece be curved? Take $E = 13{,}000$ tons/in.^{2} (R.N.E.C., Keyham, 1921.)

8. A length of 50″ of 0·232″ diameter wire is coiled to a mean diameter of 3½″ to make a closely coiled helical spring. Neglecting the inertia of the wire, find with what period a mass of 2 lbs. would oscillate when suspended on the spring. (R.N.E.C., Keyham, 1922.)

9. Estimate the length of 0·232″ diameter wire necessary to form a helical spring with a mean diameter of 3½″, whose stiffness under axial load is to be 20 lbs. per inch. (R.N.E.C., Keyham, 1923.)

10. Prove that, for a given resilience and given maximum shearing stress, the ratio of the weight of a close-coiled helical spring made of tube to that of one made of solid wire is $k^2/(1 + k^2)$, where k is the ratio of the outside diameter of the tube to the inside diameter.

11. Find the maximum safe load and deflection for a close-coiled helical spring of 0·25″ diameter wire, having 12 complete turns of 3″ mean diameter. Maximum stress 60,000 lbs./in.^{2} and $C = 12 \times 10^6$ lbs./in.^{2} (R.N.E.C., Keyham, 1922.)

12. Calculate the stiffness of a closely coiled helical steel spring consisting of 10 turns of ⅛″ diameter wire coiled on a mandrel 2″ diameter. Take $C = 12 \times 10^6$ lbs./in.^{2} (R.N.E.C., Keyham, 1925.)

13. A composite spring is made by joining two springs end to end, one spring having twice the stiffness of the other. Show that the stiffness of the composite spring is ⅓ of that of the stiffer of the two springs.

14. A close-coiled spring has a mean radius of 2 ins. and the wire is ¼″ diameter. There are 30 turns. Calculate the work done in rotating one end 90° relative to the other, by a couple whose axis coincides with the axis of the spring. (H.M. Dockyard Schools, 1931.)

CHAPTER XXV

STRESSES IN CURVED BEAMS OF LARGE CURVATURE

285. Introductory.—In all the problems of flexure which we have considered in previous chapters we have assumed that the dimensions of the cross sections of the members are small in comparison with the radius of curvature of the central axis, before and after bending. We must now investigate the flexure of members, the dimensions of whose cross sections are comparable with the radius of curvature of the central axis. The most common instances of such members are chain links, rings, and hooks. Up to the present time no really satisfactory theory of bending, applicable to such pieces, has been discovered.

Considerable light is thrown on the problem by the investigations of two-dimensional stress systems given in Chapter XXVII. Referring to pp. 430–433 it will be found that when a flat plate, in the form of a semi-circular annulus, is subjected to bending in its own plane, radial stresses exist which are not entirely negligible. It is the failure to take account of these radial stresses which renders unsatisfactory the existing theories of bending applied to such pieces as chain links and hooks.

The first important attempt to give a suitable theory of the strength of chain links was made by Winkler,* and the investigations of this chapter follow closely the lines of his memoir as quoted and corrected by Karl Pearson in his *History*. A later attempt to improve on this was made by Andrews and Pearson,† but their theory does not appear to be any better than Winkler's as we shall see presently. Winkler's theory is really a modification of the Bernoulli-Euler theory of bending to allow for the fact that the cross sections are not small compared with the radius of curvature of the central axis. We shall first give a general statement of this theory, and then apply it to a particular case and compare the results with those given by other theories.

286. Winkler's Theory of the Flexure of Curved Bars.—It is assumed that the central axis of the bar forms a plane curve before and after bending, that the cross section is uniform, and that longitudinal elements of the bar, parallel to the central axis, exert no action on each other. This last assumption is the main fallacy of the theory, and is known to be untrue.

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* *Der Civilingenieur*, Bd. IV, 1858. See also Todhunter and Pearson's *History of Elasticity*, Vol. II, Pt. i, p. 423 *et seq.*

† *Drapers' Company Research Memoirs*, I, 1904.

Referring to Fig. 330 let A_0B_0 be a small portion of the unstrained central axis, of length ds_0, the centre of curvature being C_0 and the radius of curvature

$$\rho_0 = C_0A_0 = C_0B_0.$$

Then

$$\theta_0 = \frac{ds_0}{\rho_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Relative to the cross section at A_0 let the central axis be deformed into the curve A_0B, whose centre of curvature is C, and whose radius of curvature is

$$\rho = CA_0 = CB.$$

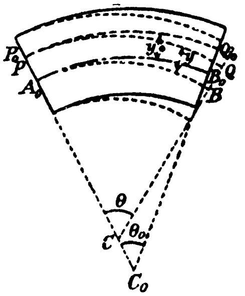

FIG. 330.

Let e_0 = the strain of the central axis, for the element A_0B_0, then the length of A_0B is

$$ds = (1 + e_0)ds_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

and the angle θ is given by

$$\theta = \frac{ds}{\rho} = \frac{(1 + e_0)ds_0}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Let P_0Q_0 be a longitudinal element of the beam, which, in the unstrained state, is at a distance y_0 from the central axis A_0B_0. After strain let P_0Q_0 become PQ, at a distance y from the central axis AB. Let e = the strain of P_0Q_0.

Then we have

$$P_0Q_0 = (\rho_0 + y_0)\theta_0 \text{ and } PQ = (\rho + y)\theta.$$

Therefore

$$e = \frac{PQ - P_0Q_0}{P_0Q_0} = \frac{(\rho + y)\theta}{(\rho_0 + y_0)\theta_0} - 1$$

Hence, from (i) and (iii)

$$e = \frac{(\rho + y)(1 + e_0)\dfrac{ds_0}{\rho}}{(\rho_0 + y_0)\dfrac{ds_0}{\rho_0}} - 1$$

$$= \frac{(1 + e_0)\left(1 + \dfrac{y}{\rho}\right)}{1 + \dfrac{y_0}{\rho_0}} - 1$$

Now Winkler assumes that the difference between y and y_0 can be neglected, an error which was pointed out by Andrews and Pearson (*loc. cit.*, p. 398), who modify the theory accordingly by an attempt to allow for the radial strain. As, however, they ignore the radial stresses referred to above, it is hardly to be expected that their results will be any more reliable than Winkler's. We shall see later that this view has a good deal to justify it, and accordingly we shall write $y = y_0$. In this case the expression for e can be written in the form

$$e = e_0 + \frac{(1 + e_0)y_0\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)}{1 + \frac{y_0}{\rho_0}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

This form will be found convenient in the subsequent work. The stress at a distance y_0 from the central axis is $p = Ee$, and the total action over a cross section consists of a normal stress-resultant

$$P = \int Ee \, . \, dS$$

and a stress-couple

$$M = \int Eey_0 dS,$$

where dS is an element of area of the cross section. Inserting the value of e from (1), these become

$$P = Ee_0 S + E(1 + e_0)\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\int \frac{\rho_0 y_0}{\rho_0 + y_0} dS \, . \quad . \quad \text{(iv)}$$

and

$$M = E(1 + e_0)\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\int \frac{\rho_0 y^2}{\rho_0 + y_0} dS, \quad . \quad . \quad . \quad \text{(v)}$$

the integrals extending over the whole area of the cross section. In deriving this expression for M we have taken $\int Ee_0 y \, . \, dS = 0$, since the central axis passes through the centroid of the section.

We shall now write

$$\int \frac{\rho_0 y_0^2}{\rho_0 + y_0} dS = Sh^2 \quad . \quad . \quad . \quad . \qquad (2)$$

Then

$$\int \frac{\rho_0 y}{\rho_0 + y_0} dS = \frac{1}{\rho_0}\int\left(\rho_0 y_0 - \frac{\rho_0 y_0^2}{\rho_0 + y_0}\right) dS = -\frac{Sh^2}{\rho_0}.$$

Hence (iv) and (v) become *

$$\mathbf{P} = \mathbf{E}\mathbf{e}_0\mathbf{S} - \mathbf{E}(1 + \mathbf{e}_0)\left(\frac{1}{\boldsymbol{\rho}} - \frac{1}{\boldsymbol{\rho}_0}\right)\frac{\mathbf{S}\mathbf{h}^2}{\boldsymbol{\rho}_0} \quad . \quad . \quad . \quad . \qquad (3)$$

$$\mathbf{M} = \mathbf{E}(1 + \mathbf{e}_0)\left(\frac{1}{\boldsymbol{\rho}} - \frac{1}{\boldsymbol{\rho}_0}\right)\mathbf{S}\mathbf{h}^2 \, . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

---

* Some writers prefer to let $\int \frac{\rho_0 dS}{\rho_0 + y_0} = S'$, and then deduce the expressions $\int \frac{\rho_0 y_0 dS}{\rho_0 + y_0} = \rho_0(S - S')$ and $\int \frac{\rho_0 y_0^2 dS}{\rho_0 + y_0} = \rho_0^2(S' - S)$. The formula for converting from one system to the other is $h^2 = \rho^2\left(\frac{S'}{S} - 1\right)$.

These are the general formulæ which we shall use in the further development of the theory.

287. Pure Bending.—When the bar is subjected to terminal couples only we must have

$$P = 0.$$

Then dividing (4) by ρ_0 and adding the result to (3) we get

$$e_0 = \frac{M}{ES\rho_0}.$$

Then, from (4), $\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\left(1 + \frac{M}{ES\rho_0}\right) = \frac{M}{ESh^2}.$

This gives $\frac{1}{\rho} - \frac{1}{\rho_0} = \frac{\rho_0 M}{h^2(M + E\rho_0 S)}$

Substituting for e_0 and $1/\rho - 1/\rho_0$ in (1) we get

$$e = \frac{M}{ES\rho_0}\left(1 + \frac{y_0}{\rho_0 + y_0} \cdot \frac{\rho_0^2}{h^2}\right)$$

Hence the tensile stress is given by

$$\mathrm{p} = \frac{\mathrm{M}}{\mathrm{S}\rho_0}\left(1 + \frac{\mathrm{y}_0}{\rho_0 + \mathrm{y}_0} \cdot \frac{\rho_0^2}{\mathrm{h}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

M being the bending moment tending to increase the curvature.

288. Formulæ for $\mathbf{h^2}$.—In the case of a rectangular section of radial depth D, and thickness B at right angles to the plane of bending, we have

$$h^2 = \frac{1}{BD}\int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{\rho_0 y^2 B dy_0}{\rho_0 + y_0} = \frac{\rho_0^3}{D}\log_e \frac{2\rho_0 + D}{2\rho_0 - D} - \rho_0^2 \quad . \quad . \quad (6)$$

For a circular cross section of diameter D,

$$\frac{\pi D^2}{4}h^2 = \int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{2\rho_0 y_0^2\sqrt{D^2/4 - y_0^2}\ dy_0}{\rho_0 + y_0}$$

which gives

$$h^2 = \frac{D^2}{16}\left\{1 + \tfrac{1}{2}\left(\frac{D}{2\rho_0}\right)^2 + \frac{3.5}{6.8}\cdot\left(\frac{D}{2\rho_0}\right)^4 + \cdots\cdot\right\} \quad . \quad . \quad (7)$$

For the trapezium shown in Fig. 331,

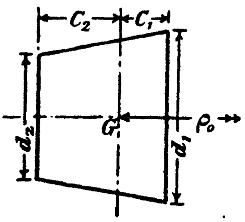

FIG. 331.

$$h^2 = \rho_0^2\left[-1 + \frac{\rho_0}{S}\left\{\left(d_2 + \frac{d_1 - d_2}{c_1 + c_2}[c_1 + \rho_0]\right)\log_e \frac{\rho_0 + c_1}{\rho_0 - c_2} - (d_1 - d_2)\right\}\right]$$

For any other section,

$$h^2 = \frac{1}{S}\int \frac{\rho_0 y_0{}^2 . dS}{\rho_0 + y_0} = \frac{1}{S}\int y_0{}^2\left(1 + \frac{y_0}{\rho_0}\right)^{-1} . dS$$

$$= \frac{1}{S}\int\left\{1 - \frac{y_0}{\rho_0} + \left(\frac{y_0}{\rho_0}\right)^2 - . . \right\} y_0{}^2 dS$$

$$= k^2 - \frac{1}{\rho_0 S}\int y_0{}^3 dS + \frac{1}{\rho_0{}^2 S}\int y_0{}^4 dS + ... \qquad (8)$$

and the values of the integrals can be found graphically ; k is the radius of gyration of the cross section about the transverse axis through its centroid.

289. Deformation of the Central Axis.—In certain cases we require to estimate the change in the length of a chord of the central axis. In Fig. 332 let $D_0A_0E_0$ be a portion of the central axis before strain, and let A_0B_0 be an element of this line of length ds_0. We wish to examine the shift of E_0 relative to D_0 on account of the strains.

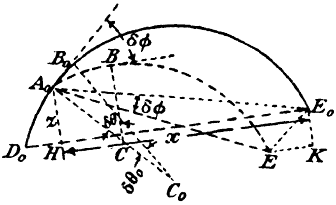

FIG. 332.

Suppose the part D_0A_0 of the axis to be unstrained, and that the element A_0B_0 is strained to A_0B, carrying B_0E_0 with it to BE, but the axis between B and E undergoing no strain. Let e_0 be the linear strain of the element A_0B_0, so that the length of A_0B is $(1 + e_0)ds_0$. Let $\delta\varphi$ be the angle between the tangents at A_0 and B, as shown. Then

$$\delta\varphi = \delta\theta - \delta\theta_0 = \frac{(1 + e_0)\delta s_0}{\rho} - \frac{\delta s_0}{\rho_0}$$

which can be arranged thus

$$\delta\varphi = (1 + e_0)\delta s_0\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) + e_0\frac{\delta s_0}{\rho_0}$$

Hence

$$\frac{d\varphi}{ds_0} = (1 + e_0)\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) + \frac{e_0}{\rho_0}$$

or

$$\frac{d\varphi}{ds_0} = \frac{M}{ESh^2} + \frac{e_0}{\rho_0}, \text{ from (4)} \qquad . . . . (vi)$$

Now draw EK parallel to D_0E_0, and E_0K perpendicular to D_0E_0. Then, as in § 203, p. 273, the displacement of E_0 in the direction E_0D_0, on account of the flexure of A_0B_0 is

$$KE = z\delta\varphi,$$

where $z = A_0H$, perpendicular to D_0E_0. Hence the total shift of E_0 in the direction E_0D_0, on account of the flexure of the axis is

$$\int_{D_0}^{E_0} z . d\varphi = \int_{D_0}^{E_0}\left(\frac{M}{ESh^2} + \frac{e_0}{\rho_0}\right) z ds_0,$$

from (vi). In addition to this there is the displacement of E_0 which arises from the linear strain e_0. In the direction D_0E_0 it amounts to

$$\int_{D_0}^{E_0} e_0 \sin\theta \,.\, ds_0$$

where θ is the angle between D_0E_0 and the normal at A_0. Hence the total displacement of E_0 in the direction E_0D_0, is u, where

$$u = \int_{D_0}^{E_0}\left\{\left(\frac{M}{ESh^2}+\frac{e_0}{\rho_0}\right)z - e_0\sin\vartheta\right\}ds_0 \quad . \quad . \quad . \quad (9)$$

Similarly we can show that the total displacement of E_0, in the direction E_0K, is v where

$$v = \int_{D_0}^{E_0}\left\{\left(\frac{M}{ESh^2}+\frac{e_0}{{}_0}\right)x + e_0\cos\theta\right\}ds_0 \quad . \quad . \quad . \quad (10)$$

in which $x = E_0H$.

290. Application to Hooks.—Consider the hook shown in Fig. 333 loaded with a weight W Let EF be the principal section of the hook, i.e. the normal cross section through the point on the central axis which is farthest from the line of action of W. Then the action on the section EF consists of a bending moment $-$ Wa, and a tension W. We can apply the formula (5), p. 401, to find the stress due to the bending moment, and add to the result the stress due to the direct pull W.

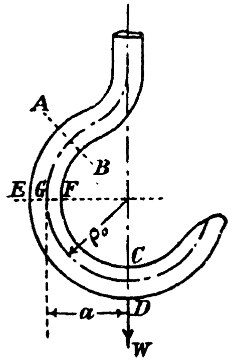

FIG. 333.

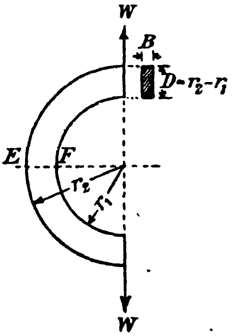

FIG. 334.

If G be the centroid of the section EF, let $GE = y_1$, and $GF = y_2$; let p_1 and p_2 be the corresponding tensile stresses. Then *

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* The formulæ given by Andrews and Pearson are

$$p_1 = \frac{W}{S} + \frac{Wa}{S\rho_0\gamma_2}\left\{\left(1+\frac{y_1}{\rho_0}\right)^{-\frac{m+1}{m}} - \gamma_1\right\}$$

and $p_2 = \frac{W}{S} + \frac{Wa}{S\rho_0\gamma_2}\left\{\left(1-\frac{y_2}{\rho_0}\right)^{-\frac{m+1}{m}} - \gamma_1\right\}$, where $\gamma_1 = \frac{1}{S}\int\left(1+\frac{y_0}{\rho_0}\right)^{-\frac{m+1}{m}} dS$

and $\gamma_3 = -\frac{1}{S}\int\left(1+\frac{y_0}{\rho_0}\right)^{-\frac{1}{m}} dS$, $\gamma_2 = \gamma_1 - \gamma_3$, $\frac{1}{m}$ being Poisson's ratio.

$$p_1 = \frac{-Wa}{S\rho_0}\left(1 + \frac{y_1}{\rho_0 + y_1} . \frac{\rho_0^2}{h^2}\right) + \frac{W}{S} = \text{tensile stress at } E \qquad (11)$$

$$p_2 = \frac{-Wa}{S\rho_0}\left(1 - \frac{y_2}{\rho_0 - y_2} . \frac{\rho_0^2}{h^2}\right) + \frac{W}{S} = \text{tensile stress at } F \qquad (12)$$

where ρ_0 is the radius of curvature of the central axis at G.

We shall now compare the values of the stresses given by these formulæ with those found by other formulæ. Now for purposes of comparison we shall take the case when the cross section of the hook is uniform, and the central axis a circular arc, over, say, the range AB to CD. Then it is clear that the stresses on the section EF of the hook (Fig. 333) will not differ appreciably from the stresses over the section EF of the semi-circular beam in Fig. 334, if the mean radius and cross section be the same for both. If, as a special case, the section of the hook be a rectangle whose width perpendicular to the plane of bending is small compared with EF, the stresses in the case of Fig. 334 are accurately given by the formulæ (14), p. 433. It is for this reason that we take this special case for purposes of comparison.

We shall first take the case when $r_2 = 2r_1$, so that the mean radius is $\rho_0 = 1{\cdot}5r_1$ and the depth of the section is $D = r_1$.

We have, from (6), p. 401,

$$h^2 = \frac{(1{\cdot}5r_1)^3}{r_1} \log_e \frac{3r_1 + r_1}{3r_1 - r_1} - (1{\cdot}5r_1)^2$$

$$= r_1^2\left(\frac{27}{8} \log_e 2 - 2{\cdot}25\right) = 0{\cdot}089r_1^2$$

$$y_1 = y_2 = \frac{D}{2} = 0{\cdot}5r_1$$

$$a = \rho_0 = 1{\cdot}5r_1.$$

Then from (11) and (12) we have

$$p_1 = \frac{W}{S} - \frac{W . 1{\cdot}5r_1}{S . 1{\cdot}5r_1}\left(1 + \frac{0{\cdot}5r_1}{2r_1} . \frac{2{\cdot}25r_1^2}{0{\cdot}089r_1^2}\right) = -6{\cdot}32\frac{W}{S}.$$

$$p_2 = \frac{W}{S} - \frac{W . 1{\cdot}5r_1}{S . 1{\cdot}5r_1}\left(1 - \frac{0{\cdot}5r_1}{r_1} . \frac{2{\cdot}25r_1^2}{0{\cdot}089r_1^2}\right) = 12{\cdot}63\frac{W}{S}.$$

If we calculate the stresses by the simple bending formulæ we get

$$p_1 = \frac{W}{S} - \frac{12Way_0}{S . D^2} = \frac{W}{S}\left(1 - \frac{12 \times 1{\cdot}5r_1 \times 0{\cdot}5r_1}{r_1^2}\right) = -8\frac{W}{S}$$

$$p_2 = \frac{W}{S} + \frac{12Way_0}{SD^2} = 10\frac{W}{S}.$$

If we apply the Andrews-Pearson formulæ (see footnote, p. 403), we find, taking $m = 10/3$,

$$\gamma_1 = \frac{1}{S}\int_{-\frac{D}{2}}^{\frac{D}{2}}\left(1+\frac{y_0}{\rho_0}\right)^{-1-\frac{1}{m}}B\,dy_0 = \frac{m\rho_0}{D}\left[\left(\frac{2\rho_0}{2\rho_0 - D}\right)^{\frac{1}{m}} - \left(\frac{2\rho_0}{2\rho_0 + D}\right)^{\frac{1}{m}}\right] = 1{\cdot}0605$$

$$\gamma_3 = -\frac{B}{S}\int_{-\frac{D}{2}}^{\frac{D}{2}}\left(1+\frac{y_0}{\rho_0}\right)^{-\frac{1}{m}}dy_0$$

$$= \frac{m\rho_0}{(m-1)D}\left[\left(\frac{2\rho_0 + D}{2\rho_0}\right)^{1-\frac{1}{m}} - \left(\frac{2\rho_0 - D}{2\rho_0}\right)^{1-\frac{1}{m}}\right] = 1{\cdot}0075$$

$$\gamma_2 = 0{\cdot}0530$$

Then

$$p_1 = \frac{W}{S} + \frac{W}{0{\cdot}0530S}\left\{\left(\frac{4}{3}\right)^{-1{\cdot}3} - 1{\cdot}0605\right\} = -6{\cdot}22\frac{W}{S}$$

$$p_2 = \frac{W}{S} + \frac{W}{0{\cdot}0530S}\left\{\left(\frac{2}{3}\right)^{-1{\cdot}3} - 1{\cdot}0605\right\} = 12{\cdot}95\frac{W}{S}$$

Finally let us apply what we know to give the correct values of the stresses, namely (14) on p. 433. The stresses p_1 and p_2 are found by putting $r = r_2$ and r_1 respectively in the formula for p_0:

$$p_1 = \frac{-6r_1 + \frac{5r_1^2}{2r_1} + \frac{4r_1^4}{8r_1^3}}{5r_1^2\log_e 2 - 3r_1^2}\cdot\frac{W}{B} = -6{\cdot}44\frac{W}{S}$$

$$p_2 = \frac{-3r_1 + \frac{5r_1^2}{r_1} + \frac{4r_1^4}{r_1^3}}{5r_1^2\log_e 2 - 3r_1^2}\cdot\frac{W}{B} = 12{\cdot}88\frac{W}{S}.$$

If we repeat the calculations for the case when $r_2 = 7r_1$ we obtain the following results:

Formulæ (11) and (12) $p_1 = -1{\cdot}43\frac{W}{S}$, $p_2 = 10\frac{W}{S}$.

Simple bending formulæ $p_1 = -3{\cdot}00\frac{W}{S}$, $p_2 = 5\frac{W}{S}$.

Andrews-Pearson formulæ $p_1 = -1{\cdot}28\frac{W}{S}$, $p_2 = 11{\cdot}4\frac{W}{S}$.

Correct formulæ (p. 433) $p_1 = -1{\cdot}67\frac{W}{S}$, $p_2 = 11{\cdot}65\frac{W}{S}$.

On studying these results it will be seen that to apply the ordinary formulæ for straight beams will lead to very serious errors; the Andrews-Pearson formulæ do not give any better results than the Winkler formulæ. It does not appear to the present author that the extra complexity of the Andrews-Wilson formulæ is justified by the results, particularly as its basis is no better than that of Winkler. In an article in *Engineering* (September 11 and 25, 1914), Prof. Morley, examining experimental results, comes to the conclusion that on the whole the Andrews-Pearson formulæ do not give such good results as Winkler's. Furthermore the determination of γ_1 and γ_3 must be done with great accuracy in order to get an approximately correct value for their difference γ_2. We shall therefore develop the rest of the investigation on the lines of Winkler's work. When the curve of the central axis is nearly semi-circular, and the section approximates to a thin rectangle, we can check our calculations by formulæ (14) on p. 433. Unfortunately these formulæ will not help us very much when the section is not approximately rectangular and the central axis not approximately a semi-circle. In certain cases something might be effected by dividing the beam into laminæ parallel to the planes of bending, applying these formulæ to each lamina, and integrating to find the total moment of resistance. But the process will be very laborious and it is doubtful if the results would justify the work.

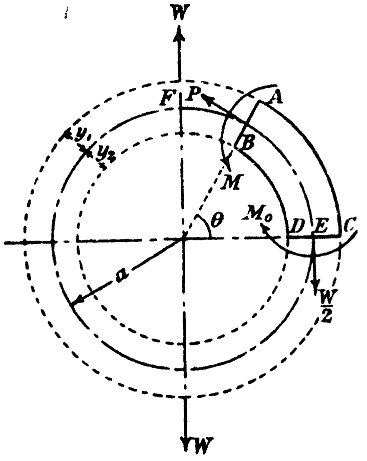


FIG. 335.

291. Chain Ring.*—We shall now consider the case of a circular ring subjected to pulls W applied at opposite ends of a diameter (Fig. 335).

Let a = the mean radius of the ring, that is, the radius of the line of centroids of the cross sections.

Let M_0 = the bending moment, or stress couple, at the ends of a diameter perpendicular to the line of action of the load.

Consider a portion of the ring $ACDB$; let M be the bending moment and P the normal force on the section AB. Then we have $\rho_0 = a$, and

$$P = \frac{W}{2}\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$M = M_0 + \frac{Wa}{2}(1 - \cos\theta) \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Hence, from (3) and (4), p. 400, we have

$$Ee_0S - E(1 + e_0)\left(\frac{1}{\rho} - \frac{1}{a}\right)\frac{Sh^2}{a} = \frac{W}{2}\cos\theta \quad . \quad . \quad . \quad \text{(iii)}$$

$$E(1 + e_0)\left(\frac{1}{\rho} - \frac{1}{a}\right)Sh^2 = \frac{Wa}{2}(1 - \cos\theta) + M_0 \quad \text{(iv)}$$

Dividing (iv) by a and adding to (iii) we get

$$Ee_0S = \frac{W}{2} + \frac{M_0}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

---

* Reference should be made to *Photo-Elasticity*, Coker and Filon, for experiments on chain links and rings.

which shows that e_0 is constant. Then, multiplying (iv) by $ad\theta$ and integrating over a quadrant of the ring gives

$$E(1+e_0)Sh^2\int^{\frac{\pi}{2}}\left(\frac{1}{\rho}-\frac{1}{a}\right)ad\theta=\frac{Wa}{2}\int_0^{\frac{\pi}{2}}(1-\cos\theta)ad\theta+\int_0^{\frac{\pi}{2}}M_0ad\theta.$$

Now $\int_0^{\frac{\pi}{2}}\frac{(1+e_0)ad\theta}{\rho}$ is the angle between the normals to the strained central axis at E and F, which is $\frac{\pi}{2}$. Hence the last equation becomes

$$\frac{\pi}{2}ESh^2-ESh^2(1+e_0)\frac{\pi}{2}=\frac{Wa^2}{2}\left(\frac{\pi}{2}-1\right)+\frac{\pi M_0a}{2}$$

whence, using the value of e_0 given by (v),

$$M_0=\frac{Wa}{2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2-h^2}-1\right)\quad.\quad.\quad.\quad.\quad.\quad(13)$$

Substituting for M_0 from (13), (ii) gives

$$M=\frac{Wa}{2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-\cos\theta\right)\quad.\quad.\quad.\quad.\quad.\quad(14)$$

The maximum value of the bending moment occurs at the points of loading, where $\theta=\frac{\pi}{2}$ and $-\frac{\pi}{2}$, and we have

$$M_{max}=\frac{Wa}{\pi}\cdot\frac{a^2}{a^2+h^2}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(15)$$

Then from (v) we have

$$e_0=\frac{a^2}{a^2+h^2}\cdot\frac{W}{\pi ES}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(16)$$

To calculate the stresses we must find e. From (iv) we get

$$E(1+e_0)\left(\frac{1}{\rho}-\frac{1}{a}\right)Sh^2=\frac{Wa}{2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-\cos\theta\right).$$

Then (1) on p. 400 gives

$$e=\frac{a^2}{a^2+h^2}\cdot\frac{W}{\pi ES}+\frac{Wa}{2ESh^2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-\cos\theta\right)\frac{ay_0}{a+y_0}$$

The normal stress at any point is $p=Ee$, therefore

$$p=\frac{W}{S}\left\{\frac{1}{\pi}\cdot\frac{a^2}{a^2+h^2}+\frac{a^2}{2h^2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-\cos\theta\right)\frac{y_0}{a+y_0}\right\}\quad.\quad(17)$$

At F, on the line of action of W, $\theta=\frac{\pi}{2}$, and

$$p=\frac{W}{\pi S}\left(\frac{a^2}{a^2+h^2}+\frac{a^2}{h^2}\cdot\frac{a^2}{a^2+h^2}\cdot\frac{y_0}{a+y_0}\right)$$

$$=\frac{W}{\pi S}\cdot\frac{a^2}{a^2+h^2}\left(1+\frac{a^2}{h^2}\cdot\frac{y_0}{a+y_0}\right).$$

The maximum stresses on this section are

$$\left.\begin{aligned} p_1 &= \frac{W}{\pi S}\cdot\frac{a^2}{a^2+h^2}\left(1+\frac{a^2}{h^2}\,\frac{y_1}{a+y_1}\right), \text{ at the outside } \quad . \quad . \\ p_2 &= \frac{W}{\pi S}\cdot\frac{a^2}{a^2+h^2}\left(1-\frac{a^2}{h^2}\cdot\frac{y_2}{a-y_2}\right), \text{ at the inside } . \quad . \quad . \end{aligned}\right\} (18)$$

For the section CD we have $\theta = 0$ and

$$\left.\begin{aligned} p_1 &= \frac{W}{S}\left\{\frac{1}{\pi}\cdot\frac{a^2}{a^2+h^2}+\frac{a^2}{2h^2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-1\right)\frac{y_1}{a+y_1}\right\} \quad . \quad . \\ p_2 &= \frac{W}{S}\left\{\frac{1}{\pi}\cdot\frac{a^2}{a^2+h^2}-\frac{a^2}{2h^2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-1\right)\frac{y_2}{a-y_2}\right\} \quad . \quad . \end{aligned}\right\} (19)$$

Which of these four stresses is numerically the greatest depends on the relative size of the mean radius of the ring and the dimensions of the cross section.

292. Ring with Stud.—Suppose now that the ring has a stud as shown in Fig. 336. Let R denote the thrust in the stud.

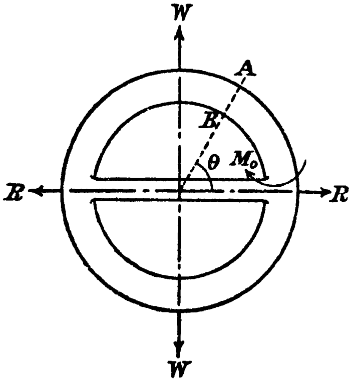


FIG. 336.

We have seen above that the strain of the central axis, on account of the load W, is independent of θ, and we deduce from (16) on p. 407 that in this case we shall have

$$e_0 = \frac{a^2}{a^2+h^2}\cdot\frac{W+R}{\pi ES}. \quad . \quad . \quad . \quad (20)$$

The bending moment due to R can be deduced from (14), p. 407, by writing R for W and $\frac{\pi}{2}-\theta$ for θ, thus

$$M = \frac{Wa}{2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-\cos\theta\right)+\frac{Ra}{2}\left(\frac{2}{\pi}\cdot\frac{a^2}{a^2+h^2}-\sin\theta\right) \quad (21)$$

Hence the stresses can be determined when we have found R, which is achieved by considering the contraction of the stud.

From (9), p. 403, we see that the contraction of the diameter of the ring, in the direction of R, is given by

$$u = 2\int_0^{\frac{\pi}{2}}\left\{\left(\frac{M}{ESh^2}+\frac{e_0}{a}\right)a\sin\theta - e_0\sin\theta\right\}ad\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left\{\frac{(W+R)a^3}{\pi ESh^2}\cdot\frac{a^2}{a^2+h^2}\sin\theta-\frac{a^3}{2ESh^2}(W\cos\theta+R\sin\theta)\sin\theta\right\}d\theta$$

whence

$$u = \frac{2Wa}{ES}\cdot\frac{a^2}{h^2}\left(\frac{1}{\pi}\cdot\frac{a^2}{a^2+h^2}-\frac{1}{4}\right)+\frac{2Ra}{ES}\cdot\frac{a^2}{h^2}\left(\frac{1}{\pi}\cdot\frac{a^2}{a^2+h^2}-\frac{\pi}{8}\right).$$

But we also have $u = \dfrac{Rl}{E_1S_1}$, where l is the length, S_1 the area of the cross section, and E_1 is Young's Modulus for the stud. We can write this

$$u = \lambda\frac{Rl}{ES}, \text{ where } \lambda = \frac{ES}{E_1S_1} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Equating the two values of u, we get

$$R = \frac{\dfrac{a^2}{h^2}\left(\dfrac{a^2}{a^2+h^2} - \dfrac{\pi}{4}\right)W}{\dfrac{\pi}{2}\cdot\dfrac{\lambda l}{a} + \dfrac{a^2}{h^2}\left(\dfrac{\pi^2}{8} - \dfrac{a^2}{a^2+h^2}\right)} = nW, \text{ say} \quad . \quad . \quad . \qquad (23)$$

Inserting this value of R in the expressions for e_0 and M, we get for the stress

$$p = Ee = \frac{W}{S}\left[\frac{n+1}{\pi}\cdot\frac{a^2}{a^2+h^2} + \frac{a^2}{h^2}\left\{\frac{n+1}{\pi}\cdot\frac{a^2}{a^2+h^2} - \tfrac{1}{2}(\cos\theta + n\sin\theta)\right\}\frac{y_0}{a+y_0}\right] \quad . \quad . \quad . \qquad (24)$$

This will be a maximum when

$$\cos\theta + n\sin\theta = 0$$

$$\text{or } \tan\theta = -\frac{1}{n}.$$

Let $n = \tan a$, then

$$\cos\theta + n\sin\theta = \frac{\cos(\theta - a)}{\cos a}.$$

Now n is positive, therefore a will be in the first quadrant, and therefore $\cos(\theta - a)$ must be positive. Therefore, for the stress to be a maximum, $\theta - a$ must be numerically as large as possible. Hence if $a < \dfrac{\pi}{4}$ the condition for a maximum is $\theta = \dfrac{\pi}{2}$, and if $a > \dfrac{\pi}{4}$, the condition is $\theta = 0$.

If $\theta = \dfrac{\pi}{2}$ we find

$$p = \frac{W}{S}\left[\frac{n+1}{\pi}\cdot\frac{a^2}{a^2+h^2}\left(1 + \frac{a^2}{h^2}\cdot\frac{y_0}{a+y_0}\right) - \frac{a^2}{2h^2}\cdot\frac{ny_0}{a+y_0}\right] \quad . \qquad (25)$$

The treatment of elliptical links will be found in Todhunter and Pearson's *History*, Vol. 2, Pt. 1, p. 439.

EXAMPLES XXV

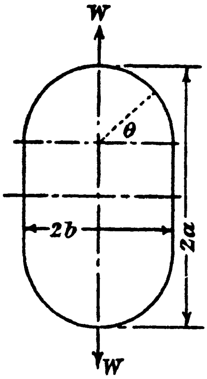


FIG. 337.

1. In the case of an oval link with straight sides and semi-circular ends, as shown in Fig. 337, show that in the semi-circular parts the bending moment is given by

$$M = \frac{Wb}{2}(n - \cos\theta),$$

whilst in the straight part it is given by $\frac{Wb}{2}(n - 1)$, where

$$n = \frac{\frac{h^2}{k^2}(a - b) + b}{\frac{h^2}{k^2}(a - b) + \frac{\pi b}{2}\left(1 + \frac{h^2}{b^2}\right)},$$

h referring to the circular part of the link, and k being the radius of gyration of the cross section. (K. Pearson.)

2. A curved beam, whose central line is a circular arc of radius 6″, is formed of a tube of radius 2″ outside, and thickness ¼″, determine the greatest tension and compression stresses set up by a bending moment of 10 tons. ins. tending to increase the curvature. (Mech. Sc. Trip., 1922, B.)

3. The bracket shown in Fig. 338 is used to support an under-running rail for an electric railway, and is subject to an upward thrust in the centre line of the section of the rail. If the middle portion of the bracket is a **T**, of dimensions 2½″ × 2½″ × ½″, bent to a circle about O as centre, find the greatest values of the tensile and compressive stresses when the upward thrust is 1,000 lbs. (Mech. Sc. Trip., B, 1914.)

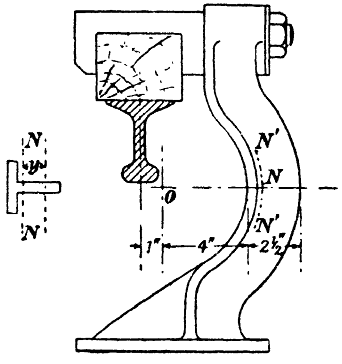


FIG. 338.

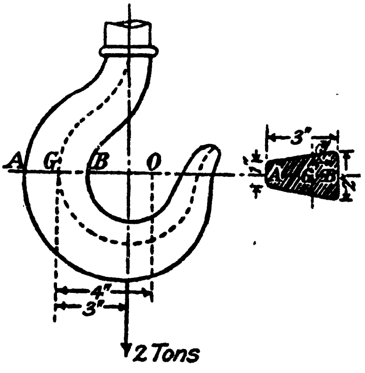


FIG. 339.

4. A ring of mean diameter 4″ is made of round steel 1″ diameter. It is subjected to four equal pulls in two directions at right angles and passing through the centre. Determine the maximum value of the pulls if the maximum tensile stress is not to exceed 5 tons/in.² (Mech. Sc. Trip., B, 1913.)

5. Calculate the greatest tensile and compressive stresses in the hook shown in Fig. 339. (Mech. Sc. Trip., B, 1911.)

CHAPTER XXVI

GENERAL ANALYSIS OF STRESS AND STRAIN

293. Need for General Analytical Methods.—Many problems in stress calculation require for their solution a method of treatment differing from any of those employed in our previous chapters. Examples of such problems are afforded by thick tubes subjected to internal or external pressure, rotating discs, the flexure of flat plates, the more exact calculation of bending stresses which is desirable in a few particular cases, the torsion of shafts of non-circular cross section, etc., etc. In general, the method consists in considering the equilibrium and deformation of an element of the body and so forming differential equations, the solution of which enables us to calculate the stresses and strains at any point in the body. To this end we shall occupy ourselves first with establishing certain general equations to which we shall afterwards make frequent reference.

STRESSES IN THREE DIMENSIONS

294. Stress Components.—If we know the forces which act on any three planes in three different directions, per unit area, we can find the force per unit area on any fourth plane. For, consider any tetrahedron *OABC* (Fig. 340), and suppose the edges *OA*, *OB*, *OC* are mutually perpendicular, and that we know the forces on the faces *OAB*, *OBC*, *OCA*. Let l, m, n be the direction cosines of the resultant force P on the face *ABC*, referred to *OA*, *OB*, *OC* as axes. Then for the equilibrium of the matter within the tetrahedron we must have

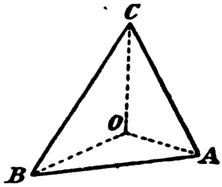


FIG. 340.

lP + the sum of the resolved parts, in direction *OA*, of the known forces on the other three faces, $= 0$,

and two similar equations. We also have the equation $l^2 + m^2 + n^2 = 1$. Thus there are four equations to find the four unknown quantities l, m, n, P. The edges *OA*, *OB*, *OC* have been taken as perpendicular to each other for convenience of expression, but the proposition would be equally true whatever the relative inclinations of the three planes for which we know the forces.

Now, to specify completely the resultant force on any one plane we must know its components in three directions, so that if we are to be able to find the force across any arbitrary plane it would appear that we must know altogether nine components. This, however, is not true, for consider the equilibrium of a small rectangular block of matter, the length of its sides being δx, δy, δz.

Let p_x, p_y, p_z be the normal stresses on the faces of the block *OABC*.... in the directions *Ox*, *Oy*, *Oz*.

Let q_{xz} and q_{yz} be the tangential components of the resultant stress on the face z = constant, i.e. the shear stresses. Similarly, let q_{zx} and q_{yx} be the shear stresses on a face x = constant; p_{xy} and q_{zy} the shear stresses on the faces y = constant.

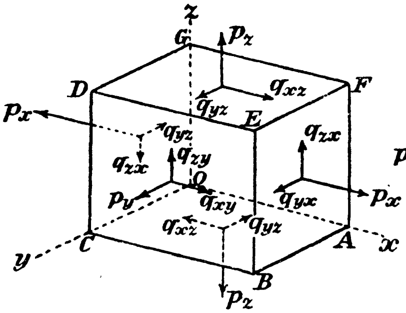


FIG. 341.

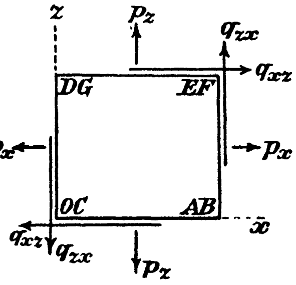


FIG. 342.

The directions of these are shown in the diagrams (Figs. 341 and 342). With regard to the notation adopted for shear stresses it will be noticed that the second letter of the suffix denotes which plane the stress is acting on, whilst the first letter of the suffix shows the axis to which the direction of the stress is parallel; thus the suffix xy denotes that the stress is in a plane y = constant and parallel to the axis of x.

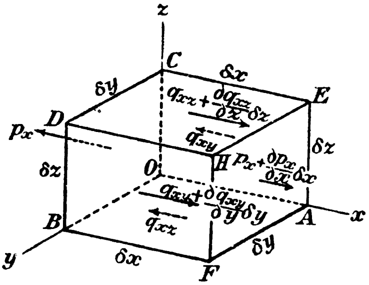
Fig. 342 is a view of the block looking along the axis of y. Evidently, for equilibrium, the couple formed by q_{xz} acting on the upper and lower faces must be equal and opposite to the couple formed by q_{zx} acting on the right and left faces. Therefore, we must have $q_{xz}\delta x \, . \, \delta y \times \delta z = q_{zx} \times \delta y \, . \, \delta z \times \delta x$, and therefore $q_{xz} = q_{zx}$. Similarly, $q_{yx} = q_{xy}$ and $q_{zy} = q_{yz}$. Thus the nine stress components are reduced to six: p_x, p_y, p_z; q_{xy}, q_{yz}, q_{zx}.

295. Stress Equations of Equilibrium.—Let the stress components, referred to a system of rectangular axes, be specified as in § 294, and consider the equilibrium of a small rectangular block whose edges are parallel to the three co-ordinate axes. Let the lengths of the edges be δx, δy, δz, as in Fig. 343, and suppose the element to be acted on by body forces X, Y, Z,* per unit volume in the positive directions of the axes.

---

* These forces may be gravitational, or due to rotation, magnetic attraction, etc., and do not refer to forces applied to the boundary of the body.

If p_x be the normal stress on the plane *OBDC*, the rate of increase of p_x in direction x will be $\frac{\partial p_x}{\partial x}$, the sign of partial differentiation being used since p_x, and all the stresses, will in general be functions of x, y and z. Then the normal stress on the plane *AEHF* will be $p_x + \frac{\partial p_x}{\partial x}\delta x$. Similarly, if q_{xv} is the shear stress on the plane *OAEC* in the direction of the axis



z
C
δx
δy
E
$q_{xz} + \frac{\partial q_{xz}}{\partial x}\delta z$
D
p_x
q_{xy}
δz
δz
O
H
$p_x + \frac{\partial p_x}{\partial x}\delta x$
$q_{xy} + \frac{\partial q_{xy}}{\partial y}\delta y$
B
q_{xz}
A
x
δy
y
δx
F

FIG. 343.

of x, the shear stress on the plane *BDHF* is $q_{xv} + \frac{\partial q_{xv}}{\partial y}\delta y$. The shear stresses on the planes *OAFB* and *CEHD*, parallel to *Ox*, are q_{xz} and $q_{xz} + \frac{\partial q_{xz}}{\partial z}\delta z$. The directions of all these stresses are shown in Fig. 343, the stresses parallel to the other axes being omitted for the sake of clearness.

The total stress acting on the area *OCDB* in the direction of the axis of x is $p_x \cdot \delta y\delta z$; the total stress on the area *CEHD* parallel to the same axis is $\left(q_{xz} + \frac{\partial q_{xz}}{\partial z}\delta z\right)\delta x\delta y$, and so on. Taking all the forces acting in this direction we see that for equilibrium we must have

$$\left(p_x + \frac{\partial p_x}{\partial x}\,\delta x\right)\delta y\delta z - p_x \cdot \delta y\delta z + \left(q_{xv} + \frac{\partial q_{xv}}{\partial y}\delta y\right)\delta x\delta z$$

$$- q_{xv} \cdot \delta x \cdot \delta z + \left(q_{xr} + \frac{\partial q_{xz}}{\partial z}\delta z\right)\delta x\delta y + X\delta x\delta y\delta z = 0$$

or

$$\frac{\partial p_x}{\partial x} + \frac{\partial q_{xv}}{\partial y} + \frac{\partial q_{xz}}{\partial z} + X = 0$$

In the same way, resolving parallel to the axes *Oy* and *Oz*, we obtain two more equations of equilibrium. Remembering (§ 294) that $q_{xz} = q_{zx}$, $q_{vz} = q_{zy}$, and $q_{yx} = q_{xy}$, we obtain:

$$\left.\begin{aligned} \frac{\partial p_x}{\partial x} + \frac{\partial q_{xv}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + X = 0 \\ \frac{\partial q_{xv}}{\partial x} + \frac{\partial p_v}{\partial y} + \frac{\partial q_{yz}}{\partial z} + Y = 0 \\ \frac{\partial q_{zx}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial p_z}{\partial z} + Z = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

296. Plane Stress with No Body Forces : Cartesian Co-ordinates.—When the body is acted on only by forces applied to its boundary, the quantities X, Y, Z are zero. If the applied forces are such that the stresses are all parallel to one plane, say the xy plane, p_z, q_{zx}, and q_{yz} being zero, the equations of equilibrium become

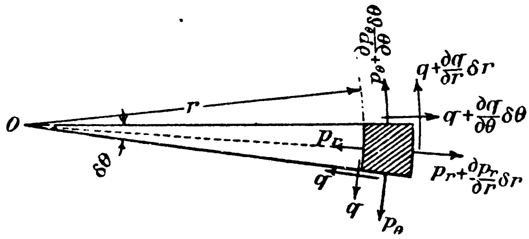
$$\left.\begin{aligned}\frac{\partial p_x}{\partial x}+\frac{\partial q}{\partial y}=0\ .\ .\ .\ .\\ \frac{\partial q}{\partial x}+\frac{\partial p_y}{\partial y}=0\ .\ .\ .\ .\end{aligned}\right\}\ .\ .\ .\ .\ .\ .\quad (2)$$

where q is written for q_{xy}, being the only shear stress. It is easily verified that the stresses can now be expressed in terms of a single function V of x and y :

$$p_x=\frac{\partial^2 V}{\partial y^2};\ p_y=\frac{\partial^2 V}{\partial x^2};\ q=-\frac{\partial^2 V}{\partial x\partial y}\ .\ .\ .\ .\quad (3)$$

for these values of p_x, p_y and q satisfy the equations (2) identically. The function V * is called the " stress function."

297. Plane Stress with No Body Forces : Polar Co-ordinates.—The above equations can readily be transformed to polar co-ordinates (r, θ) by the usual formulæ of transformation, but it will be more instructive to discover the equations independently.



$p_\theta+\frac{\partial p_\theta}{\partial\theta}\delta\theta$
$q+\frac{\partial q}{\partial r}\delta r$
$q+\frac{\partial q}{\partial\theta}\delta\theta$
O
r
$\delta\theta$
p_r
$p_r+\frac{\partial p_r}{\partial r}\delta r$
q
q
p_θ

FIG. 344.

In Fig. 344 let O be the origin of the polar co-ordinates, and consider an element of the material bounded by two radii separated by an angle $\delta\theta$, and two arcs of radius r and $r+\delta r$ with O as centre. This element is shown shaded.

Let p_r = the radial stress across the inner arc, so that $p_r+\frac{\partial p_r}{\partial r}\delta r$ is the radial stress across the outer arc.

Let p_θ and $p_\theta+\frac{\partial p_\theta}{\partial\theta}\delta\theta$ be the normal (" hoop ") stresses across the radial boundaries of the element.

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* V possesses the property that if the curves $\frac{\partial V}{\partial x}$ = const. and $\frac{\partial V}{\partial y}$ = const. be drawn for a series of values of the constants separated by equal intervals, the stresses at any point, across planes parallel to (xz) and (yz), are directed along the tangents to these curves respectively at the point in question. (J. N. Michell, *London Math. Soc. Proc.*, Vol. 32 (1901).

Let the shear stresses be as shown.

Let the material be of unit thickness perpendicular to the plane of the paper, and consider the equilibrium of the element. Resolving along the middle radius we have:

$$\left(p_r + \frac{\partial p_r}{\partial r}\delta r\right)(r + \delta r)\delta\theta - p_r \, . \, r\delta\theta + \left[\left(q + \frac{\partial q}{\partial \theta}\delta\theta\right)\delta r - q\delta r\right]\cos\frac{\delta\theta}{2}$$

$$- \left[p_\theta \delta r + \left(p_\theta + \frac{\partial p_\theta}{\partial \theta}\delta\theta\right)\delta r + \left(q + \frac{\partial q}{\partial r}\delta r\right)(r + \delta r)\delta\theta + q \, . \, r\delta\theta\right]\sin\frac{\delta\theta}{2} = 0$$

This reduces to

$$\frac{\partial p_r}{\partial r} + \frac{1}{r}\frac{\partial q}{\partial \theta} - \frac{p_\theta}{r} = 0$$

on rejecting small quantities of the second order.

Similarly, resolving at right angles to the middle radius, we find

$$\left[\left(p_\theta + \frac{\partial p_\theta}{\partial \theta}\delta\theta\right)\delta r - p_\theta\delta r\right]\cos\frac{\delta\theta}{2} + \left(q + \frac{\partial q}{\partial r}\delta r\right)(r + \delta r)\delta\theta$$

$$- q \, . \, r\delta\theta + \left[\left(q + \frac{\partial q}{\partial \theta}\delta\theta\right)\delta r + q\delta r\right]\sin\frac{\delta\theta}{2} = 0,$$

which reduces to

$$\frac{\partial q}{\partial r} + \frac{1}{r}\frac{\partial p_\theta}{\partial \theta} + \frac{2q}{r} = 0$$

Thus the stress equations in polar co-ordinates are

$$\left.\begin{aligned} \frac{\partial p_r}{\partial r} + \frac{1}{r}\frac{\partial q}{\partial \theta} - \frac{p_\theta}{r} &= 0 \\ \frac{\partial q}{\partial r} + \frac{1}{r}\frac{\partial p_\theta}{\partial \theta} + \frac{2q}{r} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It can easily be verified that p_r, p_θ and q can be expressed in terms of a stress function V by the equations *

$$\left.\begin{aligned} p_r &= \frac{1}{r^2} \cdot \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r} \\ p_\theta &= \frac{\partial^2 V}{\partial r^2} \\ q &= -\frac{\partial}{\partial r}\left(\frac{1}{r} \cdot \frac{\partial V}{\partial \theta}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where V is a function of r and θ, for on substituting these values in (4) the equations are satisfied.

298. Displacements in Cartesian Co-ordinates.—Let us consider a two-dimensional system of strain, and let the positions of all points in the body be referred to fixed rectangular axes *Ox*, *Oy* (Fig. 345).

Let (x, y) be the co-ordinates of a particle A in the body before strain, and $(x + u,\ y + v)$ the co-ordinates of the same particle after strain when it has moved to A'. Then u and v are the displacements of A in the directions of the axes of x and y. In general u and v will be functions of x and y.

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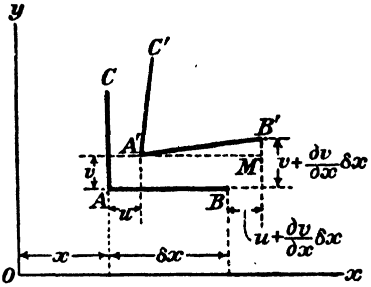
* J. H. Michell, *London Math. Soc. Proc.*, Vol. 31 (1899).

Let B be another point in the unstrained body, its co-ordinates before strain being $(x + \delta x, y)$; then after strain it will be at B', its co-ordinates being



y
C'
C
B'
A'
M
$v+\frac{\partial v}{\partial x}\delta x$
v
A
u
B
$u+\frac{\partial v}{\partial x}\delta x$
x
δx
O
x

FIG. 345.

$$\left(x + \delta x + u + \frac{\partial u}{\partial x}\delta x\ ;\ y + v + \frac{\partial v}{\partial x}\delta x\right)$$

$\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ being the rates at which u and v increase in the direction x.

Then, to the first order of small quantities, the length of $A'B'$ is

$$\delta x + \frac{\partial u}{\partial x}\delta x$$

so that the increase of length is $\frac{\partial u}{\partial x}\delta x$

Hence, if e_x represents the strain in the direction x, we have

$$e_x = \frac{\partial u}{\partial x}.$$

Similarly, the strain in the direction y is

$$e_y = \frac{\partial v}{\partial y}.$$

Again, from the figure, we see that the slope of $A'B'$ is

$$\frac{B'M}{A'M} = \frac{\frac{\partial v}{\partial x}\delta x}{\delta x + \frac{\partial u}{\partial x}\delta x} = \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}}$$

$$= \frac{\partial v}{\partial x}\left(1 + \frac{\partial u}{\partial x}\right)^{-1}$$

$$= \frac{\partial v}{\partial x},$$

to the first order of small quantities. Thus, the change in direction of AB is $\frac{\partial v}{\partial x}$. Similarly the change in direction of an elementary length AC, originally parallel to Oy, is $\frac{\partial u}{\partial y}$. Hence, the shear strain, that is the change in the angle CAB is $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$.

Thus the components of strain are given in terms of the displacements u and v by the equations

$$e_x = \frac{\partial u}{\partial x}\,; \quad e_y = \frac{\partial v}{\partial y}\,; \quad e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad . \quad . \quad . \quad (6)$$

Similarly, if e_x, e_y, e_z, e_{xy}, e_{yz}, e_{zx} denote the components of strain in a three-dimensional system, we can show that

$$e_z = \frac{\partial w}{\partial z}\,; \quad e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\,; \quad e_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad . \quad . \quad . \quad (7)$$

where w is the displacement of the point (x, y, z) in the direction z. We leave the deduction of these equations as an exercise for the student.

299. Relations between the Strain Components.—The above expressions for the strains, in terms of the displacements, show that the former are not all independent; eliminating u and v from the three equations (6) will give a relation which must hold between the strains in a two-dimensional system. Differentiating the first of (6) twice with respect to y we have

$$\frac{\partial^2 e_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \,.\, \partial y^2}.$$

Similarly the second gives

$$\frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \,.\, \partial y}.$$

Adding these together we have

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^3 u}{\partial x \,.\, \partial y^2} + \frac{\partial^3 v}{\partial x^2 \,.\, \partial y}$$

$$= \frac{\partial^2}{\partial x \,.\, \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

That is

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \,.\, \partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is usually called the equation of compatibility.

300. Relations Between the Stresses and Displacements in a Two-Dimensional Stress System.—If the only stresses in the material be p_x, p_y and q, the ordinary stress-strain relations (p. 27) give, from (16), § 29,

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= e_x = \frac{p_x}{E} - \frac{p_y}{mE} \quad . \quad . \quad . \\ \frac{\partial v}{\partial y} &= e_y = \frac{p_y}{E} - \frac{p_x}{mE} \quad . \quad . \quad .\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Solving these equations for p_x and p_y we get

$$\left.\begin{aligned}p_x &= \frac{m^2E}{m^2-1}\left(\frac{\partial u}{\partial x} + \frac{1}{m}\frac{\partial v}{\partial y}\right) \quad . \quad . \quad . \\ p_y &= \frac{m^2E}{m^2-1}\left(\frac{\partial v}{\partial y} + \frac{1}{m}\frac{\partial u}{\partial x}\right) \quad . \quad . \quad .\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also we have

$$q = Ce_{xy} = C\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

or

$$q = \frac{mE}{2(m+1)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

301. Equations for Finding the Displacements in a Two-Dimensional Stress System.—Substituting the above values of p_x and q in the first of the equilibrium equations (2) we have

$$\frac{m^2E}{m^2-1}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{1}{m}\frac{\partial v}{\partial y}\right) + \frac{mE}{2(m+1)} \cdot \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

or

$$\frac{m}{m-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{m}\frac{\partial^2 v}{\partial x . \partial y}\right) + \tfrac{1}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x . \partial y}\right) = 0$$

Similarly, from the second equilibrium equation we get

$$\frac{m}{m-1}\left(\frac{\partial^2 v}{\partial y^2} + \frac{1}{m}\frac{\partial^2 u}{\partial x . \partial y}\right) + \tfrac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x . \partial y}\right) = 0$$

These two equations may be rearranged thus :

$$\left.\begin{aligned}\frac{m}{m-1} \cdot \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \tfrac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= 0 \\ \frac{m}{m-1} \cdot \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \tfrac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

The quantities $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ and $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ are known as the " dilatation " and " rotation " respectively, and are usually denoted by Δ and $\tilde{w}$, so that

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$2\,\tilde{w} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

The equations may then be written

$$\left.\begin{aligned} \frac{m}{m-1}\cdot\frac{\partial\Delta}{\partial x}-\frac{\partial\tilde{w}}{\partial y}&=0\\ \frac{m}{m-1}\cdot\frac{\partial\Delta}{\partial y}+\frac{\partial\tilde{w}}{\partial x}&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Differentiating the first of these with respect to x and the second with respect to y, and adding the results we get, after dividing by $m/(m-1)$,

$$\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}=0$$

or

$$\nabla^2\Delta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

where ∇^2 stands for the operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Again, using the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ given by (9), we have

$$\Delta=\frac{m-1}{mE}(p_x+p_y).$$

If p_x and p_y are given by a stress function V, as in § 296, we get

$$\Delta=\frac{m-1}{mE}\left(\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x^2}\right)$$

Hence, from (14) we have

$$\nabla^2\left(\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x^2}\right)=0$$

or

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 V=0$$

which may be abbreviated to

$$\nabla^4 V=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The problem of finding the stress distribution in a two-dimensional system of stress reduces to finding a solution of this equation, which is such that the conditions obtaining at the boundary of the material are satisfied.

It should be noted that the elastic constants do not enter into the equation for V, so that the type of internal strain under given boundary conditions does not depend on the particular material used, provided it obeys Hooke's Law and is isotropic. We shall illustrate the method of solving particular problems presently.

302. Two-Dimensional Strain System.—We have considered above the case when the stresses all act parallel to the xy plane; in general this will imply *strain* in three dimensions. For example, a flat plate, acted on by tensions applied to its boundary, parallel to its central plane, and uniformly distributed over the thickness of the plate, will contract in directions perpendicular to the plate. To prevent this contraction we should have to apply equal and opposite tensions to the two surfaces of the plate. We should then have a three-dimensional stress system producing a two-dimensional state of strain. In general, if the strain system is in two dimensions the stress system must be three-dimensional, and conversely. An example of an exception to this is the torsion of a circular shaft.

Let the *strains* be entirely parallel to the xy plane. Then we have (p. 28)

$$e_x = \frac{\partial u}{\partial x} = \frac{1}{E}\left(p_x - \frac{p_y + p_z}{m}\right)$$

$$e_y = \frac{\partial v}{\partial y} = \frac{1}{E}\left(p_y - \frac{p_z + p_x}{m}\right)$$

$$0 = e_z = \frac{\partial w}{\partial z} = \frac{1}{E}\left(p_z - \frac{p_x + p_y}{m}\right).$$

Solving these simultaneous equations for p_x, p_y, p_z we get

$$\left.\begin{aligned} p_x &= \frac{mE}{(m-2)(m+1)}\left\{(m-1)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} \quad . \quad . \quad . \\ p_y &= \frac{mE}{(m-2)(m+1)}\left\{(m-1)\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right\} \quad . \quad . \quad . \\ p_z &= \frac{mE}{(m-2)(m+1)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (16)$$

and, as before,

$$q_{xy} = \frac{mE}{2(m+1)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The strains e_{yz} and e_{zx} are zero, so that q_{yz} and q_{zx} are zero also; w is zero, whilst u and v are functions of x and y only. The equations of equilibrium are

$$\frac{\partial p_x}{\partial x} + \frac{\partial q_{xy}}{\partial y} = 0\,; \quad \frac{\partial q_{xy}}{\partial x} + \frac{\partial p_y}{\partial y} = 0\,; \quad \frac{\partial p_z}{\partial z} = 0.$$

Substituting for p_x, p_y and q_{xy} in the first two of these we get, after a little rearrangement,

$$\left.\begin{aligned} &\frac{m-1}{m-2}\cdot\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \tfrac{1}{2}\cdot\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \quad . \quad . \quad . \\ \text{and} \quad & \\ &\frac{m-1}{m-2}\cdot\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \tfrac{1}{2}\ \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (18)$$

Comparing these with equations (12) it is easy to see that we shall have, in this case also,

$$\nabla^4 V = 0.$$

303. Transformation to Polar Co-ordinates.—In many problems it is more convenient to work with polar co-ordinates, and in these cases we require the equation corresponding with $\nabla^4 V = 0$.

Write $x = r\cos\theta$, and $y = r\sin\theta$.

Then

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\,.\,\frac{\partial V}{\partial x} + \sin\theta\,.\,\frac{\partial V}{\partial y} \quad . \quad . \quad . \quad \text{(i)}$$

and

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -\sin\theta\,.\,\frac{\partial V}{\partial x} + \cos\theta\,.\,\frac{\partial V}{\partial y}\,. \quad . \quad . \quad . \quad \text{(ii)}$$

Solving (i) and (ii) as simultaneous equations for $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial x} = \cos\theta\,.\,\frac{\partial V}{\partial r} - \frac{1}{r}\sin\theta\,.\,\frac{\partial V}{\partial \theta}$$

$$\frac{\partial V}{\partial y} = \sin\theta\,.\,\frac{\partial V}{\partial r} + \frac{1}{r}\cos\theta\,.\,\frac{\partial V}{\partial \theta}$$

or, in general,

$$\frac{\partial}{\partial x} = \cos\theta\,.\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\,.\,\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\,.\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\,.\,\frac{\partial}{\partial \theta}.$$

Then

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta\,.\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\,.\,\frac{\partial}{\partial \theta}\right)\left(\cos\theta\,.\,\frac{\partial V}{\partial r} - \frac{1}{r}\sin\theta\,.\,\frac{\partial V}{\partial \theta}\right)$$

$$= \cos^2\theta\,.\,\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin\theta\,.\,\frac{\partial V}{\partial \theta} + \frac{1}{r}\sin^2\theta\,\frac{\partial V}{\partial r}$$

$$- \frac{2}{r}\sin\theta\cos\theta\,\frac{\partial^2 V}{\partial r\,.\,\partial \theta} + \frac{1}{r^2}\cos\theta\sin\theta\,\frac{\partial V}{\partial \theta} + \frac{1}{r^2}\sin^2\theta\,\frac{\partial^2 V}{\partial \theta^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\,.\,\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\sin^2\theta\,.\,\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\sin^2\theta\,\frac{\partial V}{\partial r}$$

$$+ \frac{\sin 2\theta}{r^2}\cdot\frac{\partial V}{\partial \theta} - \frac{\sin 2\theta}{r}\cdot\frac{\partial^2 V}{\partial r\,.\,\partial \theta} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\,.\,\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\cos^2\theta\,.\,\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\cos^2\theta\,.\,\frac{\partial V}{\partial r}$$

$$\frac{\sin 2\theta}{r^2}\cdot\frac{\partial V}{\partial \theta} + \frac{\sin 2\theta}{r}\cdot\frac{\partial^2 V}{\partial r\,.\,\partial \theta} \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Adding together (iii) and (iv) we get

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$

Hence the equation $\nabla^4 V = 0$ becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 V = 0 \quad . \ . \ . \ . \ . \quad (19)$$

EXAMPLES XXVI

1. Prove that, with cylindrical co-ordinates (r, θ, z) the stress equations of equilibrium when there are no body forces, are:

$$\frac{\partial p_r}{\partial r} + \frac{1}{r}\cdot\frac{\partial q_{r\theta}}{\partial \theta} + \frac{\partial q_{rz}}{\partial z} + \frac{p_r - p_\theta}{r} = 0$$

$$\frac{\partial q_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial p_\theta}{\partial \theta} + \frac{\partial q_{\theta z}}{\partial z} + \frac{2q_{r\theta}}{r} = 0$$

$$\frac{\partial q_{rz}}{\partial r} + \frac{1}{r}\frac{\partial q_{\theta z}}{\partial \theta} + \frac{\partial p_z}{\partial z} + \frac{q_{rz}}{r} = 0.$$

2. Prove that in question 1 the strains are given by

$$e_r = \frac{\partial u_r}{\partial r}; \quad e_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}; \quad e_z = \frac{\partial u_z}{\partial z}$$

$$e_{\theta z} = \frac{1}{r}\cdot\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}; \quad e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r};$$

$$e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\cdot\frac{\partial u_r}{\partial \theta};$$

where u_r, u_θ, u_z are the displacements of the point (r, θ, z) in the directions r, θ, z respectively.

3. The dilatation Δ being defined as the increment of volume per unit volume, show that in a three-dimensional strain system

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{dz}, \text{ in Cartesian co-ordinates.}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}, \text{ in cylindrical co-ordinates.}$$

4. Show that, for small displacements, the extension of an element in the direction (l, m, n) is

$$e_x l^2 + e_y m^2 + e_z n^2 + e_{yz}.mn + e_{zx}nl + e_{xy}lm.$$

5. Establish the equation $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 V = 0$, from first principles, assuming the relations between the stresses and V given in § 297.

CHAPTER XXVII

SOME PROBLEMS IN TWO DIMENSIONS

304. Some Particular Solutions of the General Equation.—We shall now illustrate the application of the above principles by applying them to the solution of certain problems of practical interest. For the most part the method is indirect: we cannot usually start with the data of a particular problem and work towards an answer, but we can guess solutions to the equation $\nabla^4 V = 0$ and see what problem that solution solves. In some cases the data of the problem suggest the form of solution to take. The solutions of this equation can be made to throw considerable light on several practical problems which at present have not yielded to complete mathematical analysis, and the subject has been extensively developed lately by C. E. Inglis.* The results of his investigations are of great importance, so that we shall include most of them here. We shall, however, begin with a few simple solutions.

(i) $V = Ax + By$

satisfies $\nabla^4 V = 0$, but $\dfrac{\partial^2 V}{\partial x^2}$, $\dfrac{\partial^2 V}{\partial y^2}$, and $\dfrac{\partial^2 V}{\partial x \partial y}$ all vanish so that all the stresses are zero.

(ii) $V = Ax^2 + By^2 + Cxy$

also satisfies the equation, A, B and C being constants. We have

$$p_x = \frac{\partial^2 V}{\partial y^2} = 2B\,;\ p_y = \frac{\partial^2 V}{\partial x^2} = 2A\,;\ q = -\frac{\partial^2 V}{\partial x \partial y} = -C.$$

Thus, this solution corresponds with normal stresses, parallel to the axes of x and y, which are constant everywhere, together with a shear stress which is also constant. Hence, if we take $C = 0$, $A = 0$, and $B = \frac{1}{2}P_1$, we shall have $p_x = P_1$, $p_y = q = 0$, i.e. the case of pure tension parallel to the axis of x.

Similarly, $V = -q.xy$ gives the case of a plate subjected to constant shearing stresses q parallel to its edges.

$V = \frac{1}{2}P_2x^2$ gives pure tension parallel to the axis of $y : p_y = P_2$, while $p_x = q = 0$.

(iii) $V = Ay^3$

is a solution of the equation $\nabla^4 V = 0$. This gives

$$p_x = \frac{\partial^2 V}{\partial y^2} = 6Ay\,;\ p_y = 0\,;\ q = 0.$$

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* Inst. Naval Architects, 1922, also *Engineering*, Vol. 95, p. 415.

Hence we have a tensile stress parallel to the axis of x, proportional to the distance from that axis.

Consider a plate of thickness b, and bounded by the lines $y = \pm d/2$, as in Fig. 346, and the lines $x = 0$, $x = l$.

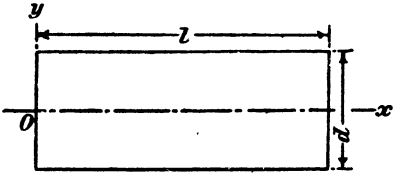

FIG. 346.

On any section parallel to Oy the stress at any point will consist of a tensile stress $p_x = 6Ay$ only.

The resultant stress across the section is

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} b\,.\,dy.p_x = \int_{-\frac{d}{2}}^{\frac{d}{2}} 6Aby.dy = 0.$$

The resultant stress couple is

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} bdy\,.\,p_x\,.\,y = 6A\int_{-\frac{d}{2}}^{\frac{d}{2}} by^2.dy = 6AI,$$

where I denotes the moment of inertia of the cross section about an axis through its centre perpendicular to the xy plane.

If we take $A = \frac{M}{6I}$, the resultant couple will be M, and we have $p_x = 6Ay = My/I$. This is the case of simple bending in the plane xy, due to forces distributed over the ends $x = 0$ and $x = l$ in accordance with the law $p_x \propto y$. Similarly, the solution $V = A'x^3$ will correspond with pure bending due to forces distributed in a similar way along the edges parallel to the axis of x.

(iv) Another solution is

$$V = Axy^3$$

where A is a constant. This gives

$$p_x = \frac{\partial^2 V}{\partial y^2} = 6Axy\ ;\ p_y = 0\ ;\ q = -\frac{\partial^2 V}{\partial x \partial y} = -3Ay^2.$$

Hence for points on the line $x =$ const., p_x varies as y, as in case (iii) above, so that the resultant action due to p_x over the section of a plate whose edges are given by $y = \pm\frac{d}{2}$, will be a couple $6AxI$ in the above notation, i.e. a couple proportional to the distance of the line from the axis of y.

Simultaneously, there will be a shear stress $q = -3Ay^2$.

The total action due to this, on the section of a plate of width d as above, is

$$-\int_{-\frac{d}{2}}^{\frac{d}{2}} 3Ay^2 \, . \, bdy = -\frac{Abd^3}{4} = \text{constant}.$$

Thus all such sections will be subjected to this shearing force.

305. Narrow Cantilever of Rectangular Section with Concentrated Load.—The results of cases (ii) and (iv) above suggest that we can probably find a solution for the case of a cantilever loaded with a single load at one end. In this case every cross section is acted on by the same shearing force and by a bending moment proportional to the distance of the section from the loaded end. In case (ii) when A and B are zero, we have a constant shearing stress at all points ; in case (iv) we have a shearing stress which gives a constant total action over the cross section of a plate of width d, as in Fig. 346, but the shearing stress does not vanish at the extremities of this section, as it must in the case of a beam. Case (iv) also gives a couple acting on every cross section proportional to the distance from this axis of y, and if we take the origin at the loaded end of the cantilever this is what is required.

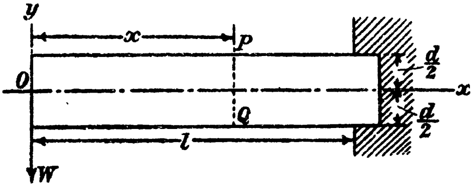

FIG. 347.

Perhaps by a combination of the two solutions we can fit all the conditions of the problem.

Let l be the free-length of the cantilever, d its depth parallel to the axis of y, and b its thickness perpendicular to the plane xy, and take the origin at the middle of the loaded end, as in Fig. 347.

Take $$V = Axy^3 + Bxy.$$

Then $$p_x = \frac{\partial^2 V}{\partial y^2} = 6Axy \, ; \; p_y = 0$$

$$q = -\frac{\partial^2 V}{\partial x \, . \, \partial y} = -3Ay^2 - B.$$

The conditions to be satisfied are that the total action on a section such as PQ should consist of a couple $= Wx$, and a shearing force $= W$, and that the shear stress should vanish when $y = \pm \frac{d}{2}$. This last condition requires

$$3A\frac{d^2}{4} - B = 0, \text{ or } B = -\frac{3}{4}Ad^2.$$

Then $$V = A(xy^3 - \tfrac{3}{4}d^2xy).$$

The resultant stress across the section PQ is

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} p_x \,.\, bdy = 6Abx \int_{-\frac{d}{2}}^{\frac{d}{2}} ydy = 0.$$

The resultant couple is

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} p_x bydy = 6Ax \int_{-\frac{d}{2}}^{\frac{d}{2}} by^2dy = 6AIx,$$

where I is the moment of inertia of the section about an axis through its centre perpendicular to the xy plane. If this is to equal Wx, we must have $A = \frac{W}{6I}$. Then

$$V = \frac{\mathrm{W}}{6I}(xy^3 - \tfrac{3}{4}d^2xy)$$

and

$$q = -\frac{W^2y}{2I} + \frac{Wd^2}{8I} = -\frac{W}{2I}\left(y^2 - \frac{d^2}{4}\right)$$

Thus q vanishes when $y = \pm\frac{d}{2}$, as required (compare p. 202).

The last thing to check is the total shear over the section PQ; it is

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} q \,.\, bdy = -\frac{Wb}{2I}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(y^2 - \frac{d^2}{4}\right)dy = \frac{Wbd^3}{12I} = W,$$

which is right. Thus the solution

$$\left.\begin{aligned} V &= \frac{W}{6I}(xy^3 - \tfrac{3}{4}d^2xy) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ p_x &= \frac{Wxy}{I},\ q = -\frac{W}{2I}\left(y^2 - \frac{d^2}{4}\right) \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (1)$$

exactly fits the conditions of the problem, provided W is distributed over the end section according the same law as q, i.e. in a parabolic manner. This shows that, with this reservation with regard to the distribution of W, for beams of rectangular section, the solution afforded by the simple theory of bending gives the true values of the stresses. Therefore, in accordance with the principle of St. Venant, we can conclude that the stresses so calculated are extremely close approximation to the truth, for all cross sections except near the ends. We can, however, show that the assumption made in the simple theory that cross sections remain plane is not true in this case.

From (9), p. 418, we have

$$\frac{\partial u}{\partial x} = \frac{p_x}{E} - \frac{P_y}{mE} = \frac{Wxy}{EI}$$

$$\frac{\partial v}{\partial y} = \frac{p_y}{E} - \frac{p_x}{mE} = -\frac{Wxy}{mEI}$$

Integrating these equations gives

$$u = \frac{Wx^2y}{2EI} + f(y)$$

$$v = -\frac{Wxy^2}{2mEI} + \varphi(x),$$

where f and φ are functions to be determined. Now,

$$q = \frac{mE}{2(m+1)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{mE}{2(m+1)}\left(\frac{Wx^2}{2EI} - \frac{Wy^2}{2mEI} + \frac{df}{dy} + \frac{d\varphi}{dx}\right)$$

But above we had

$$q = -\frac{Wy^2}{2I} + \frac{Wd^2}{8I}.$$

Hence we must have

$$\frac{d\varphi}{dx} = -\frac{Wx^2}{2EI}$$

$$-\frac{Wy^2}{4(m+1)I} + \frac{mE}{2(m+1)}\cdot\frac{df}{dy} = -\frac{Wy^2}{2I} + \frac{Wd^2}{8I}.$$

The first of these gives $\varphi = -\frac{Wx^3}{6EI} + v_0$, where v_0 is the value of v when $x = y = 0$. The second gives

$$\frac{mE}{m+1}\cdot\frac{df}{dy} = -\frac{2m+1}{2(m+1)}\cdot\frac{Wy^2}{I} + \frac{Wd^2}{4I}$$

Hence

$$\frac{mE}{m+1}f = -\frac{2m+1}{6(m+1)}\cdot\frac{Wy^3}{I} + \frac{Wd^2y}{I} + \text{a constant.}$$

or

$$f = -\frac{2m+1}{6mEI}Wy^3 + \frac{m+1}{mEI}Wd^2y + u_0$$

where u_0 is the value of u when $x = y = 0$.

Thus

$$u = \left(\frac{Wx^2}{2EI} + \frac{m+1}{mEI}Wd^2\right)y - \frac{2m+1}{6mEI}Wy^3 + u_0 \quad . \quad . \quad (2)$$

which shows that u is not linear in y as it would be if the cross sections remained plane. The sections are distorted in the manner shown in Fig. 348.

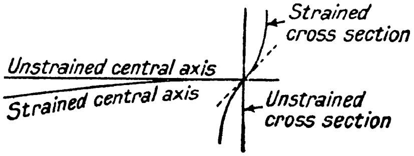

FIG. 348.

306. Narrow Cantilever of Rectangular Section with Uniformly Distributed Load.*—

Let b = the width of the section, and d the depth (Fig. 349).

,, w = the load per unit length, distributed uniformly over the top surface, so that the pressure per unit area, there, is w/b.

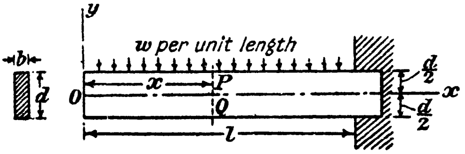

FIG. 349.

As before, take the origin on the centre line of the beam, at the free end. At any section PQ, distant x from the end, the shearing force is proportional to x, the bending moment is proportional to x^2, and the resultant tension is zero.

It is left to the reader to verify that the following is a solution of the problem.

$$V = \frac{w}{bd^3}\left(x^2y^3 - \frac{3d^2}{4}x^2y - \frac{d^3}{4}x^2 - \frac{y^5}{5} + \frac{d^2}{10}y^3\right) \quad . \quad . \quad (3)$$

giving

$$\left.\begin{aligned} p_x &= \frac{w}{bd^3}(6x^2y - 4y^3 + \tfrac{3}{5}d^2y) \quad . \quad . \quad . \\ p_y &= \frac{w}{2bd^3}(4y^3 - 3d^2y - d^3) \quad . \quad . \quad . \quad . \quad . \\ q &= \frac{6wx}{d^3}\left(y^2 - \frac{d^2}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

It will be noticed that p_x is not linear in y, as the simple theory of bending assumes, and it can be shown in the manner of § 305 that cross sections do not remain plane.

307. Solution of $\nabla^4 V = 0$ in Polar Co-ordinates.—It will be found on substitution that the equation is satisfied by the following value of V:—

$$V = (A_1r^n + B_1r^{n+2}) \cos n\theta,$$

where n may have any positive or negative value, A_1 and B_1 being constants. It will be convenient to establish at once the corresponding expressions for the stresses.

$$p_r = \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\cdot\frac{\partial V}{dr} \text{ (from (5), p. 415)}$$

$$= -n^2(A_1r^{n-2} + B_1r^n)\cos n\theta + \{nA_1r^{n-2} + (n+2)B_1r^n\}\cos n\theta$$

$$= -\{n(n-1)A_1r^{n-2} + (n+1)(n-2)B_1r^n\}\cos n\theta.$$

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[*] *Cf.* J. H. Michell, *Quart. Journal of Math.*, 1900; L. N. G. Filon, *Phil. Trans.*, Ser. A., 1903; and *Proc. R.S.*, 1904.

Similarly

$$p_\theta = \{n(n-1)A_1r^{n-2} + (n+1)(n+2)B_1r^n\}\cos n\theta$$
$$q = \{n(n-1)A_1r^{n-2} + (n+1)nB_1r^n\}\sin n\theta.$$

If we write

$$A_1n(n-1) = A \text{ and } B_1(n+1) = B,$$

the expressions for the stresses take the more simple forms :

$$\left.\begin{aligned}\mathbf{p_r} &= -\{\mathbf{Ar^{n-2} + (n-2)Br^n}\}\mathbf{\cos n\theta} \\ \mathbf{p_\theta} &= \{\mathbf{Ar^{n-2} + (n+2)Br^n}\}\mathbf{\cos n\theta} \\ \mathbf{q} &= \{\mathbf{Ar^{n-2} + nBr^n}\}\mathbf{\sin n\theta}\end{aligned}\right\} \quad (5)$$

Solutions of many interesting problems can be found by giving n different values.

Another system of stress can be formed by writing $\cos n\theta$ in place of $\sin n\theta$ and $-\sin n\theta$ in place of $\cos n\theta$.

In some problems the stresses are seen by inspection to be independent of θ, and the equation $\nabla^4 V = 0$ then reduces to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 V = 0, \quad \text{(i)}$$

which may be written

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} = 0 \quad \text{(ii)}$$

where

$$\Phi = \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} \quad \text{(iii)}$$

The solution of (ii) is

$$\Phi = A_1 + B_1 \log_e r,$$

where A_1 and B_1 are arbitrary constants. Hence (iii) becomes

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = A_1 + B_1 \log_e r$$

or

$$\frac{d}{dr}\left(r\frac{dV}{dr}\right) = A_1r + B_1r\log_e r$$

$$\therefore\ r\frac{dV}{dr} = \frac{A_1r^2}{2} + B_1\int r\log_e r \,.\, dr + C_1$$

$$= \left(\frac{A_1}{2} - \frac{B_1}{4}\right)r^2 + \frac{B_1r^2}{2}\log_e r + C_1$$

$$\therefore\ \frac{dV}{dr} = \left(\frac{A_1}{2} - \frac{B_1}{4}\right)r + \frac{B_1r}{2}\log_e r + \frac{C_1}{r}$$

$$\therefore\ V = \left(\frac{A_1}{4} - \frac{B_1}{4}\right)r^2 + \frac{B_1r^2}{4}\log_e r + C_1\log_e r + D_1.$$

By altering the constants we can write this

$$V = A + Br^2 + C\log_e r + Dr^2\log_e r \quad (6)$$

which is the general solution of (i).

The corresponding stress components are, from (5), p. 415,

$$\left.\begin{aligned} \mathbf{p_r} &= \mathbf{2B + D + \frac{C}{r^2} + 2D \log_e r} \\ \mathbf{p_\theta} &= \mathbf{2B + 3D - \frac{C}{r^2} + 2D \log_e r} \\ \mathbf{q} &= \mathbf{0.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

308. Thick Hollow Cylinder under Radial Pressures.*—If we take $n = 0$, we get from equations (6) above

$$p_r = -\frac{A}{r^2} + 2B \,; \quad p_\theta = \frac{A}{r^2} + 2B \,; \quad q = 0,$$

giving stresses which are symmetrical about the axis, with zero shear stress.

309. Incomplete Circular Plate with Terminal Couples.†—Let us now consider a beam of thin cross section, measured perpendicular to the plane of bending, the centre line being a circular arc, when the ends are acted on by couples M, as in Fig. 350. Let t be the thickness of the beam, and r_1, r_2 the radii of the bounding arcs. Evidently this is a case where the stress and strain are independent of θ, so that we shall use the forms of the stress components given by (7), above.

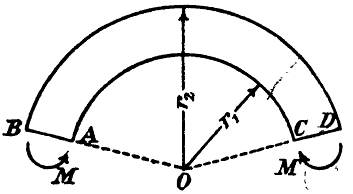

FIG. 350.

Since the arcs AC and BD are free from stress, we must have $p_r = 0$ when $r = r_1$ or r_2 ; hence :

$$2B + D + \frac{C}{{r_1}^2} + 2D \log_e r_1 = 0 \quad . \quad . \quad . \quad (\text{i})$$

$$2B + D + \frac{C}{{r_2}^2} + 2D \log_e r_2 = 0$$

Hence, by subtracting and rearranging, we get

$$C = -2D\frac{(\log_e r_1 - \log_e r_2){r_1}^2{r_2}^2}{{r_2}^2 - {r_1}^2} \quad . \quad . \quad . \quad (\text{ii})$$

Then, from (vi) substituting for C,

$$2B + D\left[1 - \frac{2{r_1}^2}{{r_2}^2 - {r_1}^2}\log_e r_1 + \frac{2{r_2}^2 \log_e r}{{r_2}^2 - {r_1}^2}\right] = 0. \quad (\text{iii})$$

Substituting for B and C in terms of D, from (ii) and (iii) in the first of (7) gives, after some simplification,

$$p_r = -\frac{2D({r_2}^2 \log_e r_2 - {r_1}^2 \log_e r_1)}{{r_2}^2 - {r_1}^2}\left[1 - \frac{({r_2}^2 - {r_1}^2)\log_e r + \dfrac{{r_1}^2{r_2}^2}{r^2}\log_e \dfrac{r_2}{r_1}}{{r_2}^2 \log_e r_2 - {r_1}^2 \log_e r_1} \;.\right]$$

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* This problem is treated independently below, p. 437.

† This result of § 309 was given by J. J. Guest, *Proc. Roy. Soc. A.*, 1918.

Denoting the quantity outside the [ ] by P, we have

$$p_r = -P\left[1 - \frac{(r_2^2 - r_1^2)\log_e r + \dfrac{r_1^2 r_2^2}{r^2}\log_e \dfrac{r_2}{r_1}}{r_2^2 \log_e r_2 - r_1^2 \log_e r_1}\right]$$

Similarly

$$p_\theta = -P\left[1 - \frac{(r_2^2 - r_1^2)(1 + \log_e r) - \dfrac{r_1^2 r_2^2}{r^2}\log_e \dfrac{r_2}{r_1}}{r_2^2 \log_e r_2 - r_1^2 \log_e r_1}\right]$$

whilst $q = 0$.

If we now evaluate $\int_{r_1}^{r_2} t p_\theta . dr$ we find the result is zero, so that there is no total tension on any cross section, whilst if we evaluate $\int t p_\theta r dr$ between the same limits we find the result is a couple, tending to increase the curvature, of magnitude

$$-\frac{Pt\left\{\left(r_1 r_2 \log_e \dfrac{r_2}{r_1}\right)^2 - \left(\dfrac{r_2^2 - r_1^2}{2}\right)^2\right\}}{r_2^2 \log_e r_2 - r_1^2 \log_e r_1}$$

where t is the thickness of the plate. If we denote this couple by M, the stresses at any point in the plate are given by

$$\left.\begin{aligned} p_r &= \frac{M}{t}\left[\frac{r_2^2 \log_e r_2 - r_1^2 \log_e r_1 - (r_2^2 - r_1^2)\log_e r - \dfrac{r_1^2 r_2^2}{r^2}\log_e \dfrac{r_2}{r_1}}{\left(r_1 r_2 \log_e \dfrac{r_2}{r_1}\right)^2 - \left(\dfrac{r_2^2 - r_1^2}{2}\right)^2}\right] \\ p_\theta &= \frac{M}{t}\left[\frac{r_2^2 \log_e r_2 - r_1^2 \log_e r_1 - (r_2^2 - r_1^2)(1 + \log_e r) + \dfrac{r_1^2 r_2^2}{r^2}\log_e \dfrac{r_2}{r_1}}{\left(r_1 r_2 \log_e \dfrac{r_2}{r_1}\right)^2 - \left(\dfrac{r_2^2 - r_1^2}{2}\right)^2}\right] \end{aligned}\right\} \quad (8)$$

$q = 0$.

It should be noted that p_θ tends to become very large as r_1 becomes very small. Prof. Inglis finds that when $r_2 = 7r_1$, the normal stress over any radial cross section is more than three times as great at the inner boundary as it is at the outer boundary, whilst the radial stress very nearly reaches the smaller of these two values at a point about one-sixth of the width from the inner arc.

310. Semi-Circular Plate Subjected to Terminal Shearing Forces.*—In Fig. 351 the beam AB is of thin rectangular section of thickness t. The tension over any radial cross section is proportional to $\sin\theta$, whilst the shearing force varies as $\cos\theta$.

In the general expressions (5) for the stresses we can write $\cos n\theta$

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* C. E. Inglis, *loc. cit.*, p. 423. See also an important paper by Coker and Haigh, *Proc. I.N.A.*, 1935.

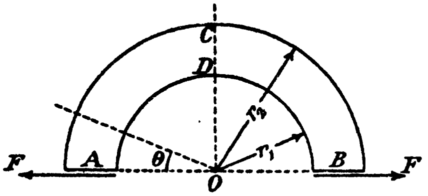

FIG. 351.

instead of $\sin n\theta$ and $-\sin n\theta$ instead of $\cos n\theta$, and so obtain another possible stress distribution:

$$\begin{aligned} p_r &= \{Ar^{n-2} + (n-2)Br^n\}\sin n\theta \\ p_\theta &= -\{Ar^{n-2} + (n+2)Br^n\}\sin n\theta \\ q &= \{Ar^{n-2} + nBr^n\}\cos n\theta \end{aligned}$$

From these we can derive two possible stress distributions by taking $n = 1$ and -1 in turn: we get

$$n = 1 \left\{\begin{aligned} p_r &= \left(\frac{A_1}{r} - B_1 r\right)\sin\theta \quad . \quad . \quad . \\ p_\theta &= -\left(\frac{A_1}{r} + 3B_1 r\right)\sin\theta \quad . \quad . \\ q &= \left(\frac{A_1}{r} + B_1 r\right)\cos\theta \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (9)$$

and

$$n = -1 \left\{\begin{aligned} p_r &= -\left(\frac{A_2}{r^3} - \frac{3B_2}{r}\right)\sin\theta \quad . \quad . \quad . \\ p_\theta &= \left(\frac{A_2}{r^3} + \frac{B_2}{r}\right)\sin\theta \quad . \quad . \quad . \quad . \quad . \\ q &= \left(\frac{A_2}{r^3} - \frac{B_2}{r}\right)\cos\theta \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad (10)$$

In the second of these systems take $B_2 = A_1$ and we get

$$\left.\begin{aligned} p_r &= -\left(\frac{A_2}{r^3} - \frac{3A_1}{r}\right)\sin\theta \quad . \quad . \\ p_\theta &= \left(\frac{A_2}{r^3} + \frac{A_1}{r}\right)\sin\theta \quad . \quad . \quad . \quad . \\ q &= \left(\frac{A_2}{r^3} - \frac{A_1}{r}\right)\cos\theta \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

Subtract (11) from (9) and we have as a possible stress system

$$\left.\begin{aligned} p_r &= \left(-B_1 r - \frac{2A_1}{r} + \frac{A_2}{r^3}\right)\sin\theta \quad . \quad . \\ p_\theta &= \left(-3B_1 r - \frac{2A_1}{r} - \frac{A_2}{r^3}\right)\sin\theta \quad . \quad . \\ q &= \left(B_1 r + \frac{2A_1}{r} - \frac{A_2}{r^3}\right)\cos\theta \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (12)$$

We will write these

$$\left.\begin{aligned} p_r &= \left(Ar + \frac{B}{r} - \frac{C}{r^3}\right)\sin\theta \quad . \quad . \quad . \\ p_\theta &= \left(3Ar + \frac{B}{r} + \frac{C}{r^3}\right)\sin\theta \quad . \quad . \quad . \\ q &= -\left(Ar + \frac{B}{r} - \frac{C}{r^3}\right)\cos\theta \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (13)$$

In this system, when $\theta = 0$ we have only shear, which fits our case. Also, since p_r and q are the same functions of r, we can make both vanish simultaneously at the two bounding arcs, $r = r_1$ and r_2. Using these conditions we find that p_r, p_θ and q can be written:

$$\left.\begin{aligned} p_r &= P\left(-r + \frac{r_1^2 + r_2^2}{r} - \frac{r_1^2 r_2^2}{r^3}\right)\sin\theta \\ p_\theta &= P\left(-3r + \frac{r_1^2 + r_2^2}{r} + \frac{r_1^2 r_2^2}{r^3}\right)\sin\theta \\ q &= P\left(r - \frac{r_1^2 + r_2^2}{r} + \frac{r_1^2 r_2^2}{r^3}\right)\cos\theta \end{aligned}\right\}$$

If we integrate these over a cross section we find that the total tension and shear on a radial section are:

$$Pt\left\{(r_2^2 - r_1^2) - (r_2^2 + r_1^2)\log_e\frac{r_2}{r_1}\right\}\cos\theta$$

and

$$Pt\left\{(r_2^2 + r_1^2)\log_e\frac{r_2}{r_1} - (r_2^2 - r_1^2)\right\}\sin\theta.$$

Therefore in our case the stresses will be given by

$$\left.\begin{aligned} p_r &= \frac{F}{t}\cdot\frac{-r + \dfrac{r_1^2 + r_2^2}{r} - \dfrac{r_1^2 r_2^2}{r^3}}{(r_1^2 + r_2^2)\log_e\dfrac{r_2}{r_1} - (r_2^2 - r_1^2)}\,.\sin\theta \\ p_\theta &= \frac{F}{t}\cdot\frac{-3r + \dfrac{r_1^2 + r_2^2}{r} + \dfrac{r_1^2 r_2^2}{r^3}}{(r_1^2 + r_2^2)\log_e\dfrac{r_2}{r_1} - (r_2^2 - r_1^2)}\,.\sin\theta \\ q &= \frac{F}{t}\cdot\frac{r - \dfrac{r_1^2 + r_2^2}{r} + \dfrac{r_1^2 r_2^2}{r^3}}{(r_1^2 + r_2^2)\log_e\dfrac{r_2}{r_1} - (r_2^2 - r_1^2)}\,.\cos\theta \end{aligned}\right\} \quad (14)$$

Where $r_2 = 7r_1$, the normal stress on the cross section DC (Fig. 351) is found to be seven times as great at D as it is at C, whilst the maximum radial stress is slightly greater than the smaller of these.

By combining the cases just considered the general case of such a beam bent by any concentrated loads can be analysed.*

Example.†—A flat plate cut in the form shown in Fig. 352 is acted on by the forces W, $\frac{1}{2}W$, $\frac{1}{2}W$. It is required to investigate the stresses on the cross sections *AB*, *AC*, *DE*, *DF*.

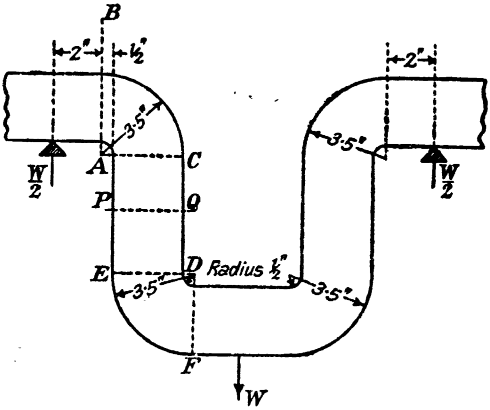

FIG. 352.

The section *AB* is acted on by a shearing force $\frac{1}{2}W$, and a bending moment $-\frac{W}{2} \times 2 = -W$. For estimating the effects of the former the part of the plate between *AB* and *AC* may be treated as half the semi-circular plate of § 309, and for estimating the effects of the latter the results of § 310 can be applied.

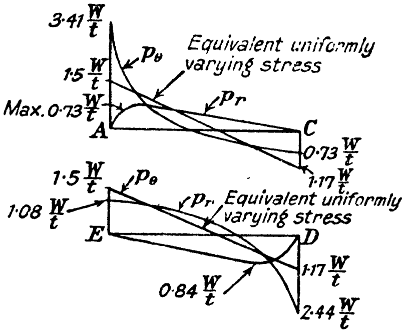

FIG. 353.

The section *DF* is subjected to a shearing force $\frac{W}{2}$ combined with a bending moment $\frac{W}{2} \times 6 = 3W$.

The stresses for the sections *DE* and *AC* are shown in Fig. 353. It will be noticed that they are very different from those which would be found by the ordinary simple theory of bending. Thus we should, in applying the simple theory, regard the section AC as acted on by a bending moment $2W$ and a tension $\frac{1}{4}W$; the former would produce maximum normal stresses on the section equal to $\pm\frac{4}{3}\cdot\frac{W}{t}$, whilst the latter would give a normal stress $\frac{1}{6}\cdot\frac{W}{t}$, so that the normal stresses at A and C would be $\frac{3}{2}\times\frac{W}{t}$ and $-\frac{7}{6}\times\frac{W}{t}$. Actually the maximum tensile stress is $3{\cdot}41\frac{W}{t}$, or 2·28 times what the simple theory would have it. This result should be carefully noticed, for it will be seen that the problem here treated has a strong resemblance to the problem of a crankshaft. The figures given by the simple theory will be more nearly approached on the section PQ, between AC and ED. The theory can also be applied to calculate the stresses due to the bent **T** or **I** sections.

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* For stress in plates with holes, i.e. in multiple connected regions, see a paper by L. N. G. Filon, *Brit. Assoc. Report*, 1921.

† Taken from the paper by Prof. Inglis (see footnote, p. 423).

EXAMPLES XXVII

1. A force P is applied at a point in the edge of a flat plate; the line of action of the force makes angles a and $\pi - a$ with the edge of the plate and is in the plane of the plate. Show that, within the limits $a > \theta > -(\pi - a)$, the stress distribution is given by $p_r = (2P/\pi r)\cos\theta$.* (Michell.)

2. A flat plate is bounded at one corner by two straight lines OA and OB inclined at an angle γ. A force P acts at O in the plane of the plate in a direction making an angle φ with OA. The direction of the maximum radial stress makes an angle a with OA. Show that a is given by

$$\tan a = \frac{\gamma\sin\varphi - \sin\gamma\sin(\gamma-\varphi)}{\gamma\cos\varphi - \sin\gamma\cos(\gamma-\varphi)}$$

Show also that the condition that the radial stress shall be one-signed is approximately that the line of action of P lies within the middle third of the angle γ, provided $\gamma<\pi/2$. (Michell, *loc. cit.*, below.)

3. A tapering flat plate as shown in Fig. 354 is acted on by couples M acting in its plane and applied to its ends. Show that the stresses are given by †

$$p_r = -\frac{2M\sin 2\theta}{t(\sin 2a - 2a\cos 2a)r^2},$$

$$p_\theta = 0;\quad q = \frac{M(\cos 2\theta - \cos 2a)}{t(\sin 2a - 2a\cos 2a)r^2}.$$

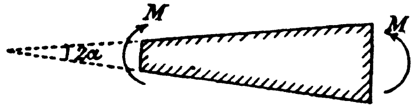

FIG. 354.

4. A semi-infinite flat plate is bounded on one side by a straight line. A thrust w per unit length acts, in the plane of the plate, at right angles to the straight edge, over a length $AB = 2a$. The position of any point

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* The effects of a pressure uniformly distributed along a finite length of a straight edged plate can be found by integration (Michell, *Proc. London Math. Soc.*, 1902).

† C. E. Inglis.

P in the plate is defined by the angles $PAB = \theta$ and $PBA = \varphi$. The thickness of the plate is $2h$. Prove that the stress components at P are given by

$$p_x = -\frac{w}{2\pi h}\{\pi - \theta - \varphi + \sin(\theta + \varphi)\cos(\theta - \varphi)\}$$

$$p_y = -\frac{w}{2\pi h}\{\pi - \theta - \varphi - \sin(\theta + \varphi)\cos(\theta - \varphi)\}$$

$$q = \frac{w}{2\pi h}\sin(\theta + \varphi)\sin(\varphi - \theta)$$

the origin being at the middle of AB, which is taken as the axis of y. (Michell.)

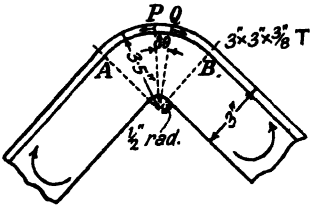

FIG. 354a.

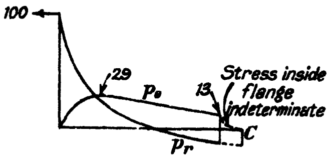

FIG. 354b.

5. Fig. 354a shows a bent T section which is subjected to bending moments as indicated by the arrows. Assuming that the stress in the flange is constant and equal to the stress p_θ at the top of the web, show that the stress distribution over a radial cross section between A and B is given approximately by Fig. 354b, where the stresses are expressed in terms of the maximum hoop stress in the web.

CHAPTER XXVIII

THICK CYLINDRICAL AND SPHERICAL SHELLS

311. Thick Cylindrical Shell under Radial Pressures.*

—We shall consider the case of a thick, hollow, circular cylinder subjected to fluid pressure on its inner and outer surfaces (Fig. 355).

Let r_1 = the inside radius
r_2 = „ outside „
p_1 = „ internal pressure per unit area.
p_2 = „ external pressure „ „

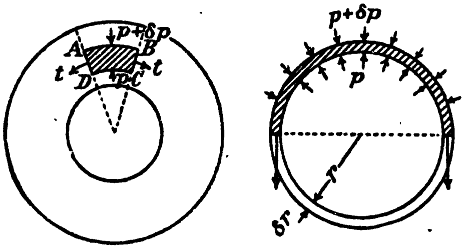

FIG. 355.

Symmetry shows that the stresses and strains are independent of θ, the angular position of any radius. Consider an element *ABCD*: the faces *AB* and *CD* will be under a radial pressure, whilst the faces *AD* and *BC* will experience a "hoop" tension.

Let p = the radial compressive stress at radius r
„ t = „ hoop tensile stress at radius r.

Consider now an annular ring of the tube, of internal radius r and external radius $r + \delta r$. The radial pressure on the inner surface will be p, and on the outer surface $p + \delta p$.

The nett pressure tending to burst the ring across any diameter (Fig. 355) is $2pr - 2(p + \delta p)(r + \delta r)$. This is balanced by a tensile stress of total amount $2t.\delta r$, both per unit axial length of tube. Hence

$$2pr - 2(p + \delta p)(r + \delta r) = 2t\delta r.$$

Multiplying out and neglecting products of small quantities this gives

$$p\delta r + r\delta p + t\delta r = 0$$

---

* Pipes with eccentric bore have been dealt with by Jeffery. See *Brit. Assoc. Report*, 1921.

Hence, in the limit

$$p + r\frac{dp}{dr} + t = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

We shall now assume that plane cross sections of the tube, perpendicular to the axis, remain plane and perpendicular to the axis, and that all cross sections suffer the same distortion. This implies that the strain parallel to the axis is either zero or constant, and must be very nearly true for sections remote from the ends.

Let e_z and p_z be the longitudinal strain and stress respectively, positive when tensile, then

$$e_z = \frac{1}{E}\left(p_z + \frac{p - t}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Let u = the radial displacement at distance r from the axis, then as on p. 420.

$$\frac{du}{dr} = -\frac{1}{E}\left(p + \frac{p_z + t}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

$$\frac{u}{r} = \frac{1}{E}\left(t - \frac{p_z - p}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

Multiply (ii) by r, differentiate and subtract from (i): we get

$$(p + t)\left(1 + \frac{1}{m}\right) + \frac{r}{m}\frac{dp}{dr} + r\frac{dt}{dr} = \frac{r}{m}\frac{dp}{dr} \, . = \frac{-r}{m^2}\frac{d}{dr}(p - t),$$

from (2) since e_z is constant. But from (1) $p + t = -r\frac{dp}{dr}$. Hence we get

$$r\frac{dt}{dr} - \left(1 + \frac{1}{m}\right)r\frac{dp}{dr} + \frac{r}{m}\frac{dp}{dr} = \frac{r}{m^2}\frac{d}{dr}(p - t), \text{ or } \left(1 - \frac{1}{m^2}\right)\frac{d}{dr}(p - t) = 0.$$

Thus $p-t$ is constant, and it follows from (2) above that p_z must be constant.

Let $p - t = 2a$ . . . . (iii)
Then from (1) we have

$$r\frac{dp}{dr} = 2a - 2p.$$

$$\therefore \frac{dp}{a - p} = \frac{2dr}{r}$$

$$\therefore -\log_e(a - p) = 2\log_e r + \text{const.}$$

$$\log_e[r^2(a - p)] = \text{const.}$$

$$\therefore r^2(a - p) = \text{const} = \beta, \text{ say.}$$

Hence we can write

$$\mathbf{p} = \alpha - \frac{\beta}{\mathbf{r}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Then it follows from (i) that

$$\mathbf{t} = -\alpha - \frac{\beta}{\mathbf{r}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At the outer surface we must have $p = p_2$ and at the inner surface $p = p_1$. Therefore, from (3)

$$p_1 = a - \frac{\beta}{r_1^2} \text{ and } p_2 = a - \frac{\beta}{r_2^2}.$$

These equations give

$$\left.\begin{aligned} a &= \frac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} \quad . \quad . \quad . \\ \beta &= \frac{(p_2 - p_1) r_1^2 r_2^2}{r_2^2 - r_1^2} \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

Substituting in (3) and (4) we get finally:

$$\left.\begin{aligned} \mathbf{p} &= \frac{1}{\mathbf{r}_2^2 - \mathbf{r}_1^2}\left[\mathbf{p}_2\mathbf{r}_2^2 - \mathbf{p}_1\mathbf{r}_1^2 + \frac{(\mathbf{p}_1 - \mathbf{p}_2)\mathbf{r}_1^2\mathbf{r}_2^2}{\mathbf{r}^2}\right] \quad . \quad . \quad . \\ \mathbf{t} &= \frac{1}{\mathbf{r}_2^2 - \mathbf{r}_1^2}\left[\mathbf{p}_1\mathbf{r}_1^2 - \mathbf{p}_2\mathbf{r}_2^2 + \frac{(\mathbf{p}_1 - \mathbf{p}_2)\mathbf{r}_1^2\mathbf{r}_2^2}{\mathbf{r}^2}\right] \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad (6)$$

312. External Pressure Negligible.—The most important practical applications of the above formulæ occur when the external pressure is atmospheric and negligible compared with the internal pressure. In such cases we can put $p_2 = 0$, and the formulæ become

$$p = \frac{p_1 r_1^2}{r_2^2 - r_1^2} \cdot \frac{r_2^2 - r^2}{r^2}; \quad t = \frac{p_1 r_1^2}{r_2^2 - r_1^2} \cdot \frac{r_2^2 + r^2}{r^2} \quad . \quad . \quad (7)$$

Both p and t have their maximum values at the inner surface of the tube, where $r = r_1$:

$$p = p_1; \quad t_1 = \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} p_1 . = \frac{k^2 + 1}{k^2 - 1} p_1, \quad . \quad . \quad . \quad (8)$$

where $k = r_2/r_1$.

Let us consider what will be the maximum permissible internal pressure, according to the various theories set out in Chapter VI.

Let f = the elastic limit of the material in simple tension. Then :—

(i) MAXIMUM PRINCIPAL STRESS THEORY.—The elastic limit will be reached when $t = f$, the elastic limit in simple tension, that is when

$$\frac{p_1}{f} = \frac{k^2 - 1}{k^2 + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

(ii) MAXIMUM SHEAR STRESS THEORY.—The principal stresses are p and t, which are compressive and tensile respectively, so that failure will occur when (p. 91) $p + t = f$; this gives, from (8),

$$\frac{\mathbf{p}_1}{\mathbf{f}} = \frac{\mathbf{k}^2 - 1}{2\mathbf{k}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(iii) STRAIN ENERGY THEORY.—In this case we must have * (p. 92)

$$p^2 + t^2 + \frac{2}{m}pt = f^2,$$

which gives

$$\frac{p_1}{f} = \frac{k^2 - 1}{\sqrt{2\frac{m+1}{m}k^4 + 2\frac{m-1}{m}}} \quad . \quad . \quad . \quad . \quad (11)$$

An important series of experiments was conducted by Messrs. Cook and Robertson † on thick-walled tubes of cast iron and mild steel, the results of which are summarized in Figs. 356 and 357. The experimental points are marked with small circles, and the theoretical values of p_1/f have been plotted against k according to the above three theories. In the case of the ductile material, mild steel, the strain-energy theory comes very close to the mark, whilst in the case of the more brittle cast iron the maximum stress theory fits the facts best. In this latter case f has been taken as the breaking strength of the cast iron. Strictly speaking these experiments belong to the class of three-dimensional

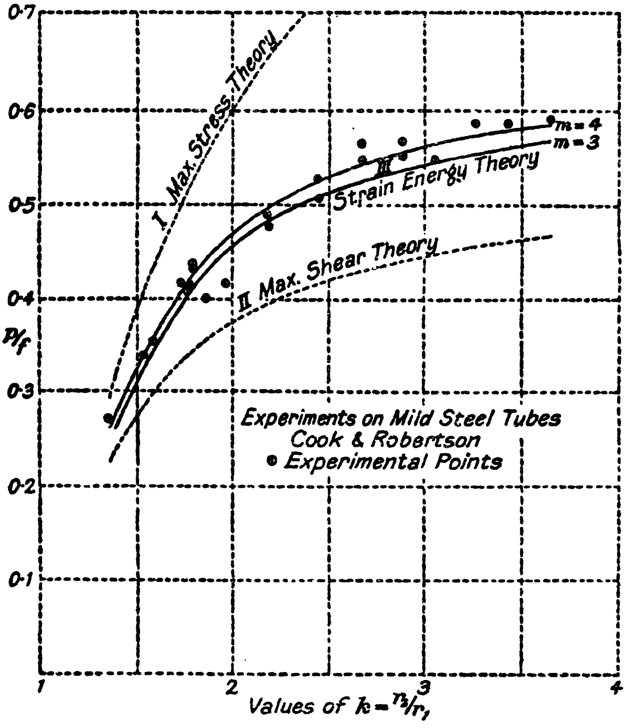

FIG. 356.

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* Neglecting the longitudinal stress.

† *Engineering*, Dec. 15, 1911. Further experiments on thick tubes are described in a paper to the Physical Society by G. A. Wedgwood, May 10, 1929, and the Brit. Ass. report on stresses in overstrained materials (1931).

stresses, but the longitudinal stress is small, and they afford the strongest confirmation of Haigh's theory for ductile materials (*cf.* Chapter VI).

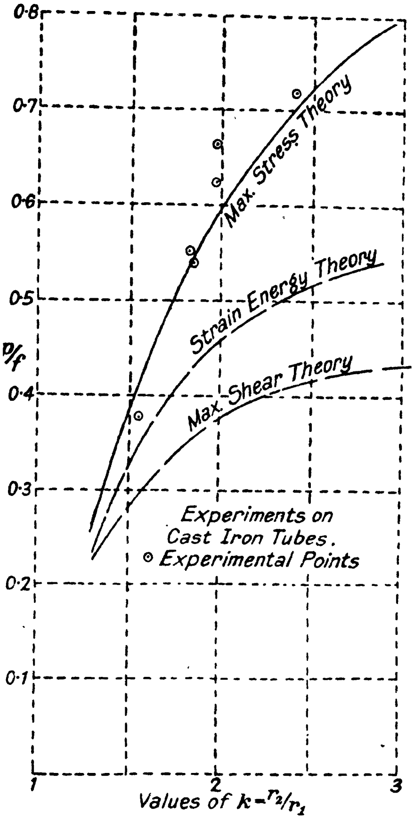

FIG. 357.

Adopting the maximum stress theory for brittle materials we have

$$\frac{k^2-1}{k^2+1}=\frac{p_1}{f}$$

$$\therefore\ k^2=\frac{f+p_1}{f-p_1} \qquad \ldots\ldots\ldots \quad (12)$$

Therefore if $p_1 = f$, k will have to be infinite, which means that we cannot make a cast-iron tube sufficiently thick to withstand an internal pressure equal to the breaking stress of the material.

For ductile materials like mild steel we have from (11) above

$$\frac{(k^2-1)^2}{2\left(\frac{m+1}{m}k^4+\frac{m-1}{m}\right)}=\frac{p_1^2}{f^2}$$

or $$\left(1-\frac{2p_1^2}{f^2}\cdot\frac{m+1}{m}\right)k^4-2k^2+\left(1-\frac{2p_1^2}{f^2}\frac{m-1}{m}\right)=0.$$

Solving this equation for k^2 we get

$$k^2 = \frac{1 \pm 2\lambda\sqrt{1 - \dfrac{m^2 - 1}{m^2}\lambda^2}}{1 - 2\lambda^2\dfrac{m + 1}{m}} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where λ stands for p_1/f. Now $\dfrac{m^2 - 1}{m^2}$ is < 1, and clearly λ must * be < 1, so that the quantity under the radical in the numerator is positive. Taking the upper sign we see the condition that k^2 may be real is

$$\lambda^2 < \frac{m}{2(m + 1)}$$

or

$$\frac{p_1}{f} < \sqrt{\frac{m}{2(m + 1)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If $m = 4$ this shows that we cannot make a steel tube to take an internal pressure greater than $0{\cdot}633f$, and if $m = 3$ we must have

$$p_1 < 0{\cdot}613f.$$

313. Longitudinal Stress.—We have from (2) of § 311,

$$e_s = \frac{p_s}{E} + \frac{p - t}{mE}.$$

If the ends are constrained in such a way that e_s is zero, we must have

$$P_s = -\frac{(p - t)}{m} = \frac{2(p_1 r_1^2 - p_2 r_2^2)}{m(r_2^2 - r_1^2)} \quad . \quad . \quad . \quad . \quad (15)$$

If one end is free so that there is no longitudinal stress, then

$$e_s = \frac{1}{mE}(p - t) = \frac{2(p_2 r_2^2 - p_1 r_1^2)}{mE(r_2^2 - r_1^2)} \quad . \quad . \quad . \quad . \quad (16)$$

Example 1.—The cylinder of an hydraulic ram is 8″ diameter inside. The factor of safety is to be 2, the internal pressure 2·5 tons/in.² The material is steel having an elastic limit of 20 tons/in.² in simple tension. Calculate the outside diameter, (i) on the maximum shear-stress theory, (ii) on the strain-energy theory, taking $m = \dfrac{10}{3}$.

(i) We must have $p + t \not> 10$ tons/in.², i.e. we take $f = 10$ tons/in².
From (10) we have

$$2{\cdot}5 = \frac{1}{2}\left(1 - \frac{1}{k^2}\right)10$$

$$1 - \frac{1}{k^2} = \frac{1}{2}$$

$$\therefore\ k = \sqrt{2}$$

$$r_1 = 4'', \ \therefore\ r_2 = 4\sqrt{2} = 5{\cdot}66'', \text{ say } 5{\cdot}7''.$$

---

* Since m lies between 3 and 4 for most metals, k^2 would be negative and k imaginary if $\lambda \not< 1$.

(ii) From (11) we have

$$2{\cdot}5 = \frac{\frac{1}{\sqrt{2}}(k^2 - 1).10}{\sqrt{\frac{13}{10}k^4 + \frac{7}{10}}}$$

$$\begin{aligned} k^4 - 2{\cdot}39k^2 + 1{\cdot}09 &= 0 \\ k^2 &= 1{\cdot}78 \text{ or } 0{\cdot}615 \\ k &= 1{\cdot}336 \text{ or } 0{\cdot}785 \end{aligned}$$

A value of k less than unity is obviously inadmissible, so we take $k = 1{\cdot}336$, which gives $r_2 = 5{\cdot}35''$.

Example 2.—Calculate the initial difference of diameter for a tube of external diameter 12″ to be shrunk on to a solid shaft 8″ diameter so that the pressure between the tube and the shaft is 2 tons/in.² Take $E = 13{,}700$ tons/in.², $m = \frac{10}{3}$. (Mech. Sc. Trip., 1920.)

The problem is to find the initial difference between the inside diameter of the tube and the outside diameter of the shaft. Since the difference will be very small we can take both diameters as 8″ for the purpose of finding the stresses. We shall here neglect the compressibility of the shaft (cf. p. 449.)

For the tube we have

$$p_1 = 2 \text{ tons/in.}^2; \; r_1 = 4''; \; p_2 = 0; \; r_2 = 6''.$$

Hence, from (6), p. 439, at the inside we have

$$\begin{aligned} t &= \frac{1}{36 - 16}\left[32 + \frac{2 \times 16 \times 36}{16}\right] \\ &= \frac{104}{20} = 5{\cdot}2 \text{ tons/in.}^2 \end{aligned}$$

Hence the inside fibres of the tube will be under a tension of 5·2 tons/in.² and a pressure of 2 tons/in.² Therefore their strain will be,

$$e = \frac{5{\cdot}2}{E} + \frac{2}{mE} = \frac{1}{13{,}700}(5{\cdot}2 + 0{\cdot}6) = \frac{5{\cdot}8}{13{,}700}$$

This strain is neutralized by the shaft and the radius is changed. Suppose the unstrained radius is r_0. Then we must have

$$r_0(1 + e) = 4''$$

$$\therefore\; 4'' - r_0 = r_0 e = 4e\,{}^* = \frac{4 \times 5{\cdot}8}{13{,}700} = 0{\cdot}0017.''$$

This is the initial difference of radius which must exist. The difference of diameter will be 0·0034″.

Example 3.—A gun tube B of external diameter 40″ is to be shrunk on an inner tube A of external diameter 32″ and internal diameter 16″; the pressure between the two tubes is required to be 5 tons/in.² Calculate the proper initial difference between the inner radius of B and the outer radius of A, if $E = 13{,}000$ tons/in.² and $m = \frac{10}{3}$. (H.M. Dockyard Schools, 1924.)

For A we have

$$\begin{aligned} &p_1 = 0, \; p_2 = 5 \text{ tons/in.}^2 \\ &r_1 = 8'', \; r_2 = 16''. \end{aligned}$$

---

[*] See p. 3, § 5.

Hence, at the outside, we have

$$t = -\frac{p_2(r_1^2 + r_2^2)}{r_2^2 - r_1^2} = -\frac{5 \times 320}{192} = -8{\cdot}35 \text{ tons/in.}^2$$

$$p = 5 \text{ tons/in.}^2$$

Therefore the circumferential strain at the outside of A is

$$\frac{1}{E}\left(-8{\cdot}35 + \frac{5 \times 3}{10}\right) = -\frac{6{\cdot}85}{E} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

For B we have

$$p_1 = 5 \text{ tons/in.}^2; \; p_2 = 0$$

$$r_1 = 16'', \; r_2 = 20''.$$

Hence, at the inside, we have

$$t = \frac{p_1(r_2^2 + r_1^2)}{r_2^2 - r_1^2} = \frac{5 \times 656}{144} = 22{\cdot}8 \text{ tons/in.}^2$$

$$p = 5 \text{ tons/in.}^2$$

Therefore the circumferential strain at the inside of B is

$$\frac{1}{E}\left(22{\cdot}8 + \frac{5 \times 3}{10}\right) = \frac{24{\cdot}3}{E}.$$

If B were taken off, the outer radius of A would increase by $\dfrac{16'' \times 6{\cdot}85}{E}$ and the inside radius of B would decrease by $\dfrac{16'' \times 24{\cdot}3}{E}$.

Therefore the initial difference of radius should be

$$\frac{16 \times 31{\cdot}15}{E} = \frac{16 \times 31{\cdot}15}{13{,}000} = 0{\cdot}038'' \text{ nearly.}$$

314. Compound Tubes.*—It will be seen from the formulæ (6) and (7), p. 439, that the stresses decrease rapidly as the radius increases. For instance, if $r_2/r_1 = 2$, the hoop stress t, when outside pressure is zero, will be two-and-a-half times as great on the inner surface as it is on the outer. Thus, the material is not used economically. Again, it was pointed out in § 312 that, for a given strength of material, there is a limit to the internal pressure which a tube can stand, however thick the walls are made. For these reasons, various methods have been devised for strengthening thick tubes, one of which is to form the tube by shrinking one tube on the outside of another. Then the inner tube is subjected to external pressure by the cooling of the outer one, which itself is subjected to an equal internal pressure. It will be seen from Example 3 above that the hoop stress of the inner tube is negative, i.e. is compressive, whilst for the outer one it is positive, i.e. tensile. If now the inside of the compound tube be subjected to fluid pressure a tensile hoop stress will be superimposed throughout the material. As the inner tube is initially in a state of compression, the final tensile stress set up will not be so great as if the tube were initially free from stress. Similarly, since the outer tube is initially in a state of tension, the final tensile stress will be greater than if the tube had been initially unstressed. In this way the stresses are more or less equalized throughout the walls of the compound tube, particularly if three or four tubes are shrunk on to each other, and thus the tube can take a greater fluid pressure.

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* For a paper on the longitudinal strength of guns, see *Engineering*, Feb. 16, 1917.

Nothing is gained by giving a general analysis of this, and we shall content ourselves with working out an illustrative example.

Example.—A gun tube is built up of four tubes,* as shown in Fig. 358; it is required to calculate the shrinkage pressures necessary between the tubes, for an explosion pressure of 8·25 tons/in.², the factor of safety being 3, and the elastic limit of the material being 30 tons/in.² in simple tension.

(i) We shall first work the problem on the maximum shear-stress theory. We must have $p + t \not> 10$ tons/in.²

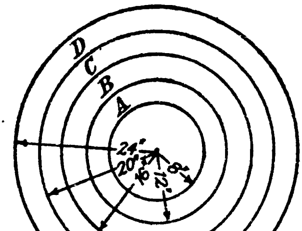

FIG. 358.

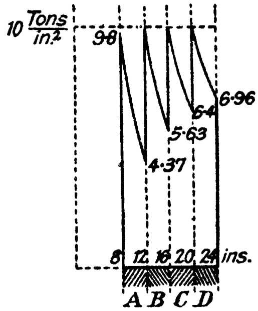

FIG. 359.

For the explosion pressure, treating the whole tube as solid, we have with $p_2 = 0$,

$$p + t = \frac{2p_1 r_1^2 r_2^2}{(r_2^2 - r_1^2)r^2}$$

where $p_1 = 8\cdot25$ tons/in.²; $r_1 = 8''$, $r_2 = 24''$. Hence

$$p + t = \frac{16\cdot5 \times 64 \times 576}{512 \times r^2} = \frac{1{,}190}{r^2} \quad . \quad . \quad . \quad \text{(i)}$$

We now commence filling in the table shown.

	1	2	3	4	5	6
	r (ins.).	r^2	$p + t$ due to explosion.	$p + t$ due to shrinking.	Radial pressures due to shrinking.	Actual values of $p + t$
Inside of A .	8	64	18·5	— 8·70	⑥ 0	9·80
Outside of A .	12	144	8·25	— 3·88	2·422	4·37
Inside of B .	12	144	8·25	1·75	2·422	⑦ 10·00
Outside of B .	16	256	4·65	0·98	2·040	5·63
Inside of C .	16	256	4·65	5·35	2·040	10·00
Outside of C .	20	400	2·97	3·43	1·075	6·40
Inside of D .	20	400	2·97	7·03	1·075	10·00
Outside of D .	24	576	2·06	4·9	0	6·96
				④	⑤	

Stresses are given in tons/in.²

Columns 1 and 2 are filled in from the data, and column 3 from equation (i).

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* These figures are purely fictitious.

Next find the initial stresses, due to shrinking, required to make $p + t = 10$ tons/in.^{2} at the insides of B, C, D and A. These are the figures underlined in column 4. We now have to find the radial pressures between the tubes, necessary to produce these values of $p + t$. At the inside of any tube we have

$$\boxed{p + t = \frac{2(p_1 - p_2)r_2^2}{r_2^2 - r_1^2}} \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

Thus for D, we have $p_2 = 0, p + t = 7{\cdot}03$ tons/in.^{2}, $r_2 = 24''$ and $r_1 = 20''$. Therefore

$$7{\cdot}03 = \frac{2p_1 \times 576}{176}.$$

Hence for D

$$p_1 = \frac{7{\cdot}03 \times 176}{1{,}152} = 1{\cdot}075 \text{ tons/in.}^2 \quad . \quad . \qquad \text{(iii)}$$

Then, for C we have

$$p_2 = 1{\cdot}075,\ p + t = 5{\cdot}35,\ r_2 = 20'',\ r_1 = 16''.$$

Hence, from (ii)

$$5{\cdot}35 = \frac{2(p_1 - 1{\cdot}075)400}{144}$$

$$\therefore\ p_1 - 1{\cdot}075 = \frac{5{\cdot}35 \times 144}{800} = 0{\cdot}965 \text{ tons/in.}^2$$

$$p_1 = 2{\cdot}040 \text{ tons/in.}^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

For B:

$$1{\cdot}75 = \frac{2(p_1 - 2{\cdot}040)256}{112}$$

which gives

$$p_1 = 2{\cdot}422 \text{ tons/in.}^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

The values of p_1 given by (iii), (iv) and (v) are now inserted in column 5 of the table.

From column 5 we can complete column 4:

At the outside of any tube

$$\boxed{p + t = \frac{2(p_1 - p_2)r_1^2}{r_2^2 - r_1^2}} \quad . \quad . \quad . \quad . \qquad \text{(vi)}$$

We have then

At outside of D, ($p_2 = 0$, $p_1 = 1{\cdot}075$, $r_2 = 24''$, $r_1 = 20''$)

$$p + t = \frac{2{\cdot}150 \quad 400}{176} = 4{\cdot}9 \text{ tons/in.}^2 \quad . \quad . \qquad \text{(vii)}$$

At outside of C, ($p_2 = 1{\cdot}075$, $p_1 = 2{\cdot}040$, $r_2 = 20''$, $r_1 = 16''$)

$$p + t = \frac{2 \times 0{\cdot}965 \times 256}{144} = 3{\cdot}43 \text{ tons/in.}^2 \quad . \qquad \text{(viii)}$$

At outside of B, ($p_2 = 2{\cdot}040$, $p_1 = 2{\cdot}422$, $r_2 = 16''$, $r_1 = 12''$)

$$p + t = \frac{2 \times 0{\cdot}382 \times 144}{112} = 0{\cdot}98 \text{ tons/in.}^2 \quad . \qquad \text{(ix)}$$

At outside of A, ($p_2 = 2{\cdot}422$, $p_1 = 0$, $r_2 = 12''$, $r_1 = 8''$).

$$p + t = -\frac{2 \times 2{\cdot}422 \times 64}{80} = -3{\cdot}88 \text{ tons/in.}^2\ . \qquad \text{(x)}$$

At inside of A we have from (ii)

$$p + t = -\frac{2 \times 2{\cdot}422 \times 144}{80} = -8{\cdot}70 \text{ tons/in.}^2 \qquad \text{(xi)}$$

The figures (vii), (viii), (ix), (x), (xi) are now inserted in column 4.

Adding together columns 4 and 5 we get the final stresses as shown in column 6. It will be seen that by building up the tube we have practically halved the stress at the bore, and have rendered possible what would have been impossible in the case of a solid tube, namely to withstand a pressure of 8¼ tons/in.^{2} without $p + t$ exceeding 10 tons/in.^{2} The stress distribution is shown in Fig. 359.

(ii) We shall treat the same problem on the principle of maximum strain-energy, and the work will be rather more tedious and not so direct, as we must work the stresses separately.

Here we have to make

$$F(p, t) = p^2 + t^2 + 0{\cdot}6pt \not> 10^2.$$

As before we first find the stresses due to the explosion pressure, treating the tube as solid and unstressed beforehand, with $p_2 = 0$. From equations (7) we obtain the values given in columns 1 and 2 of the table:

	1	2	3	4	5	6
	due to explosion		shrinking	final values		
	p	t	p'	p	t	$F(p, t)$
Inside A . .	8·25	10·25	0	8·25	— 2·25	61·9
Outside A . .	3·10	5·15	3·47	6·57	— 3·92	43·9
Inside B . .	3·10	5·15	3·47	6·57	5·81	100
Outside B . .	1·29	3·36	2·57	3·86	3·12	31·8
Inside C . .	1·29	3·36	2·57	3·86	8·18	100
Outside C . .	0·46	2·51	1·24	1·70	5·97	44·7
Inside D . .	0·46	2·51	1·24	1·70	9·39	100
Outside D . .	0·00	2·06	0	0	7·72	59·5

For each tube we have :—

At the inside, $t = t_1 = \dfrac{p_1(r_1^2 + r_2^2) - 2p_2r_2^2}{r_2^2 - r_1^2}$; $p = p_1$

„ outside, $t = t_2 = \dfrac{2p_1r_1^2 - p_2(r_1^2 + r_2^2)}{r_2^2 - r_1^2}$; $p = p_2$.

Tube D :—

Let p_1 = the radial pressure on the inside due to shrinking.

Let p_1 and t_1 be the final stresses at the inside, then

$$p_1 = p_1' + 0{\cdot}46$$

$$t_1 = 2{\cdot}51 + \frac{976p_1'}{176} = 2{\cdot}51 + 5{\cdot}55p_1'.$$

Substituting these values in the equation

$$p_1^2 + t_1^2 + 0{\cdot}6p_1t_1 = 100$$

we get

$$p_1'^2 + 0{\cdot}915p_1' - 2{\cdot}67 = 0$$

which gives $p_1' = 1{\cdot}24$ for the positive root.

Then

$$p_1 = 1{\cdot}70 \text{ tons/in.}^2$$
$$t_1 = 9{\cdot}39 \text{ tons/in.}^2$$

Also we find, at the outside, $t_2 = 7{\cdot}72$.

We now proceed in exactly the same way for each of the other tubes, C, B, A in turn, and thus the table above is completed. It will be seen that according to this theory greater shrinking pressures between the tubes are demanded.

315. Driving Fits on Solid Shafts.*—Consider the case where a tube of external diameter $2r_2$ and internal diameter $2r_1 - \delta$ is forced on to a solid shaft of external diameter $2r_1$.

Let E_1 and m_1 refer to the shaft, and E_2 and m_2 to the tube.

As in the case considered in § 311 we shall have

$$p = a - \frac{\beta}{r^2}$$

$$t = -a - \frac{\beta}{r^2},$$

where p and t are the radial pressure and hoop tension at radius r in the shaft.

Now, clearly the stress in the shaft is not infinite at the centre, where $r = 0$. Therefore we must have $\beta = 0$. Then

$$p = -t = a = \text{constant}.$$

Thus the material of the shaft is subjected at all points to both radial and circumferential compressive stress of intensity p_1, where p_1 denotes the pressure between the shaft and the tube. Therefore the circumferential *compressive* strain at the outside of the shaft is

$$-\left(\frac{t}{E_1} + \frac{p_1}{m_1 E_1}\right) = \frac{p_1}{E_1}\left(1 - \frac{1}{m_1}\right) \quad . \quad . \quad . \quad \text{(i)}$$

since $t = -p_1$.

At the inside of the tube the hoop tension is given by (8), p. 439,

$$t_1 = \frac{p_1(r_2^2 + r_1^2)}{r_2^2 - r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

neglecting the small quantity δ in comparison with r_1. Also the circumferential tensile strain there is

$$\frac{t_1}{E_2} + \frac{p_1}{m_2 E_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

From (i) and (iii) we have, equating the sum of the changes in diameters of the shaft and the inside of the tube to their original difference :

$$2r_1\left[\frac{p_1}{E_1}\left(1 - \frac{1}{m_1}\right) + \frac{p_1}{m_2 E_2} + \frac{t_1}{E_2}\right] = \delta \quad . \quad . \quad \text{(iv)}$$

Now what we usually wish to know is the relation between δ and t_1, the latter being the maximum tensile stress in the tube.

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* See also an article by Morley in *Engineering*, Aug. 11, 1911.

Substituting for p_1 from (ii) in (iv) we get

$$2r_1t_1\left[\left(1-\frac{1}{m_1}\right)\frac{r_2^2-r_1^2}{r_2^2+r_1^2}\cdot\frac{1}{E_1}+\frac{r_2^2-r_1^2}{r_2^2+r_1^2}\cdot\frac{1}{m_2E_2}+\frac{1}{E_2}\right]=\delta$$

or

$$t_1\left[\left(\frac{m_1-1}{m_1E_1}+\frac{1}{m_2E_2}\right)\frac{r_2^2-r_1^2}{r_2^2+r_1^2}+\frac{1}{E_2}\right]=\frac{\delta}{2r_1} \qquad . \quad . \quad (17)$$

From this we can calculate t_1 in terms of δ, or vice versa.

From (ii) we have

$$p_1=\frac{r_2^2-r_1^2}{r_2^2+r_1^2}t_1 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Combining (17) and (18) we can test the strength of the tube by either the shear-stress theory, or the strain-energy theory. In practice $\delta/2r_1$ will usually be of the order 1/2,000.

Example.—We shall now apply these formulæ to the example worked out on p. 443, where we neglected the compressibility of the shaft. We have

$$m_1 = m_2 = \frac{10}{3}$$

$$E_1 = E_2 = 13{,}700 \text{ tons/in.}^2$$

$$r_1 = 4'',\ r_2 = 6''.$$

$$p_1 = 2 \text{ tons/in.}^2$$

From (18) we have $t_1 = 2\cdot6p_1 = 5\cdot2$ tons/in.^{2}

Then (17) gives

$$\delta = \frac{8'' \times 5\cdot2}{13{,}700}\left(\frac{20}{52}+1\right) = \frac{8 \times 5\cdot2 \times 72}{13{,}700 \times 52} = 0\cdot0042'',$$

which is 23 per cent. greater than the value we found when we neglected the compressibility of the shaft. If we take the previous value of δ (0·0034″) the pressure due to shrinking will be only 1·62 tons/in.^{2} instead of 2, an error of nearly 20 per cent.

TUBES STRENGTHENED BY WIRE WINDING

316. Purpose of Wire Winding.—We have seen above that a tube may be made to withstand a higher internal pressure if it be built of several tubes shrunk upon each other than if it be a solid homogeneous tube. To be quite sure that the greatest stresses in each tube are kept within the desired limits requires great accuracy of turning and boring in order to produce the correct shrinking pressures. This accuracy is more easily desired than obtained in such large tubes as those used for gun construction. An alternative method of strengthening a tube is to wind it, like a reel of cotton, with steel ribbon or "wire," applying a suitable tension to the wire during the winding process. The chief difficulty, in formulating a theory of wire wound tubes, is to decide how the wire behaves. In what follows we suppose that the wire winding acts as if it were a homogeneous tube, the theory being due to C. E. Inglis.

317. General Equations.—Let the internal and external radii of the main tube be r_1 and r_2, and let the overall radius of the finished winding be r_3, as shown in Fig. 360.

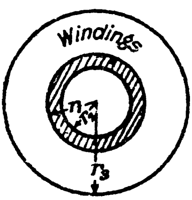

FIG. 360.

Let P be the internal pressure due to the explosion or other cause.

„ Let p_1 and t_1 be the stresses due to the internal pressure.

„ p_0 and t_0 be the initial stresses.

„ T be the winding tensile stress in the wire at radius r.

Then $p_0 + p_1$ and $t_0 + t_1$ are the final stresses.

As in § 311 we shall have, for the general equation of equilibrium before the explosion occurs :

$$r\frac{dp_0}{dr} + p_0 + t_0 = 0 \text{ or } \frac{d}{dr}(rp_0) + t_0 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Inside the winding, p_0 represents the radial pressure, per unit area, due to all the layers of wire between $r = r$ and $r = r_3$. These outer layers, being wound on afterwards, will reduce the hoop stress in the wire between $r = r_2$ and $r = r$. Regarding the tube as homogeneous from $r = r_1$ to $r = r$, and under an external pressure p_0, the diminution of hoop tensile stress is given by (8), p. 439 : it is $p_0(r^2 + r_1^2)/(r^2 - r_1^2)$. Hence we have

$$t_0 = T - p_0\frac{r^2 + r_1^2}{r^2 - r_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Equations (1) and (19) will obtain throughout our theory.

318. Shear Stress, or Stress-Difference (p + t), Limited Throughout.—We shall first work out what must be the tension in the wire, at any radius, during winding, in order that $p + t$ may be constant all through the winding after the explosion within the tube. That is, we wish to make

$$p_1 + t_1 + p_0 + t_0 \leqslant f,$$

where f is the maximum permissible stress under simple tension.

We have to find r_0, and T as a function of r. Within the winding we must have

$$p_1 + t_1 + p_0 + t_0 = f \quad . \quad . \quad . \quad . \quad . \quad (i)$$

From (7), p. 439, we have

$$p_1 + t_1 = \frac{2Pr_1^2r_3^2}{(r_3^2 - r_1^2)r^2} = \frac{cr_1^2}{r^2} \quad . \quad . \quad . \quad . \quad (ii)$$

where

$$c = \frac{2Pr_3^2}{r_3^2 - r_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

Then, from (i) and (ii) we get

$$p_0 + t_0 = f - c\frac{r_1^2}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (iv)$$

Hence from (19)

$$p_0 + T - p_0\frac{r^2 + r_1^2}{r^2 - r_1^2} = f - c\frac{r_1^2}{r^2}$$

$$\therefore T = f - c\frac{r_1^2}{r^2} + \frac{2p_0 r_1^2}{r^2 - r_1^2} \quad . \quad . \quad . \quad . \quad (\text{v})$$

This enables us to find T when we know p_0.

Again, from (i) and (iv) we have:

$$\frac{dp_0}{dr} = c\frac{r_1^2}{r^3} - \frac{f}{r}$$

$$\therefore p_0 = -\frac{cr_1^2}{2r^2} - f \log_e r + A$$

where A is a constant.

We must have $p_0 = 0$ when $r = r_3$,

$$\therefore A = \frac{cr_1^2}{2r_3^2} + f \log_e r_3.$$

$$\therefore p_0 = \frac{cr_1^2}{2}\left(\frac{1}{r_3^2} - \frac{1}{r^2}\right) + f \log_e \frac{r_3}{r} \quad . \quad . \quad . \quad (\text{vi})$$

Let p_2 be the value of p_0 when $r = r_2$, i.e. where the windings touch the inner tube. Then

$$p_2 = \frac{cr_1^2}{2}\left(\frac{1}{r_3^2} - \frac{1}{r_2^2}\right) + f \log_e \frac{r_3}{r_2} \quad . \quad . \quad . \quad . \quad (\text{vii})$$

At the inside of the tube, where $r = r_1$, this will produce a hoop stress $\dfrac{-2p_2r_2^2}{r_2^2 - r_1^2}$. This is the initial hoop stress at the bore of the tube due to the windings; the initial radial stress there is zero. On account of the explosion the value of $p_1 + t_1$ when $r = r_1$ is c, from (ii). Therefore the final value of $p + t$, at radius r_1, is

$$c - \frac{2p_2r_2^2}{r_2^2 - r_1^2},$$

and we require that this should have the value f.

That is

$$\frac{2p_2r_2^2}{r_2^2 - r_1^2} = c - f.$$

$$\therefore p_2 = \frac{r_2^2 - r_1^2}{2r_2^2}(c - f).$$

Inserting this value of p_2 in (vii) gives

$$\frac{r_2^2 - r_1^2}{2r_2^2}(c - f) = \frac{cr_1^2}{2r_2^2r_3^2}(r_2^2 - r_3^2) + f \log_e \frac{r_3}{r_2}$$

Putting in the value of c from (iii) this leads to the equation

$$\log_e \frac{r_3}{r_2} = \frac{P}{f} - \frac{r_2^2 - r_1^2}{2r_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

From this we can find r_3.

Again, from (v) and (vi), eliminating p_0, we get

$$T = f - c\frac{r_1^2}{r^2} + \frac{2r_1^2}{r^2 - r_1^2}\left[\frac{cr_1^2}{2}\left(\frac{r^2 - r_3^2}{r_3^2 r^2}\right) + f\log_e\frac{r_3}{r}\right]$$

$$= f\left[1 + \frac{2r_1^2}{r^2 - r_1^2}\cdot\log_e\frac{r_3}{r}\right] + c\left[\frac{r_1^4(r^2 - r_3^2)}{r^2 r_3^2(r^2 - r_1^2)} - \frac{r_1^2}{r^2}\right]$$

Substituting in this the value of c given by (iii) we get, after simplification,

$$\mathbf{T = f\left[1 + \frac{2r_1^2}{r^2 - r_1^2}\log_e\frac{r_3}{r}\right] - \frac{2Pr_1^2}{r^2 - r_1^2}}. \quad . \quad . \quad . \quad (21)$$

The overall radius r_3 is given by (20), and then T is given by (21). These two equations form the complete solution of our problem :

$$\left.\begin{array}{l} \text{At the inside, } r = r_2\text{, we find } T = f\left(1 - \dfrac{r_1^2}{r_2^2}\right) \\ \text{At the outside, } r = r_3\text{, we find } T = f - \dfrac{2Pr_1^2}{r_3^2 - r_1^2} \end{array}\right\} \quad . \quad . \quad . \quad (22)$$

319. Shear Stress Limited in Tube : Tensile Stress Limited in Windings.—We next propose to solve the same problem under these conditions.

$p + t \not> f$ in gun.

$t =$ constant $= T_0$, in windings,

after the internal (explosion or other) pressure is applied, p and t being the final stresses. Consider a diametral section of the whole gun, as in Fig. 361. Let p_2 be the pressure between the windings and the tube. Then for the equilibrium of the windings as a whole we must have

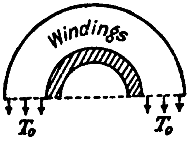

FIG. 361.

$$2r_2p_2 = 2T_0(r_3 - r_2)$$

$$\therefore p_2 = \frac{T_0(r_3 - r_2)}{r_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Then, from (7), at the inner surface of the tube ($r = r_1$), we have

$$p + t = \frac{2(P - p_2)r_2^2}{r_2^2 - r_1^2},$$

Putting in the value of p_2 from (23), this becomes

$$p + t = \frac{2r_2^2}{r_2^2 - r_1^2}\left(P - \frac{r_3 - r_2}{r_2}T_0\right)$$

Hence we must have

$$\frac{2r_2^2P}{r_2^2 - r_1^2} - \frac{2r_2(r_3 - r_2)}{r_2^2 - r_1^2}T_0 = f,$$

from which we find

$$\mathbf{r_3 = \frac{2r_2^2(P + T_0) - f(r_2^2 - r_1^2)}{2r_2T_0}}. \quad . \quad . \quad . \quad (24$$

This gives the outside radius of the windings. Regarding the whole gun as homogeneous, the tension in the windings due to the explosion is

$$t_1 = \frac{Pr_1^2}{r_3^2 - r_1^2}\left(\frac{r_3^2}{r^2} + 1\right)$$

Since the final tension is to be T_0, the initial tension t_0 at any radius r must be

$$t_0 = T_0 - t_1 = T_0 - \frac{Pr_1^2}{r_3^2 - r_1^2}\left(\frac{r_3^2}{r^2} + 1\right)$$

$$t_0 = T_0 - c\left(\frac{r_3^2}{r^2} + 1\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where

$$c = \frac{Pr_1^2}{r_3^2 - r_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Now, as before, we have (p. 450, equation 1) for the windings before explosion,

$$\frac{d}{dr}(rp_0) + t_0 = 0.$$

$$\therefore \frac{d}{dr}(rp_0) = c - T_0 + c\frac{r_3^2}{r^2}$$

from (i).

$$\therefore rp_0 = A + cr - T_0 r - c\frac{r_3^2}{r}$$

where A is a constant; hence

$$p_0 = \frac{A}{r} - T_0 + c\left(1 - \frac{r_3^2}{r^2}\right)$$

We must have $p_0 = 0$ when $r = r_3$,

$$\therefore \quad A = t_0 r_3,$$

and

$$p_0 = T_0\left(\frac{r_3}{r} - 1\right) + c\left(1 - \frac{r_3^2}{r^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

As before the initial hoop tension in the wire at radius r is, from (19) above,

$$t_0 = T - p_0\frac{r^2 + r_1^2}{r^2 - r_1^2},$$

where T is the winding tensile stress.

$$\therefore T = t_0 + \frac{r^2 + r_1^2}{r^2 - r_1^2}\left\{T_0\left(\frac{r_3}{r} - 1\right) + c\left(1 - \frac{r_3^2}{r^2}\right)\right\}, \text{ from (25).}$$

$$= T_0 - c\left(\frac{r_3^2}{r^2} + 1\right) + \frac{r^2 + r_1^2}{r^2 - r_1^2}\left\{T_0\left(\frac{r_3}{r} - 1\right) + c\left(1 - \frac{r_3^2}{r^2}\right)\right\} \text{ from (i).}$$

After simplifying, this equation reduces to

$$(r^2 - r_1^2)\,T = T_0\left\{\frac{r_3}{r}(r^2 + r_1^2) - 2r_1^2\right\} - 2c(r_3^2 - r_1^2).$$

Substitute in this the value of c from (ii) and r_2 from (24) and we get, after a little simplification,

$$T = \frac{P + T_0}{r^2 - r_1^2}\left\{\frac{r_2}{r}(r^2 + r_1^2) - 2r_1^2\right\} - \frac{(r_2^2 - r_1^2)(r^2 + r_1^2)}{2rr_2(r^2 - r_1^2)}f \qquad (26)$$

This gives the winding tension at any radius, and (24) the amount of winding required.

320. Wire Winding at Constant Tension.—We shall next suppose that the winding is done at constant stress T, and investigate the stresses due to the pressure P inside the tube.

As before, we have from (19)

$$T - t_0 = p_0\frac{r^2 + r_1^2}{r^2 - r_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

where T is the tensile stress in the wire while winding, and t_0 and p_0 are hoop and radial stresses in the winding, at any radius r, before the explosion.

We also have, from (1)

$$p_0 + r\frac{dp_0}{dr} = -t_0 = p_0\frac{r^2 + r_1^2}{r^2 - r_1^2} - T$$

$$\therefore \frac{dp_0}{dr} + p_0\left\{\frac{-2r_1^2}{(r^2 - r_1^2)r}\right\} = -\frac{T}{r} \quad . \quad . \quad . \quad (ii)$$

This equation is made integrable on multiplying by

$$e^{\int \frac{-2r_1^2 dr}{(r^2 - r_1^2)r}} = \frac{r^2}{r^2 - r_1^2}$$

Multiplying (i) by this integrating factor it becomes

$$\frac{r^2}{r^2 - r_1^2}\frac{dp_0}{dr} - \frac{2r^2r_1^2}{(r^2 - r_1^2)^2r}p_0 = -\frac{rT}{r^2 - r_1^2}$$

i.e.

$$\frac{d}{dr}\left\{\frac{r^2p_0}{r^2 - r_1^2}\right\} = -\frac{rT}{r^2 - r_1^2}$$

$$\therefore \frac{r^2p_0}{r^2 - r_1^2} = -\frac{T}{2}\log_e(r^2 - r_1^2) + A.$$

Since $p_0 = 0$ when $r = r_3$, we must have

$$A = \frac{T}{2}\log_e(r_3^2 - r_1^2)$$

$$\therefore \frac{r^2p_0}{r^2 - r_1^2} = \frac{T}{2}\log_e\frac{r_3^2 - r_1^2}{r^2 - r_1^2}.$$

$$\therefore p_0 = \frac{(r^2 - r_1^2)T}{2r^2}.\log_e\frac{r_3^2 - r_1^2}{r^2 - r_1^2} \quad . \quad . \quad . \quad . \quad (27)$$

Then from (i)

$$t_0 = T\left[1 - \frac{r^2 + r_1^2}{2r^2}\log_e\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right] \quad . \quad . \quad . \quad . \quad (28)$$

Equations (27) and (28) give the initial stresses in the winding.

At the surface of the tube, we have from (27), putting $r = r_2$,

$$p_2 = \frac{(r_2^2 - r_1^2)}{2r_2^2} T \log_e \frac{r_3^2 - r_1^2}{r_2^2 - r_1^2}.$$

At the inside of the tube we have $p_1 = 0$. Then the initial stresses in the tube can be calculated from (6). The extra stresses due to the internal pressure P are then found from (7) putting $p_1 = P$, $p_2 = 0$ and $r_2 = r_3$. The final stresses are then found by addition.

It will be seen from (28) that t_0 decreases from the outside, where $r = r_3$, as r decreases to r_2. But the hoop stress due to the explosion pressure decreases outwards. Thus it is possible to keep the final hoop stress in the winding moderately constant; it can be made to have the same value at any two desired radii, and the same applies to the stress-difference $p + t$. This can be effected by giving the proper value to T if r_3 is fixed, or vice versa. It will be possible to make the maximum stress, or any function of the stresses, have equal values at the inner surface of the tube and at any two desired radii in the winding, by choosing both T and r_3 to fit the necessary conditions.

Example.—A steel tube has its inner radius 2″ and its outer radius 6″. A radial depth of 4″ of wire, of the same steel, is wound on the tube in thin layers under a constant tension of 12,000 lbs./in.^{2} Find expressions for the ultimate value of the maximum stress-difference at any radius, (i) in the wire, (ii) in the tube, assuming that the wire, when wound, acts as a solid tube. (Mech. Sc. Trip., 1909, B.)

We have

$$r_1 = 2'', \; r_2 = 6'', \; r_3 = 10'', \; T = 12{,}000 \text{ lbs./in.}^2$$
$$r_1^2 = 4 \text{ in.}^2, \; r_2^2 = 36 \text{ in.}^2, \; r_3^2 = 100 \text{ ins.}^2$$

The initial stresses in the winding are given by (27) and (28), p. 454.

$$p_0 = \frac{(r^2 - 4)12{,}000}{2r^2} \log_e \frac{96}{r^2 - 4} = 13{,}800 . \frac{r^2 - 4}{r^2} \log_{10} \frac{96}{r^2 - 4}$$

$$t_0 = 12{,}000\left[1 - \frac{r^2 + 4}{2r^2} \log_e \frac{96}{r^2 - 4}\right] = 12{,}000\left[1 - 1{\cdot}15\frac{r^2 + 4}{r^2} \log_{10} \frac{96}{r^2 - 4}\right]$$

Hence, by addition,

$$p_0 + t_0 = 12{,}000 - \frac{110{,}400}{r^2} \log_{10} \frac{96}{r^2 - 4} \quad . \quad . \quad . \quad (i)$$

The external pressure on the tube, due to the winding, is obtained by putting $r = r_2 = 6''$.

We have

$$p_0 \text{ on surface of tube} = 13{,}800 \times \frac{32}{36} \log_{10} \frac{96}{32} = 5{,}860 \text{ lbs./in.}^2$$

The initial internal pressure on the tube is zero. Hence, from (6), p. 439, putting $p_1 = 0$ and $p_2 = 5{,}860$, we have in the tube

$$p_0 + t_0 = -\frac{2p_2 r_1^2 r_2^2}{r^2(r_2^2 - r_1^2)} = -\frac{11{,}720 \times 4 \times 36}{r^2 \times 32}$$

or

$$p_0 + t_0 = -\frac{52{,}800}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

At any point in the compound tube, the extra stress-difference due to an internal pressure P is obtained from (7), putting $p_1 = P$, $r_2 = 10''$, $r_1 = 2''$.

$$p_1 + t_1 = \frac{2P \times 4 \times 100}{r^2 \times 96} = \frac{8{\cdot}33P}{r^2} \quad . \quad . \quad . \qquad \text{(iii)}$$

The ultimate stress-difference is given by adding together (i) and (iii) for the winding, (ii) and (iii) for the tube. We get:

In the winding, $p + t = 12{,}000 + \frac{1}{r^2}\left(8{\cdot}33P - 110{,}400 \log_{10} \frac{96}{r^2 - 4}\right)$.

In the tube, $p + t = \frac{1}{r^2}\,(8{\cdot}33P - 52{,}800)$.

321. Temperature Stresses in Thick Tubes.*—We shall now investigate the stresses in a thick circular tube when the inner and outer surfaces are maintained at constant temperatures τ_1 and τ_2 respectively. We assume that the end conditions are such that the axial strain is constant, and that the tube is not subjected to any external forces.

As before, let r_1 and r_2 denote the internal and external radii.

Let κ = the coefficient of conductivity.
a = the coefficient of linear expansion.
τ = the temperature at radius r.
u = the radial displacement of a point originally at radius r.
p_r, p_θ, p_z = the radial, circumferential and axial stresses.

All the quantities involved will be independent of the angular coordinate θ and the axial co-ordinate z.

First we must express τ as a function of r. The flow of heat across the walls of an elementary tubular cylinder, of thickness δr and axial length unity, will be given by

$$\text{flow per unit time} = \kappa\frac{\delta\tau}{\delta r} \,.\, 2\pi r,$$

$\delta\tau$ being the difference between the temperatures at the inner and outer surfaces of the elementary cylinder.

If the temperatures are constant with respect to time the rate of flow of heat must be constant, and from this it follows that $r\frac{d\tau}{dr}$ must be constant, so that we can write

$$r\frac{d\tau}{dr} = A,$$

whence

$$\tau = A \log_e r + B \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

If τ_1 and τ_2 are the temperatures when $r = r_1$ and r_2 respectively, we have

$$\tau_1 = A \log_e r_1 + B$$
$$\tau_2 = A \log_e r_2 + B.$$

From these equations we find A and B, and so finally we get

$$\tau = \frac{\tau_1 \log_e \frac{r}{r_2} + \tau_2 \log_e \frac{r_1}{r}}{\log_e \frac{r_1}{r_2}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

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* See also C. H. Lees, *Proc. R. Soc.*, Vol. 101 (1922); *Engineering*, Vol. 124 (1927), p. 443, Vol. 121 (1926), p. 447, Vol. 127 (1929), p. 282.

We must now separate the strains which arise on account of the stresses from the thermal strains which arise on account of an increase of temperature. The latter are $\alpha\tau$ in all directions.

The total strains are $\frac{du}{dr}$, $\frac{u}{r}$ and e_z in the radial, circumferential and axial directions. Hence the strains due to the stresses are $\frac{du}{dr} - \alpha\tau$; $\frac{u}{r} - \alpha\tau$; and $e_z - \alpha\tau$. The stress-strain equations are, then,

$$E\left(\frac{du}{dr} - \alpha\tau\right) = p_r - \frac{1}{m}(p_\theta + p_z) \quad . \; . \; . \; . \quad \text{(ii)}$$

$$E\left(\frac{u}{r} - \alpha\tau\right) = p_\theta - \frac{1}{m}(p_z + p_r) \quad . \; . \; . \quad \text{(iii)}$$

$$E(e_z - \alpha\tau) = p_z - \frac{1}{m}(p_r + p_\theta) \quad . \; . \; . \quad \text{(iv)}$$

Also we have the equation of equilibrium * as before:

$$p_r - p_\theta + r\frac{dp_r}{dr} = 0 \quad . \; . \; . \; . \; . \; . \quad \text{(v)}$$

From (iii) we have

$$Eu = E\alpha r\tau + rp_\theta - \frac{p_z}{m}r - \frac{p_r}{m}r.$$

Differentiating this with respect to r we have

$$E\frac{du}{dr} = E\alpha\tau + E\alpha r\frac{d\tau}{dr} + p_\theta + r\frac{dp_\theta}{dr} - \frac{p_z}{m} - \frac{r}{m}\frac{dp_z}{dr} - \frac{p_r}{m} - \frac{r}{m}\cdot\frac{dp_r}{dr} \; . \quad \text{(vi)}$$

Also from (ii):

$$E\frac{du}{dr} = E\alpha\tau + p_r - \frac{p_\theta}{m} - \frac{p_z}{m} \quad . \; . \; . \; . \quad \text{(vii)}$$

and differentiating (iv) with respect to r, remembering that e_z is by hypothesis constant,

$$\frac{dp_z}{dr} = -E\alpha\frac{d\tau}{dr} + \frac{1}{m}\frac{dp_r}{dr} + \frac{1}{m}\frac{dp_\theta}{dr} \quad . \; . \; . \; . \quad \text{(viii)}$$

Subtract (vii) from (vi) and substitute for $\frac{dp_z}{dr}$ from (viii) and we get:

$$0 = E\alpha r\frac{d\tau}{dr} - p_r + \frac{p_\theta}{m} + p_\theta + r\frac{dp_\theta}{dr} + \frac{E\alpha}{m}r\frac{d\tau}{dr} - \frac{r}{m^2}\frac{dp_r}{dr} - \frac{r}{m^2}\frac{dp_\theta}{dr} - \frac{p_r}{m} - \frac{r}{m}\frac{dp_r}{dr}$$

or

$$p_r\left(1 + \frac{1}{m}\right) - p_\theta\left(1 + \frac{1}{m}\right) - r\left(1 - \frac{1}{m^2}\right)\frac{dp_\theta}{dr} + \frac{r}{m}\left(1 + \frac{1}{m}\right)\frac{dp_r}{dr} = E\alpha r\left(1 + \frac{1}{m}\right)\frac{d\tau}{dr}$$

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* p_θ and p_r are now both tensile stresses, so that the sign of p in equation (i), p. 438, must be changed.

that is

$$p_r - p_\theta - r\left(1 - \frac{1}{m}\right)\frac{dp_\theta}{dr} + \frac{r}{m}\cdot\frac{dp_r}{dr} = Ear\frac{d\tau}{dr}.$$

Substitute in this for $p_r - p_\theta$ from (v) and we get

$$r\left(1 - \frac{1}{m}\right)\frac{d}{dr}(p_r + p_\theta) + Ear\frac{d\tau}{dr} = 0$$

$$\therefore \quad \frac{d}{dr}(p_r + p_\theta) + \frac{mEa}{m-1}\cdot\frac{d\tau}{dr} = 0.$$

$$\therefore p_r + p_\theta + \frac{mEa}{m-1}\tau = c, \text{ a constant.}$$

Adding this to (v) we get

$$2p_r + r\frac{dp_r}{dr} + \frac{mEa}{m-1}\tau = c,$$

or

$$\frac{1}{r}\frac{d}{dr}(r^2p_r) = c - \frac{mEa}{m-1}\tau \quad . \quad . \quad . \quad . \quad . \quad \text{(ix)}$$

Now from (29) we have

$$r\tau = \frac{r(\tau_1 - \tau_2)\log_e r + r(\tau_2 \log_e r_1 - \tau_1 \log_e r_2)}{\log_e \frac{r_1}{r_2}}$$

$$\therefore \int r\tau dr = \frac{\frac{r^2}{2}(\tau_1 - \tau_2)(\log_e r - \frac{1}{2}) + \frac{r^2}{2}(\tau_2 \log_e r_1 - \tau_1 \log_e r_2)}{\log_e \frac{r_1}{r_2}}$$

$$= \frac{r^2\tau}{2} - \frac{r^2(\tau_1 - \tau_2)}{4\log_e \frac{r_1}{r_2}}$$

Hence, integrating (ix) we have

$$r^2p_r = \frac{cr^2}{2} - \frac{mEa}{m-1}\left(\frac{r^2\tau}{2} - \frac{r^2(\tau_1 - \tau_2)}{4\log_e \frac{r_1}{r_2}}\right) + \beta$$

Hence, where β is a constant

$$p_r = \left(\frac{c}{2} + \frac{mEa(\tau_1 - \tau_2)}{4(m-1)\log_e \frac{r_1}{r_2}}\right) + \frac{\beta}{r^2} - \frac{mEa}{2(m-1)}\tau\ .$$

or, say,

$$p_r = A + \frac{B}{r^2} - \frac{mEa}{2(m-1)}\tau \quad . \quad . \quad . \quad . \quad . \quad \text{(x)}$$

If there are no external pressures applied to the surface of the tube we must have $p_r = 0$ when $r = r_1$ or r_2; therefore

$$A + \frac{B}{r_1^2} - \frac{mEa}{2(m-1)}\tau_1 = 0$$

$$A + \frac{B}{r_2^2} - \frac{mEa}{2(m-1)}\tau_2 = 0$$

From these we find

$$\left.\begin{aligned} A &= \frac{mEa}{2(m-1)} \cdot \frac{\tau_2 r_2^2 - \tau_1 r_1^2}{r_2^2 - r_1^2} \\ B &= \frac{mEa}{2(m-1)} \cdot \frac{r_1^2 r_2^2(\tau_1 - \tau_2)}{r_2^2 - r_1^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(xi)}$$

Again, from (v) and (x) we have

$$\begin{aligned} p_\theta &= p_r + r\frac{dp_r}{dr} \\ &= A + \frac{B}{r^2} - \frac{2B}{r^2} - \frac{mEa}{2(m-1)}\tau - \frac{mEar}{(2m-1} \cdot \frac{d\tau}{dr} \\ &= A - \frac{B}{r^2} - \frac{mEa}{2(m-1)}\left(\tau + \frac{\tau_1 - \tau_2}{\log_e \frac{r_1}{r_2}}\right), \text{from (29)} \quad . \quad \text{(xii)} \end{aligned}$$

Substituting for A and B in (x) and (xii) we get

$$p_r = \frac{mEa}{2(m-1)}\left[\frac{\tau_2 r_2^2 - \tau_1 r_1^2 + \frac{r_1^2 r_2^2}{r^2}(\tau_1 - \tau_2)}{(r_2^2 - r_1^2)} - \tau\right] \quad . \quad . \quad . \quad (30)$$

$$p_\theta = \frac{mEa}{2(m-1)}\left[\frac{\tau_2 r_2^2 - \tau_1 r_1^2 - \frac{r_1^2 r_2^2}{r^2}(\tau_1 - \tau_2)}{r_2^2 - r_1^2} - \frac{\tau_1 - \tau_2}{\log_e \frac{r_1}{r_2}} - \tau\right] \quad (31)$$

where τ is given by (29).

It remains to find p_z. From (iv) we have

$$p_z = E(e_z - a\tau) + \frac{1}{m}(p_r + p_\theta).$$

The value of p_z depends on the end conditions.

First suppose the ends are fixed, so that $e_z = 0$. Then

$$p_z = -Ea\tau + \frac{1}{m}(p_r + p_\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where p_r and p_θ are given by (30) and (31).

If the ends are free we must have

$$\int_{r_1}^{r_2} 2\pi r p_z dr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii)}$$

Putting the values of p_r and p_θ in the above expression for p_z we get

$$p_z = \frac{2A}{m} - \frac{Ea}{2(m-1)} \cdot \frac{\tau_1 - \tau_2}{\log_e \frac{r_1}{r_2}} - \frac{mEa}{m-1}\tau.$$

The first two terms of this being constant, it will simplify the subsequent work if we write

$$p_z = D - \frac{mEa}{m-1}\tau \quad . \quad . \quad . \quad . \quad . \quad (\text{xiv})$$

Then we must have from (xiii),

$$\int_{r_1}^{r_2}\left(Dr - \frac{mEa}{m-1}r\tau\right)dr = 0$$

or

$$\frac{D}{2}(r_2^2 - r_1^2) = \frac{mEa}{m-1}\left[\frac{r_2^2\tau_2 - r_1^2\tau_1}{2} - \frac{(r_2^2 - r_1^2)(\tau_1 - \tau_2)}{4\log_e \frac{r_1}{r_2}}\right]$$

using the value of $\int r\tau dr$ found above.

$$\therefore D = \frac{mEa}{m-1}\left[\frac{r_2^2\tau_2 - r_1^2\tau_1}{r_2^2 - r_1^2} - \frac{\tau_1 - \tau_2}{2\log_e \frac{r_1}{r_2}}\right]$$

Then from (xiv)

$$\mathbf{p}_z = \frac{\mathbf{m}E\alpha}{\mathbf{m}-1}\left[\frac{\mathbf{r}_2\tau^2_2 - \mathbf{r}_1^2\tau_1}{\mathbf{r}_2^2 - \mathbf{r}_1^2} - \frac{\tau_1 - \tau_2}{2\log_e \frac{\mathbf{r}_1}{\mathbf{r}_2}} - \tau\right] \quad . \quad . \quad (33)$$

where τ is given by (29).

Equations (29), (30), (31), (32) and (33) give a complete solution of the problem under the assumed conditions.

322. Thick Spherical Shells.—Let us consider the stresses in a thick spherical shell subjected to internal pressure. Let r_1 and r_2 denote the internal and external radii, and p_1 and p_2 the internal and external pressures. Let u be the radial displacement of a point at radius r, then the radial and circumferential strains will be $\frac{du}{dr}$ and $\frac{u}{r}$ respectively, both being the same in all directions by symmetry. Let p be the radial compressive stress, and t the circumferential hoop stress, then

$$E\frac{du}{dr} = -p - \frac{2t}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

$$E\frac{u}{r} = \frac{p}{m} + t - \frac{t}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

Now consider an elementary shell of radii r and $r + \delta r$. The radial pressure on the outer surface will be $p + \delta p$, and on the inner surface p. The nett pressure tending to burst this shell across a diametral plane is

$$\pi r^2 p - \pi(r + \delta r)^2(p + \delta p).$$

This is resisted by a total hoop tension $t(2\pi r\,.\,\delta r)$. Hence

$$\pi r^2 p - \pi(r + \delta r)^2(p + \delta p) = t(2\pi r\,.\,\delta r).$$

Neglecting small quantities of a higher order than the first this reduces to

$$r^2\,.\,\delta p + 2pr\,.\,\delta r + 2t\,.\,r\,.\,\delta r = 0$$

or, in the limit

$$r\frac{dp}{dr} + 2p + 2t = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Differentiate (ii), after multiplying by r, and subtract the result from (i), then

$$-r\frac{dp}{dr} + (m-1)r\frac{dt}{dr} + (m+1)(p+t) = 0 \quad . \ \text{(iii)}$$

From (34) we have

$$t = -\frac{r}{2}\frac{dp}{dr} - p$$

$$\frac{dt}{dr} = -\frac{r}{2}\frac{d^2p}{dr^2} - \frac{3}{2}\cdot\frac{dp}{dr}.$$

Hence from (iii), after some simplification,

$$\frac{d^2p}{dr^2} + \frac{4}{r}\frac{dp}{dr} = 0$$

Multiplying by r^4 and integrating, we get

$$r^4\frac{dp}{dr} = \text{constant, } A \text{ say.}$$

$$\therefore \frac{dp}{dr} = \frac{A}{r^4}$$

$$\therefore p = -\frac{A}{3r^3} + B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Then, from (34)

$$t = -\frac{r}{2}\frac{dp}{dr} - p = -\frac{A}{6r^3} - B \quad . \quad . \quad . \quad . \quad . \quad (36)$$

We must have $p = p_1$ when $r = r_1$ and $p = p_2$ when $r = r_2$, hence from (35),

$$-\frac{A}{3r_1^3} + B = p_1$$

$$-\frac{A}{3r_2^3} + B = p_2$$

From these we find

$$A = \frac{3r_1^3r_2^3(p_2 - p_1)}{r_2^3 - r_1^3}\,;\quad B = \frac{p_2r_2^3 - p_1r_1^3}{r_2^3 - r_1^3}.$$

Substituting in (35) and (36) we get

$$\mathrm{p} = \frac{1}{\mathrm{r}_2^3 - \mathrm{r}_1^3}\left[\mathrm{p}_2\mathrm{r}_2^3 - \mathrm{p}_1\mathrm{r}_1^3 + \frac{\mathrm{r}_1^3\mathrm{r}_2^3}{\mathrm{r}^3}(\mathrm{p}_1 - \mathrm{p}_2)\right] \quad . \quad . \quad (37)$$

$$\mathrm{t} = \frac{1}{\mathrm{r}_2^3 - \mathrm{r}_1^3}\left[\mathrm{p}_1\mathrm{r}_1^3 - \mathrm{p}_2\mathrm{r}_2^3 + \frac{\mathrm{r}_1^3\mathrm{r}_2^3}{2\mathrm{r}^3}(\mathrm{p}_1 - \mathrm{p}_2)\right] \quad . \quad . \quad (38)$$

When $p_2 = 0$ these reduce to

$$p = \frac{p_1 r_1^3}{r_2^3 - r_1^3}\left(\frac{r_2^3}{r^3} - 1\right) \text{ and } t = \frac{p_1 r_1^3}{r_2^3 - r_1^3}\left(\frac{r_2^3}{2r^3} + 1\right) \quad . \quad . \quad (39)$$

Both p and t are greatest when $r = r_1$. We then have, writing $\frac{r_2}{r_1} = k$,

$$\text{the max. stress difference for } (p + t) = \frac{p_1}{k^3 - 1} \cdot \frac{3k^3}{2};$$

whilst the strain-energy function $\left(p^2 + t^2 + \frac{2}{m}pt\right)$

$$= \frac{p_1^2}{(k^3 - 1)^2}\left[\left(\frac{5}{4} + \frac{1}{m}\right)k^6 - (k^3 - 2)\frac{m - 1}{m}\right].$$

From these the strength of the shell can be calculated from the maximum shear-stress theory, or the maximum strain-energy theory.

The theory of thermal stresses in spherical shells has been given by C. H. Lees.*

EXAMPLES XXVIII

1. A thick steel tube, 3″ bore and 12″ external diameter, is subjected to internal pressure. What values of the pressure will produce (*a*) a maximum hoop tension of 15 tons/in.², (*b*) a maximum shear stress of 10 tons/in.² ? (R.N.C., Greenwich, 1921.)

2. Find the thickness of a cast-iron hydraulic cylinder 12″ inside diameter. The safe stress in the material is 1 ton per in.², and the internal pressure is 1,000 lbs./in.² Also sketch the curves of circumferential and radial stress existing under these conditions. (H.M. Dockyard Schools, 1921.)

3. Water is transmitted through a pipe 4″ inside diameter and 8″ outside diameter, at a pressure of 200 lbs./in.² Find the hoop stress on the inside of the pipe. (H.M. Dockyard Schools, 1923.)

4. Calculate the thickness of the shell of a bomb calorimeter of spherical form 4″ inside diameter, if the allowable working stress is 4 tons/in.² and the internal pressure is 2 tons/in.² (Mech. Sc. Trip., B., 1910.)

5. If a long bar of material in direct tension can be safely loaded up to 10 tons/in.², find what internal pressures should be regarded as the limiting load for a cylinder in which $r_2 = 2r_1$, on the assumption that the criterion of safety is (i) the maximum principal stress, (ii) the maximum slide, (iii) the maximum principal strain. Show that the values obtained are approximately in the ratio 100 : 62 : 85 if $m = 10/3$. (Mech. Sc. Trip., 1905.)

6. A steel tube 10′ long, 3″ internal diameter and 9″ external diameter, is to be subjected to internal hydraulic pressure. Assuming that the tube remains elastic, estimate the volume of water that must be forced into it to raise the internal pressure to 15 tons/in.² (R.N.C., Greenwich, 1921.) (For steel $E = 12{,}000$ tons/in.²; $m = 10/3$; for water $K = 150$ tons/in.²)

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* *Proc. Royal Soc.*, A., Vol. 100.

7. A steel tube, 8″ external diameter, is to be shrunk on another tube 4″ internal diameter. Both are of steel for which $E = 12{,}000$ tons/in.^{2} The external diameter of the inner tube on being measured is found to be uniformly 6·002″. To what diameter must the outer tube be bored if it is required that after shrinking the pressure between the surfaces in contact shall be 2 tons/in.^{2}? Find also the greatest tensile stress in the compound tube when subjected to an internal pressure of 10 tons/in.^{2} (R.N.C., Greenwich, 1922.)

8. A steel tyre 1″ thick is shrunk on to a cast-iron rim 24″ outside diameter and 3″ thick. For the steel $E = 30 \times 10^6$ lbs./in.^{2}; for the cast iron $E = 14 \times 10^6$ lbs./in.^{2}, and $m = 4$ for both. Find the inside diameter of the steel tyre to the nearest thousandth of an inch, if, after shrinking on, the tyre is to press on the rim with a force of 2·5 tons/in.^{2} (Mech. Sc. Trip., 1906.)

9. One cylinder is shrunk on to another to form a compound tube of diameters 5″, 8″, 10″, and on cooling a radial stress of 1 ton/in.^{2} is produced at the common surface. Calculate the initial difference of radii, and the least difference of temperature for shrinking on. Find also the greatest radial and hoop stresses when there is an internal pressure of 3 tons/in.^{2} and the corresponding internal and external diameters. Take $E = 12{,}500$ tons/in.^{2}, coeff. of expansion $= 11 \times 10^{-6}$, $m = 10/3$. (Mech. Sc. Trip., B., 1919.)

10. A steel collar of length 1″ is to be shrunk on a steel shaft of 2″ diameter and is to resist a twisting moment of just 6 tons. ins. Taking the coefficient of friction as 0·2, and assuming plane sections perpendicular to the axis remain plane, find the external diameter of the collar if the maximum shearing stress due to shrinking is not to exceed 7·5 tons/in.^{2} If $E = 13{,}400$ lbs./in.^{2} and $m = \dfrac{10}{3}$, what should be the inside diameter of the collar before shrinking?

If a pull of 25 tons be applied at the two ends of the shaft, find approximately the reduction in the twisting couple on the collar which can be resisted by friction. (Mech. Sc. Trip., B., 1921.)

11. A gunmetal cylinder is to be shrunk on a steel shaft 12″ diameter, the outside diameter of the cylinder being 24″. What allowance should be made in boring the cylinder, so that the pressure between the jacket and the shaft may be 1·5 tons/in.^{2}? What maximum stress will this produce in the cylinder?

What minimum increase of temperature over atmospheric must be maintained throughout the cylinder and the shaft so that the cylinder may be slipped on?

E for steel $= 13{,}500$ tons/in.^{2}; $\dfrac{1}{m} = 0{\cdot}28$; coeff. of exp. $= 11 \times 10^{-6}$ per °C.

E „ G.M. $= 5{,}500$ „ ; $\dfrac{1}{m} = 0{\cdot}35$; „ $= 18 \times 10^{-6}$ „

(R.N.C., Greenwich, 1921.)

12. In the case of a wire-wound tube, if the maximum stress everywhere is to be limited to the value f, show that the total amount of wire, and the tension of the wire during winding, are given by

$$r_3 = \frac{r_1^2 + r_2^2}{2r_2}\left(1 + \frac{P}{f}\right)$$

$$T = \frac{\dfrac{r_3}{r}(r^2 + r_1^2)f - 2r_1^2(f + P)}{r^2 - r_1^2},$$

the notation being that of § 317.

13. A steel tube 12″ internal diameter, with walls $\frac{1}{2}''$ thick, is exposed to an internal temperature of 800° F. and an external temperature of 100° F. Calculate the maximum longitudinal stress in the tube, assuming that it is long and that the ends are free from pull. Take $E = 14{,}000$ tons/in.^{2}, $a = 1/160{,}000$ per °F., $m = 10/3$. (Mech. Sc. Trip., B., 1915.)

14. A steel sleeve whose external diameter is 8 in. and whose length is two or three times its diameter is to be shrunk on to a steel shaft of 6 in. diameter. The sleeve is made $\frac{6}{1000}$ in. smaller than the shaft. Assuming that there is no slipping between the sleeve and shaft during the shrinking process, show that the maximum hoop stress in the sleeve near the middle point of its length is approximately 15 tons per sq. in. and the longitudinal stress 11 tons per sq. in. Take $E = 13{,}500$ tons per sq. in. and $\sigma = 0{\cdot}3$.

Describe the distribution of stress near the ends of the sleeve, and discuss the amount and distribution of the tangential forces exerted by the sleeve on the shaft. (Mech. Sc. Trip., B., 1914.)

15. A thick cylinder with closed ends is subjected to internal pressure. Assuming that the longitudinal stress due to the pressure on the ends is uniformly distributed, show that, according to the strain-energy theory, the maximum permissible pressure is given by

$$\frac{p^2}{f^2} = \frac{(k^2 - 1)^2}{2k^4\left(1 + \frac{1}{m}\right) + 3\left(1 - \frac{2}{m}\right)}.$$

16. A tube A is to be shrunk on to a tube B, and the pressure between them is such that the hoop tension at the inside of A is t, and at the outside of B it is $-t'$. Prove that the shrinkage allowance on diameter is independent of Poisson's Ratio and $= d(t + t')/E$, where d is the diameter of the surfaces in contact.

CHAPTER XXIX

STRESSES DUE TO ROTATION

323. General Equations.[*]—We are now going to consider the stresses set up in circular discs and cylinders on account of rotation about their axes of symmetry. In general we shall not be able to find exact solutions which fit all the conditions of a particular problem, but by making suitable assumptions we can find solutions which are sufficiently accurate for all practical cases.

Let w = the weight of the material per unit volume.
ω = the angular velocity.
p_r = the radial tensile stress.
p_θ = the hoop tensile stress.
p_z = the longitudinal tensile stress.
u = the radial displacement of a point originally at radius r, and distant z from a plane of reference at right angles to the axis of rotation.
w = the axial displacement of the same point.

Clearly the stresses, strains and displacements will all be symmetrical about the axis of rotation.

The strains are given by

$$\left.\begin{aligned} e_r &= \text{radial strain} = \frac{\partial u}{\partial r} \\ e_\theta &= \text{hoop} \quad ,, \quad = \frac{u}{r} \\ e_z &= \text{axial} \quad ,, \quad = \frac{\partial w}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$e_{\theta z} = e_{r\theta} = 0, \text{ from symmetry.}$$

$$e_{rz} = \text{shear strain in axial planes} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \quad . \quad . \quad (2)$$

We shall assume throughout that the axis of z, i.e. the axis of rotation, is the direction of one principal stress; it then follows that the radial and hoop stresses are principal stresses, and that e_{rz} is zero. Therefore from (2)

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[*] We are concerned here only with steady rotation; for a treatment of the problem of the vibrations of a rotating disc, see a paper by R. V. Southwell, *Proc. Royal Soc.*, A., 1921.

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial r}$$

$$\therefore \frac{\partial^2 u}{\partial z^2} = -\frac{\partial}{\partial z}\cdot\frac{\partial w}{\partial r} = -\frac{\partial}{\partial r}\cdot\frac{\partial w}{\partial z},$$

i.e.

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\partial e_z}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also the stress-strain equations are:

$$\begin{cases} E\frac{\partial u}{\partial r} = p_r - \frac{p_\theta + p_z}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4) \\ E\frac{u}{r} = p_\theta - \frac{p_z + p_r}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5) \\ Ee_z = p_z - \frac{p_r + p_\theta}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6) \end{cases}$$

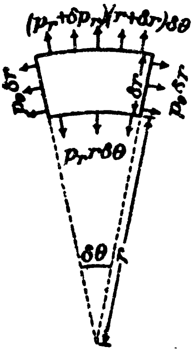

FIG. 362.

Now consider the motion of an element of the body.

Let the element have unit thickness in the direction of the axis of z, and for the rest let it be bounded by arcs of radius r and $r + \delta r$, and subtending an angle $\delta\theta$ at the axis, as shown in Fig. 362. The centrifugal force acting on the element is

$$\frac{w}{g}(r\delta\theta\,.\,\delta r)r\omega^2.$$

Hence, resolving radially we must have

$$(p_r + \delta p_r)(r + \delta r)\delta\theta + \frac{wr^2\omega^2}{g}\delta\theta\delta r - p_r r\delta\theta - 2p_\theta\delta r\frac{\delta\theta}{2} = 0$$

Simplifying this and rejecting the term $\delta p_r \delta r\,.\,\delta\theta$, we get

$$p_\theta - p_r - r\frac{\partial p_r}{\partial r} = \frac{wr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

324. Rotating Disc of Uniform Thickness.*—We shall first apply the above equations to the case of a flat circular disc, of uniform thickness. We shall assume that p_z is zero, that is, that the stresses are two-dimensional. This implies free axial strain.

Equations (4), (5) and (6) become

$$E\frac{\partial u}{\partial r} = p_r - \frac{p_\theta}{m}$$

$$E\frac{u}{r} = p_\theta - \frac{p_r}{m}$$

$$Ee_z = -\frac{1}{m}(p_r + p_\theta).$$

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* See also *The Effect of Axial Restraint on the Stress in a Rotating Disc*, W. G. Green, *Phil. Mag.*, June, 1926.

Eliminating u from the first two we get

$$p_r - \frac{p_\theta}{m} = \frac{\partial}{\partial r}\left(rp_\theta - \frac{rp_r}{m}\right) = r\frac{\partial p_\theta}{\partial r} + p_\theta - \frac{p_r}{m} - \frac{r}{m}\frac{\partial p_r}{\partial r}.$$

or

$$p_\theta - p_r + \frac{m}{m+1}r\frac{\partial p_\theta}{\partial r} - \frac{1}{m+1}r\frac{\partial p_r}{\partial r} = 0$$

Combining this with (7) we have

$$\frac{m}{m+1}\left(r\frac{\partial p_\theta}{\partial r} + r\frac{\partial p_r}{\partial r}\right) = -\frac{wr^2\omega^2}{g}$$

or

$$\frac{\partial}{\partial r}(p_\theta + p_r) = -\frac{m+1}{m}\cdot\frac{wr\omega^2}{g}$$

$$\therefore p_\theta + p_r = A - \frac{m+1}{2m}\cdot\frac{wr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad (8)$$

where A is, in general, *a function of* z.

Subtracting (7) from (8) we get

$$2p_r + r\frac{\partial p_r}{\partial r} = A - \frac{3m+1}{2m}\cdot\frac{wr^2\omega^2}{g}$$

Multiplying by r we get

$$\frac{\partial}{\partial r}(r^2p_r) = Ar - \frac{3m+1}{2m}\cdot\frac{wr^3\omega^2}{g}$$

$$\therefore r^2p_r = \frac{Ar^2}{2} - \frac{3m+1}{8m}\cdot\frac{wr^4\omega^2}{g} + B,$$

where B is also *a function of* z. Hence

$$p_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3m+1}{8m}\cdot\frac{wr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\text{A})$$

From (8) and (8A) we have

$$p_\theta = \frac{A}{2} - \frac{B}{r^2} - \frac{m+3}{8m}\cdot\frac{wr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad (9)$$

It remains to find A and B.

325. Case 1. Thin Solid Disc.—Suppose the disc is so thin that the variations of stress through the thickness can be neglected, and that the disc extends to the centre whilst the outside radius is r_2. Since the stresses cannot be infinite at the centre for all values of ω, it is clear that $B = 0$. We must also have $p_r = 0$ when $r = r_2$. Therefore

$$0 = \frac{A}{2} - \frac{3m+1}{8m}\cdot\frac{wr_2^2\omega^2}{g},$$

or

$$A = \frac{3m+1}{4m}\cdot\frac{wr_2^2\omega^2}{g}.$$

Substituting in (8A) and (9) gives

$$p_r = \frac{3m+1}{8m} \cdot \frac{w\omega^2}{g} \cdot (r_2^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$p_\theta = \frac{w\omega^2}{g}\left[\frac{3m+1}{8m} r_2^2 - \frac{m+3}{8m} r^2\right] \quad . \quad . \quad . \quad . \quad (11)$$

Both are a maximum at the centre, where each has the value

$$\frac{3m+1}{8m} \cdot \frac{w\omega^2 r_2^2}{g}.$$

Adopting the strain-energy theory of failure, the maximum safe speed is given by

$$p_r^2 + p_\theta^2 - \frac{2}{m} p_r p_\theta = f^2$$

where f is the elastic limit in simple tension. This gives

$$\omega^2 = \frac{gf}{wr_2^2} \cdot \frac{8m}{3m+1} \sqrt{\frac{m}{2(m-1)}} \quad . \quad . \quad . \quad . \quad (12)$$

326. Case 2. Thin Hollow Disc.—Again suppose we can neglect the variation of stress through the thickness of the disc. Let the disc be bounded by an internal radius r_1 and an external radius r_2. We must have $p_r = 0$ when $r = r_1$ or r_2. Substituting in (8A) and (9) we obtain two equations from which to find A and B; we find

$$A = \frac{3m+1}{4m} \cdot \frac{w\omega^2}{g} \cdot (r_2^2 + r_1^2)$$

$$B = -\frac{3m+1}{8m} \cdot \frac{w\omega^2}{g} \cdot r_2^2 r_1^2$$

Inserting these values in (8A) and (9) we get for the stresses

$$p_r = \frac{3m+1}{8m} \cdot \frac{w\omega^2}{g}\left[r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2\right] \quad . \quad . \quad . \quad (13)$$

$$p_\theta = \frac{3m+1}{8m} \cdot \frac{w\omega^2}{g}\left[r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \frac{m+3}{3m+1} r^2\right] \quad . \quad (14)$$

It will be seen that p_θ is a maximum at the inner radius, whilst p_r is a maximum where $r = \sqrt{r_1 r_2}$. When $r = r_1$ we have

$$p_\theta = \frac{3m+1}{4m} \cdot \frac{w\omega^2}{g}\left(r_2^2 + \frac{m-1}{3m+1} r_1^2\right).$$

If r_1 is very small, so that r_1^2 is negligible in comparison with r_2^2, this gives for the maximum value of p_θ

$$\frac{3m+1}{4m} \cdot \frac{w\omega^2 r_2^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is double the value at the centre of a solid disc. Thus the maximum hoop stress in a rotating disc is doubled by making a small hole at the centre. If the thickness of the disc be $2h$, and the origin be on the central plane, the stresses at a distance z from the middle plane are given by:

$$p_r = \frac{3m+1}{8m} \cdot \frac{w\omega^2}{g}\left[r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2\right] + \frac{(m+1)(h^2-3z^2)}{6m(m-1)} \cdot \frac{w\omega^2}{g} \qquad (16)$$

$$p_\theta = \frac{3m+1}{8m} \cdot \frac{w\omega^2}{g}\left[r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \frac{m+3}{3m+1} r^2\right] + \frac{(m+1)(h^2-3z^2)}{6m(m-1)} \cdot \frac{w\omega^2}{g} \qquad (17)$$

327. Rotating Circular Cylinder.—We suppose that the cylinder is rotating about its axis, and that its length is so great that the effects of the ends may be neglected over the greater part of the length. It follows that, except near the ends, the radial and hoop stresses will be the same for all cross sections; we shall also assume that e_z is constant (except near the ends), i.e. that cross sections remain plane.

From (5) we have

$$Eu = rp_\theta - \frac{r}{m}p_z - \frac{r}{m}p_r$$

$$\therefore E\frac{\partial u}{\partial r} = p_\theta - \frac{1}{m}p_z + r\frac{\partial p_\theta}{\partial r} - \frac{r}{m}\frac{\partial p_z}{\partial r} - \frac{p_r}{m} - \frac{r}{m}\frac{\partial p_r}{\partial r}$$

Hence, from (4)

$$(p_r - p_\theta)\left(1 + \frac{1}{m}\right) - r\frac{\partial p_\theta}{\partial r} + \frac{r}{m}\frac{\partial p_z}{\partial r} + \frac{r}{m}\frac{\partial p_r}{\partial r} = 0$$

Again, from (6), since e_z is constant,

$$\frac{\partial p_z}{\partial r} = \frac{1}{m} \cdot \frac{\partial p_r}{\partial r} + \frac{1}{m}\frac{\partial p_\theta}{\partial r}$$

Eliminating p_z from these two equations we get

$$(p_r - p_\theta)\left(1 + \frac{1}{m}\right) - r\frac{\partial p_\theta}{\partial r} + \frac{r}{m^2}\frac{\partial p_r}{\partial r} + \frac{r}{m^2}\frac{\partial p_\theta}{\partial r} + \frac{r}{m}\frac{\partial p_r}{\partial r} = 0.$$

Since p_r and p_θ are independent of z we can write the total, instead of the partial, differential sign, and we have, after a little simplification,

$$p_r - p_\theta + \frac{r}{m}\frac{dp_r}{dr} - r\frac{m-1}{m} \cdot \frac{dp_\theta}{dr} = 0 \quad . \quad . \quad . \quad (i)$$

From (7) and (i) we get

$$\frac{m-1}{m} \cdot r\frac{dp_r}{dr} + \frac{m-1}{m} r\frac{dp_\theta}{dr} + \frac{wr^2\omega^2}{g} = 0$$

$$\therefore \frac{d}{dr}(p_r + p_\theta) = -\frac{m}{m-1}\frac{wr\omega^2}{g}$$

$$\therefore p_r + p_\theta = A - \frac{m}{m-1} \cdot \frac{wr^2\omega^2}{2g} \quad . \quad . \quad . \quad (ii)$$

where A is a constant.

From (7) and (ii) we get

$$2p_r + r\frac{dp_r}{dr} = A - \frac{3m-2}{2(m-1)}\frac{wr^2\omega^2}{g}$$

Multiplying this by r we have

$$\frac{d}{dr}(r^2p_r) = Ar - \frac{3m-2}{2(m-1)}\cdot\frac{wr^3\omega^2}{g},$$

whence

$$p_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3m-2}{8(m-1)}\cdot\frac{wr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad (18)$$

Then, from (ii)

$$p_\theta = \frac{A}{2} - \frac{B}{r^2} - \frac{m+2}{8(m-1)}\cdot\frac{wr^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad (19)$$

As in the case of a disc, the values of A and B depend on whether the cylinder be solid or hollow.

328. Case 1. Solid Cylinder.—As the stresses cannot be infinite at the centre B must vanish. We must also have $p_r = 0$ on the outside, where $r = r_2$.

$$\therefore \frac{A}{2} = \frac{3m-2}{8(m-1)}\cdot\frac{wr_2^2\omega^2}{g}$$

Hence

$$p_r = \frac{3m-2}{8(m-1)}\frac{w\omega^2}{g}(r_2^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$p_\theta = \left[\frac{3m-2}{8(m-1)}r_2^2 - \frac{m+2}{8(m-1)}r^2\right]\frac{w\omega^2}{g} \quad . \quad . \quad . \quad (21)$$

At the centre

$$p_r = p_\theta = \frac{3m-2}{8(m-1)}\frac{w\omega^2 r_2^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If $m = \frac{10}{3}$, this gives $p_r = p_\theta = 0{\cdot}43w\omega^2r_2^2/g$, whereas in the case of a thin disc (p. 468) we get $p_r = p_\theta = 0{\cdot}41w\omega^2r_2^2/g$, so that the stresses differ but slightly.

If the ends are constrained so that e_z is zero, we have from (6)

$$p_z = \frac{1}{m}(p_r + p_\theta) = \left[\frac{3m-2}{4m(m-1)}r_2^2 - \frac{r^2}{2(m-1)}\right]\frac{w\omega^2}{g}. \quad . \quad (23)$$

If the ends are free we must have *

$$2\pi\int_0^{r_2} rp_z dr = 0$$

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* It must be noted that the condition $p_z = 0$ when $z = \pm h$ is not satisfied by this approximate solution, but the point is of no practical consequence. This also applies to (16) and (17), p. 469.

i.e.
$$\int_0^{r_2} r\left[Ee_z + \frac{1}{m}(p_r + p_\theta)\right]dr = 0$$

$$\int_0^{r_2}\left[Ee_z r + \frac{w\omega^2}{g}\left\{\frac{3m-2}{4m(m-1)}r_2^2 r - \frac{r^3}{2(m-1)}\right\}\right]dr = 0$$

$$Ee_z\frac{r_2^2}{2} = \left\{\frac{r_2^4}{8(m-1)} - \frac{(3m-2)r_2^4}{8m(m-1)}\right\}\frac{w\omega^2}{g} = -\frac{r_2^4}{4m}\cdot\frac{w\omega^2}{g}$$

$$\therefore\ Ee_z = -\frac{wr_2^2\omega^2}{2gm}.$$

Hence from (6)

$$p_z = -\frac{wr_2^2\omega^2}{2gm} + \frac{3m-2}{4m(m-1)}r_2^2\frac{w\omega^2}{g} - \frac{1}{2(m-1)}\cdot\frac{wr^2\omega^2}{g}$$

or
$$p_z = \frac{1}{4(m-1)}(r_2^2 - 2r^2)\frac{w\omega^2}{g} \quad . \quad . \quad . \quad . \quad (24)$$

At the centre this gives $p_z = wr_2^2\omega^2/4g(m-1)$, which, if $m = 10/3$, is $3wr_2^2\omega^2/28g$, which is one-quarter of the corresponding values of p_r and p_θ.

329. Case 2. Hollow Cylinder.—In this case B does not vanish and we must have $p_r = 0$ when $r = r_1$ or r_2. Hence, from (18) and (19),

$$\frac{A}{2} + \frac{B}{r_1^2} = \frac{3m-2}{8(m-1)}\cdot\frac{wr_1^2\omega^2}{g}$$
$$\frac{A}{2} + \frac{B}{r_2^2} = \frac{3m-2}{8(m-1)}\cdot\frac{wr_2^2\omega^2}{g}.$$

Solving these equations for A and B, and substituting back in (18) and (19), we get finally

$$p_r = \frac{w\omega^2}{g}\cdot\frac{3m-2}{8(m-1)}\left(r_1^2 + r_2^2 - \frac{r_1^2r_2^2}{r^2} - r^2\right) \quad . \quad . \quad . \quad (25)$$

$$p_\theta = \frac{w\omega^2}{g}\cdot\frac{3m-2}{8(m-1)}\left(r_1^2 + r_2^2 + \frac{r_1^2r_2^2}{r^2} - \frac{m+2}{3m-2}r^2\right) \quad . \quad (26)$$

and, when the ends are free,

$$p_z = \frac{w\omega^2}{g}\cdot\frac{1}{4(m-1)}(r_1^2 + r_2^2 - 2r^2) \quad . \quad . \quad . \quad . \quad (27)$$

It will be seen from the above that the results for a long cylinder do not differ greatly from those we found for a thin disc, when we assume that cross sections remain plane.

Example 1.—Calculate the maximum safe speed of a flat circular disc 12″ diameter with a hole 2″ diameter at the centre, the thickness of the disc being 1·5″. The elastic limit of the material in simple tension is 15 tons/in.^{2}, $m = 10/3$, and $w = 490$ lbs./ft.^{3}

We shall use the formulæ (16) and (17), p. 469.

$$r_1 = 1'' ;\quad r_2 = 6'' ;\quad h = 0{\cdot}75''.$$

Then

$$p_r = \frac{33}{80} \times \frac{490\,\text{lbs./ft.}^3.\,\omega^2}{32{\cdot}2\ \text{ft./sec.}^2}\left[37 - \frac{36}{r^2} - r^2\right]\text{ins.}^2$$

$$+ \frac{13(0{\cdot}562 - 3z^2)\ \text{in.}^2\ 490\ \text{lbs./ft.}^3.\omega^2}{140 \times 32{\cdot}2\ \text{ft./sec.}^2}$$

$$= \left[6{\cdot}29\left(37 - \frac{36}{r^2} - r^2\right) + 1{\cdot}41(0{\cdot}562 - 3z^2)\right]\omega^2.\frac{\text{lbs. sec.}^2\ \text{in.}^2}{\text{ft.}^4}$$

$$p_r = \left[6{\cdot}29\left(37 - \frac{36}{r^2} - r^2\right) + 1{\cdot}41(0{\cdot}562 - 3z^2)\right]\frac{\omega^2}{144^2}.\frac{\text{lbs. sec.}^2}{\text{in.}^2}$$

$$p_\theta = \left[6{\cdot}29\left(37 + \frac{36}{r^2} - \frac{19}{33}r^2\right) + 1{\cdot}41(0{\cdot}562 - 3z^2)\right]\frac{\omega^2}{144^2}.\frac{\text{lbs. sec.}^2}{\text{in.}^2}$$

Let us first examine the importance of the second ( ).

The hoop stress p_θ is greatest at the inner circumference where $r = 1''$ We then have

$$p_\theta(r = 1'') = \left[6{\cdot}29 \times 72{\cdot}4 + 1{\cdot}41(0{\cdot}562 - 3z^2)\right]\frac{\omega^2}{144^2}.\frac{\text{lbs. sec.}^2}{\text{in.}^2}$$

$$= \left[454 + (0{\cdot}793 - 42{\cdot}3z^2)\right]\frac{\omega^2}{144^2}.\frac{\text{lbs. sec.}^2}{\text{in.}^2}$$

Now the value of z ranges from 0 to $\pm\,0{\cdot}75''$, so that it appears that the second ( ) term is practically negligible, its contribution being about 2 per cent. in this case. We shall therefore take

$$p_r = \frac{6{\cdot}29\omega^2}{144^2}\left(37 - \frac{36}{r^2} - r^2\right)\text{lbs. sec.}^2/\text{in.}^2$$

$$p_\theta = \frac{6{\cdot}29\omega^2}{144^2}\left(37 + \frac{36}{r^2} - 0{\cdot}575r^2\right)\text{lbs. sec.}^2/\text{in.}^2$$

If we adopt the strain-energy criterion for failure, we must have

$$p_r^2 + p_\theta^2 - \frac{2}{m}p_r p_\theta = f^2$$

or

$$p_r^2 + p_\theta^2 - 0{\cdot}6p_r p_\theta = f^2$$

Substituting the above values of p_r and p_θ in this we get

$$\left(\frac{6{\cdot}29\omega^2}{144^2}\right)^2\left(0{\cdot}985r^4 - 81{\cdot}6r^2 + 1{,}956 + \frac{3{,}379}{r^4}\right) = f^2 \quad . \quad . \quad \text{(i)}$$

The left-hand side is a maximum when

$$3{\cdot}94r^3 - 163{\cdot}2r - \frac{13{,}480}{r^5} = 0$$

that is

$$r^8 - 41{\cdot}5r^6 - 3{,}420 = 0.$$

This equation has no positive root between 0 and 6.

The greatest value of the left-hand side of (i) is when $r = 1''$, when we get

$$5{,}215 \times \left(\frac{6{\cdot}29\omega^2}{144^2}\right)(\text{lbs. sec.}^2/\text{in.}^2)^2 = (15 \times 2{,}240\ \text{lbs./in.}^2)^2$$

which gives $\omega = 1{,}240$ radians/sec., equivalent to 11,850 r.p.m.

Example 2.—Calculate the safe speed of the above disc if there were no hole in the centre.

From (12), p. 468,

$$\omega^2 = \frac{32{\cdot}2 \text{ ft./sec.}^2 \times 15 \times 2{,}240 \text{ lbs./in.}^2}{490 \text{ lbs./ft.}^3 \times 36 \text{ in.}^2} \times \frac{80}{33} \times \sqrt{\frac{5}{7}}$$

$$= \frac{32{\cdot}2 \times 15 \times 2{,}240 \times 144^2}{490 \times 36} \times \frac{80}{33} \times 0{\cdot}845 \times \frac{1}{\text{secs.}^2}$$

$$\omega = 1{,}620 \text{ radians/sec.}$$
$$= 15{,}500 \text{ r.p.m.}$$

330. Disc of Varying Thickness.*—Since we have seen that the stresses in a long cylinder do not differ greatly from those in a thin disc of the same radial dimensions, we shall be justified in assuming in the present case that there is no axial stress, and in neglecting the variation of p_r and p_θ through the thickness of the disc. Equation (7) will be slightly modified in this case.

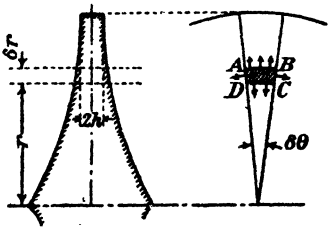

FIG. 363.

Let $2h$ be the thickness at radius r, and consider an element $ABCD$ as shown in Fig. 363. The area of the curved surface DC is $2hr\delta\theta$, so that the total radial stress across it is $2hp_r r\delta\theta$. The area of the curved surface AB is $2(h + \delta h)\ (r + \delta r)\ \delta\theta$,† and the total radial stress across it is $2(h + \delta h)(r + \delta r)(p_r + \delta p_r)\delta\theta$.

The centrifugal force acting on the element is $\frac{\text{w}\,.\,2h\,.\,r^2\delta\theta\delta r\omega^2}{g}$. Hence we have, resolving radially,

$$2(h + \delta h)(r + \delta r)(p_r + \delta p_r)\delta\theta - 2hp_r r\delta\theta - 2h\,.\,\delta r\,.\,p_\theta\,.\,\delta\theta$$
$$+ \frac{2h\text{w}r^2\omega^2\delta\theta\,.\,\delta r}{g} = 0.$$

This reduces to

$$p_r - p_\theta + r\frac{dp_r}{dr} + \frac{rp_r}{h}\,.\,\frac{dh}{dr} = -\frac{\text{w}\omega^2r^2}{g} \quad . \quad . \quad . \quad \text{(i)}$$

Again, since p_z is supposed zero, we have from (4) and (5),

$$p_r - \frac{1}{m}p_\theta = E\frac{du}{dr} \text{ and } p_\theta - \frac{1}{m}p_r = E\frac{u}{r}.$$

Solving for p_r and p_θ we get

$$p_r = \frac{mE}{m^2 - 1}\left(\frac{u}{r} + m\frac{du}{dr}\right)$$

$$p_\theta = \frac{mE}{m^2 - 1}\left(m\frac{u}{r} + \frac{du}{dr}\right).$$

---

* For experiments see *Engineering*, May 9, 1913, and *Trans. American Soc. M.E.*, May 17, 1912; also *Rotating Discs of Conical Profile*, *Engineering*, Jan. 26, 1923.

† In the figure δh is negative.

Substituting these values in (i), we get

$$\frac{d^2u}{dr^2}+\left(\frac{1}{h}\cdot\frac{dh}{dr}+\frac{1}{r}\right)\frac{du}{dr}+\frac{1}{r}\left(\frac{1}{mh}\cdot\frac{dh}{dr}-\frac{1}{r}\right)u=-\frac{w\omega^2}{g}\cdot\frac{m^2-1}{m^2E}\cdot r \qquad (28)$$

This is a differential equation for u, and we cannot proceed unless we know h in terms of r.

A case which readily admits of integration is when

$$h=\frac{c}{r^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

We have $\frac{dh}{dr}=-\frac{nc}{r^{n+1}}$, and (28) becomes

$$r^2\frac{d^2u}{dr^2}-(n-1)r\frac{du}{dr}-\frac{m+n}{m}u=-\frac{w\omega^2}{g}\cdot\frac{m^2-1}{m^2E}r^3 \qquad \text{(ii)}$$

This is a homogeneous equation. To find the complementary function assume $u=Ar^a$. Substitute in (ii), making the right-hand side zero, and we get

$$a(a-1)-(n-1)a-\frac{n+m}{m}=0$$

or

$$a^2-na-\frac{n+m}{m}=0.$$

If the roots of this be a_1 and a_2, the complementary function is $C_1ra_1+C_2ra_2$, where C_1 and C_2 are constants. A particular integral is

$$-\frac{w\omega^2}{g}\cdot\frac{m^2-1}{mE}\cdot\frac{r^3}{8m-3mn-n}$$

Hence the complete solution of (ii) is

$$u=C_1r\alpha_1+C_2r\alpha_2-\frac{w\omega^2}{g}\cdot\frac{m^2-1}{mE}\cdot\frac{r^3}{8m-3mn-n} \quad . \quad . \qquad (30)$$

The constants C_1 and C_2 have to be found from the boundary conditions.* For instance, consider a wheel of the type shown in section in Fig. 364, where the curved portions AB are given by an equation of the form (29). There are three portions to consider: the rim, the part AB, and the hub. The general expressions for the stresses in each portion involve two constants of integration, so that there are six constants to be determined. The necessary six equations are formed as follows.

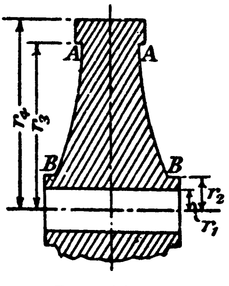

Fig. 364.

(1) When $r=r_1$, $p_r=0$.
(2) When $r=r_4$, $p_r=0$.
(3) When $r=r_2$, u is the same for the boss and for AB.
(4) When $r=r_3$, u „ „ „ „ AB and the rim.

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* A useful set of curves for estimating stresses in turbine wheels was given in *Engineering*, Aug. 30, 1912; see also Aug. 8, 1918, and Aug. 3, 1917.

(5) The total radial stress across the section AA of the disc part is equal to the total radial stress across the section AA of the rim.

(6) Similarly the total radial stress across BB is the same for the disc and the hub.

If the outer surface of the rim is not free from stress, for instance if it be acted on by the radial pull of a series of turbine blades, we can approximately replace (2) above by taking the load due to the blades as uniformly distributed round the circumference. This will be sufficiently correct for estimating the stresses in the disc part of the wheel.

The stresses in ordinary disc flywheels with heavy rims can be estimated in a similar manner.*

EXAMPLES XXIX

1. If a rotating disc is to be shaped so that the radial and hoop stresses are everywhere equal to f, show that the radial section should be such that

$$h \text{ varies as } \varepsilon^{-\mathrm{w}r^2\omega^2/2gf}.$$

2. A circular disc of uniform thickness is made in two concentric parts, the division occurring along a circle of radius a. The outer portion is shrunk on so as to exert a pressure on the inner. Prove that the hoop tension at the inside and outside of the disc will be equalized for an angular velocity ω, if the shrinkage pressure, when the disc is stationary, has the value

$$\frac{\mathrm{w}\omega^2}{g}\cdot\frac{(m+1)(a^2-r_1^2)(r_2^2-a^2)}{4ma^2},$$

in the notation of this chapter. (Mech. Sc. Trip., B., 1922.)

3. A thin circular disc of external radius r_2 is forced on to a rigid shaft of radius r_1; prove that when the speed is ω the pressure between the disc and the shaft will be reduced by

$$\frac{\mathrm{w}\omega^2}{g}\cdot\frac{(r_2^2-r_1^2)\{(3m+1)r_2^2+(m-1)r_1^2\}}{4\{(m+1)r_2^2+(m-1)r_1^2\}}.$$

(Mech. Sc. Trip., B., 1923.)

4. A steel turbine disc is to be designed so that between radii of 10″ and 16″ the radial and circumferential stresses are to be constant and both equal to 4 tons/in.^{2}, when running at 3,000 r.p.m. If the axial thickness is ½″ at the outer edge of this zone, what should it be at the inner edge? Assume $\mathrm{w} = 490$ lbs./ft.^{3} (R.N.C., Greenwich, 1923.)

5. A circular saw 0·2″ thick, 36″ diameter, is secured upon a 4″ shaft. The steel of which the saw is composed has a weight of 500 lbs./ft.^{3}, and $m = 3{\cdot}5$. Determine the permissible speed if the allowable hoop stress is 35,000 lbs./in.^{2}, and find the maximum value of the radial stress. (Mech. Sc. Trip., B., 1909.)

6. A bronze gyro-wheel has a moment of inertia = 10 lbs. ft.^{2}, and can be run with safety at 3,000 r.p.m. A larger steel wheel of similar form is required to give an angular momentum of 10^6 lbs. ft.^{2}/secs. at 1,800 r.p.m. In what ratio must all the linear dimensions be increased, and what elastic limit is required so that the steel wheel may have the same factor of safety? Weight of bronze = 530 lbs./ft.^{3}; weight of steel = 490 lbs./ft.^{3}; elastic limit of bronze = 5 tons/in.^{2} (R.N.C., Greenwich, 1923.)

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* For a method of calculating the stresses in flywheels with spokes, see an article by Prof. A. J. Sutton Pippard, *Inst. Mech. Eng.*, 1923.

CHAPTER XXX

THE TORSION OF NON-CIRCULAR SHAFTS

331. Physical Discussion.—In the case of the torsion of a circular rod of uniform section we found that a strain, such that radii of all cross sections perpendicular to the axis remain straight, and such that radii originally parallel undergo a relative rotation proportional to the distance between the sections containing them, whilst the sections remain plane, could be maintained by the sole action of suitable twisting couples applied to the ends of the rod. We found also that the stress system consists only of shear stresses in the cross sections in directions perpendicular to the radii, accompanied by the equal complementary shear stresses in axial planes, and that the intensities of these stresses, at any point, are proportional to the distance of the point from the axis of the rod.

When the rod is not of circular cross section, the state of strain described above cannot be maintained by the action of terminal twisting couples alone. In Fig. 365 let AB be a portion of the boundary of a cross section of the rod, and let O be the centroid of the section. Consider any point P on the boundary. With the supposed state of strain, the strain at P is a shear in a direction at right angles to OP; let it be represented by PS. Then the shear PS can be resolved into shears PM in the direction of the tangent to AB at P, and PN along the normal. The shear in the direction PM demands shearing stresses in the plane of the section and parallel to the tangent, accompanied by the complementary shear stress in a plane perpendicular to the section and passing through the normal NPN'; these stresses correspond with those which occur in the case of a circular rod. The shear in the direction PN demands shearing stresses in the plane of the section and parallel to the normal, accompanied by the complementary shear stress at right angles to this plane and in the tangent plane to the boundary of the rod. The production of these stresses on the bounding surface requires the application of external forces tangential to the surface. Thus the assumed state of strain cannot be maintained by the action of terminal couples only.

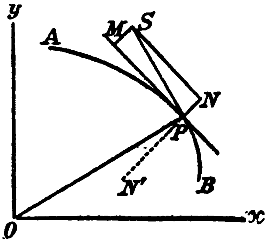

Fig. 365.

We conclude that, if this system of surface forces is absent, the cross sections of the rod cannot remain plain; in other words, the strain at P will involve displacements parallel to the axis of the rod.

Now it is clear that the distribution of stress and strain over the cross section will be the same for all cross sections, so that, if we take the axis of the rod as axis of z, the displacement w of the point (x, y) in a cross section will be independent of z, and it may be expected to be proportional to the twist per unit length.

Also, since the strains are constant along the rod, the displacements (u, v) of a point (x, y), parallel to the axes of x and y, will be proportional to the distance of the section under consideration from the section of reference, that is proportional to z.

332. Mathematical Analysis.—The general equations for the torsion of a rod lead to so few useful results that we do not give them here.* It will suffice to quote the following results:—

(i) *Elliptic Section.*—If the cross section of the rod be an ellipse, whose semi-axes are a and b $(a > b)$, the stresses are given by

$$q_{yz} = \frac{2b^2x}{2^2 + b^2}C\theta\,; \quad q_{zx} = -\frac{2a^2y}{a^2 + b^2}C\theta.$$

The maximum shearing stress occurs at the ends of the minor axis and is given by

$$q_{max} = \frac{2a^2bC\theta}{a^2 + b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The relation between torque and twist is

$$T = \frac{\pi a^3b^3C\theta}{a^2 + b^2} = CK\theta, \text{ say}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where θ is the twist per unit length, and K $[= \pi a^3b^3/(a^2 + b^2)]$ is called the *torsion constant for the section.*

The cross sections do not remain plane, alternate quadrants being depressed and elevated.

If the section be hollow, the boundaries being two similar and similarly situated ellipses, i.e. if their axes, a_1, b_1, a_2, b_2, are such that

$$a_1/b_1 = a_2/b_2,$$

we have

$$T = C\theta\left(\frac{a_1^3b_1^3}{a_1^2 + b_1^2} - \frac{a_2^3b_2^3}{a_2^2 + b_2^2}\right). \quad . \quad . \quad . \quad . \qquad (3)$$

$$q_{max} = \frac{2a_1^2b_1C\theta}{a_1^2 + b_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

(ii) *Triangular Section.*—If the section be an equilateral triangle of side a, the torsion constant, K, is $a^4.\sqrt{3}/80$. The greatest shearing stress occurs at the middle points of the sides and is given by $20T/a^3$.

(iii) *Rectangular Shaft.*—When the section is rectangular the results cannot be expressed so simply. The torsion constant, K, is given by $K = ab^3\beta$, where β is a coefficient depending on the ratio of the sides. Let the lengths of the sides be $2a$ and $2b$ $(b < a)$.

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* The reader is referred to Love's *Theory of Elasticity,* Prescott's *Applied Elasticity,* or Kelvin and Tait's *Natural Philosophy.*

The shearing stresses at the middle points of the sides of the boundary are given by

$$\left.\begin{aligned} q_1 &= \gamma_1 bC\theta \text{ at the middle of the long side} \\ q_2 &= \gamma_2 aC\theta \quad ,, \qquad ,, \qquad ,, \quad \text{short } ,, \end{aligned}\right\} \quad . \quad . \quad . \qquad (5)$$

The constants β, γ_1 and γ_2 have the following values, calculated by St. Venant *:

a/b	1·0	1·1	1·2	1·25	1·3	1·4	1·5	1·6	1·7	1·75
β	2·249	2·464	2·658	2·748	2·833	2·990	3·132	3·260	3·375	3·429
γ_1	1·351	1·440	1·518	1·553	1·585	1·644	1·695	1·739	1·777	1·793
γ_2	1·351	—	—	1·138	—	—	0·9708	0·9149	—	0·8410
a/b	1·80	2·0	2·25	2·5	3	4	5	6	8	20
β	3·479	3·659	3·842	3·990	4·213	4·493	4·662	4·773	4·913	5·165
γ_1	1·809	1·860	1·905	1·936	1·971	1·994	1·999	2·000	2·0	2·
γ_2	—	0·7395	—	0·5935	—	0·3712	0·2970	—	0·1860	0·0734

St. Venant also gave the following empirical formulæ for the constants, which agree with the exact values within 4 per cent.:

$$\left.\begin{aligned} \beta &= \frac{16}{3} - 3{\cdot}36\frac{b}{a}\left(1 - \frac{1}{12}\frac{b^4}{a^4}\right) \\ \gamma_1 &= \frac{3}{8}\left(1 + 0{\cdot}6\frac{b}{a}\right)\beta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

This completes the useful results which have been obtained by methods of pure analysis; the problem has been solved for certain other sections,† but the solutions are not readily adaptable to calculation. In all cases of irregular sections, other than those given above, it is preferable to use one of the other methods to be given presently.

St. Venant gave an empirical formula for finding the angle of twist in the case of symmetrical section with no concavities in the boundary, $\theta = 40JT/CS^4$, where J is the moment of inertia of the section about an axis through its centroid and perpendicular to its section. This formula has very little practical value as it does not help us to find the stresses.

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* A more complete table will be found in Todhunter and Pearson's *History*.

† *Quart. J. of Math.*, 1879. *Cambridge Phil. Soc. Proc.*, 1893. Love's *Theory of Elasticity*. *Messenger of Mathematics*, 1878 and 1880. See Todhunter and Pearson's *History*.

333. Torsion of Thin Tubes of any Section.—When the member undergoing twist is a straight tube of uniform cross section, the walls being thin compared with the least diameter of the tube, we may employ the following method, due to C. Batho.*

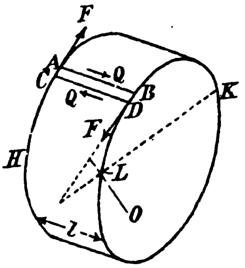

FIG. 366.

Consider the tube shown in Fig. 366, LBK and HCA being parallel cross sections, separated by an axial distance l. Let AB and CD be two sections of the tube parallel to the axis, such that $CA = DB = \delta s$.

Let F = the total shear stress on CA or BD.
„ Q = „ „ „ „ AB or CD, supposed constant.

Then for the equilibrium of the element $ABCD$ we must have

$$Q \, . \, \delta s = Fl \text{ and therefore } F = \frac{Q \, . \, \delta s}{l} \quad . \quad . \quad . \quad \text{(i)}$$

Now let us find the total couple due to F acting all round the tube. Take any point O in the plane of the cross section, and let h denote the length of the perpendicular from O on to BD. Then the moment of F about O is Fh, and the couple T, due to F acting all round the boundary is given by

$$\begin{aligned} T &= \Sigma Fh \\ &= \frac{Q}{l}\Sigma h.\delta s = \frac{Q}{l}\int h \, . \, ds, \end{aligned}$$

the integral being taken all round the boundary. The value of this integral is $2S$, where S denotes the area $LBKL$. Hence we have

$$T = \frac{Q}{l} \times \text{twice the area enclosed by the section.}$$

$$\therefore Q = \frac{lT}{2S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where T is the applied torque.

Let q be the average shearing stress on a longitudinal section such as AB, and let t be the thickness of the tube, then $Q = qlt$. Therefore, from (7)

$$qlt = \frac{lT}{2S}$$

$$\therefore q = \frac{T}{2St} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which gives the shearing stress in terms of the applied torque. The twist can be found as follows: Let θ be the twist per unit length, measured

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[*] *Engineering*, Oct. 15, 1915, and Nov. 24, 1916.

in radians as usual. Then the work done by the applied couple is $\frac{1}{2}T\theta l$; equating this to the strain energy we have

$$\tfrac{1}{2}T\theta l = \int \frac{q^2}{2C} ltds$$

the integral extending round the whole boundary. Therefore

$$\theta = \frac{1}{CT}\int q^2 tds$$

Hence, from (8),

$$\theta = \frac{T}{4S^2C}\int \frac{ds}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If t be constant this gives

$$\theta = \frac{sT}{4S^2Ct} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where s denotes the perimeter of the section.

334. Solid Sections of Irregular Shape[*].—We have now dealt with all those sections which have yielded to direct mathematical treatment in such a way as to produce practical formulæ. Such sections as **H**, **T**, **L**, **[**, etc., have not yet been brought within the range of mathematical analysis, and, prior to 1917, the engineer had not been in a position to make calculations on the torsion of rods of irregular section. Members designed as pure torsion members are not usually made with these sections, but we may sometimes require to calculate the stresses in a member which, though primarily designed for thrust or bending, is yet subjected to torsion; again, in order to apply the results of Chapter XXIII we must be able to calculate the torsion constant. The most practical method of dealing with solid rods of any section whatsoever is that given by A. A. Griffiths and G. I. Taylor,[†] which gives results of considerable accuracy by rapid drawing office processes. The method is based on the mathematical similarity which exists between the torsion problem and the problem of finding the deflection of a thin membrane under pressure. We shall not discuss the analogy here, and the reader is referred to the original paper for a description of the experimental methods. The formulæ and rules given below are based on the researches of Griffiths and Taylor, were first given by them, and have been found to give exceedingly accurate results.

We shall consider first the calculation of the angle of twist, which presents less difficulty than the estimation of the stresses. For any section we can write $T = CK\theta$, where K is a constant which depends only on the shape and dimensions of the section. The quantity K is of four dimensions in length, and we can write, in the case of a circle, $K = \frac{1}{2}Sr^2$, where S is the area and r the radius of the section. In general, let us take

$$K = \tfrac{1}{2}Sk^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

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[*] See also footnote, p. 105, and *Engineering*, June 24, 1927, for a description of the work of Prandtl and Nadai and the use of powder allowed to fall on a plate the shape of the cross-section.

[†] *Proc. Inst. Mech. Engineers*, 1917, ii; Aeronautical Research Committee, *Reports and Memoranda*, Nos. 333, 334.

and call k the "equivalent torsional radius" of the section. To determine the twist due to a given torque we must find k for the section under consideration.

Now projecting corners add very little to the torsional stiffness of a section, for instance, we can easily show from the formulæ of § 332 that for an equilateral triangle k is only 10 per cent. greater than it is for the inscribed circle, whilst the area of the triangle is 65 per cent. greater. Consequently the first step towards finding k for any given section is to round off any corners there may be. Let us consider the quadrilateral section shown in Fig. 367. We must construct a new figure by rounding off each corner A, B, etc., with an arc of suitable radius. The radius of this arc depends upon the angle through which the tangent to the boundary must be turned in passing round the corner; it also depends upon the radius of the largest circle which can be drawn within the boundary, touching it at more than two points, as shown in Fig. 367. If b be the radius of this circle, and r the radius for rounding off any particular corner, whilst a denotes the angle through which the tangent passes in going round the corner, then r/b is given by the following table or by Fig. 368.

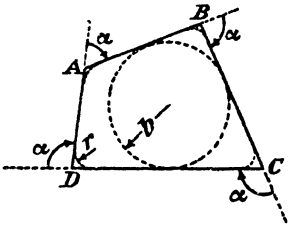

Fig. 367.

$\frac{a^\circ}{180^\circ}$	$\frac{r}{b}$	$\frac{a^\circ}{180^\circ}$	$\frac{r}{b}$
0·0	1·000	0·6	0·375
0·1	0·930	0·7	0·270
0·2	0·850	0·8	0·210
0·3	0·750	0·9	0·170
0·4	0·625	1·0	0·155
0·5	0·500	—	—

In this way a new figure is obtained, with all outward corners rounded off.

Let S_1 = the area of this new figure,
and P_1 = the perimeter of this new figure.
Then a first approximation to the value of k is

$$\mathrm{k} = \frac{2\mathrm{S}_1}{\mathrm{P}_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

A second approximation is obtained as follows :—

Let P = the perimeter of the original section, and S its area,
„ $h = \frac{2S}{P}$
„ b = the radius of the largest inscribed circle as above.

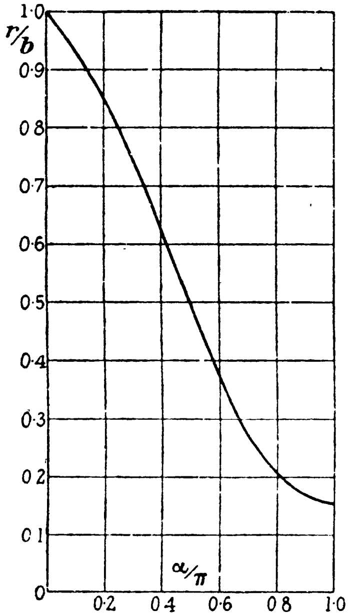

Fig. 368.

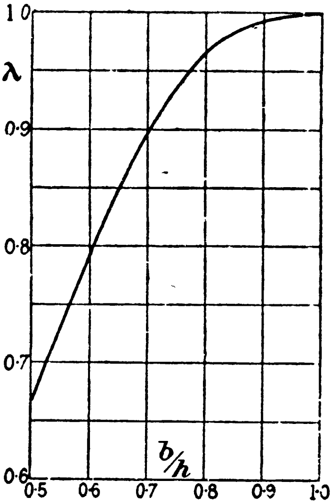

Fig. 369.

Then the value of k^2 found from (ii) must be multiplied by a factor λ, which is given by the following table, or by Fig. 369.

$\frac{b}{h}$	λ	$\frac{b}{h}$	λ
1·0	1·000	0·70	0·897
0·95	0·998	0·65	0·848
0·90	0·994	0·60	0·793
0·85	0·984	0·55	0·732
0·80	0·966	0·50	0·667
0·75	0·938	—	—

We have, finally,

$$K = \tfrac{1}{2}\lambda S\left(\frac{2S_1}{P_1}\right)^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

In the case of sections in which we can draw more than one "maximum" inscribed circle—i.e., more than one circle touching the boundary at three points—special treatment is necessary. Such cases arise with **H** section or channel section beams. These sections must be divided into component "simple" sections, and the value of K found by treating each part separately and adding the results: the above treatment is applied to each part, but the *perimeter of each component must be under-*

stood to include only that portion which forms part of the perimeter of the original section. Fig. 370 illustrates the manner in which sections are divided up: the **I** section is divided into seven components, and, in calculating K for the component A, only the part shown thick would be included in estimating P_1 and P. The rules for drawing the dividing lines are as follows: Imagine a circle of varying radius, always touching the boundary at at least two points, to move inside the section. Then there will be some positions when the circle has three or more points of contact, and between each pair of such positions there will be a position of the circle where its radius is a minimum: the division lines are to be drawn through the points of contact of these minimum circles. When there are long parallel portions, such as the web of an **I** beam, the division lines should be drawn at a distance from the commencement of the parallel portion equal to half the thickness; the tapering flanges of **I** beams should be treated in the same way. The manner of applying these rules should be clear from Fig. 370.

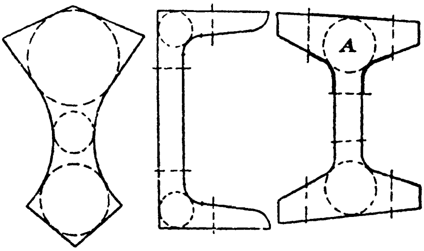

FIG. 370.

In the case of long narrow sections, the value of K may be evaluated from the formula

$$K = \frac{\dfrac{1}{3}\displaystyle\int_0^l y^3 ds}{1 + \dfrac{4}{3Sl^2}\displaystyle\int_0^l y^3 ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where y is the width of the section, measured perpendicular to the median line, at a distance s from one end, measured along the median line, and l is the length of the median line. This formula is exact for all ellipses, and is a very close approximation for all sections of a similar type down to a fineness ratio of two, at least. The integral can be evaluated in any given case by plotting y^3 against x and finding the area of the curve.

The calculation of the stresses cannot be formulated quite so explicitly as the calculation of K and the angle of twist. We can, however, arrive at a very good idea of the distribution and magnitude of the stresses from the following considerations.

(i) The *mean* value of the stress round the boundary of any section or any component of a compound section, is accurately given by $2C\theta S/P$.

(ii) The maximum stress occurs at or near one of the points of contact of the largest inscribed circle. An exception may occur if, at some other part, the boundary is more concave than at these points of contact. When there are no re-entrant angles the maximum stress is usually given with sufficient accuracy by the formula

$$q = \frac{2bC\theta}{1 + \frac{\pi^2 b^4}{S^2}}\left[1 + 0{\cdot}15\left(\frac{\pi^2 b^4}{S^2} - \frac{b}{\rho}\right)\right] \quad . \quad . \quad . \quad (15)$$

where b is the radius of the maximum inscribed circle, and ρ is the radius of curvature of the boundary at the point in question.

For compound sections the formula may be applied to each component separately.

Where the boundary is concave—i.e., where ρ is negative—the following formula is better :

$$q = \frac{2bC\theta}{1 + \frac{\pi^2 b^4}{S^2}}\left[1 + \left\{0{\cdot}118 \log_e\left(1 - \frac{b}{\rho}\right) - 0{\cdot}238\frac{b}{\rho}\right\} \tanh \frac{2a}{\pi}\right] \quad (16)$$

where a is the angle turned through by the tangent in passing round the re-entrant portion.

(iii) The stress at any point of the boundary is never less than the boundary stress in a circular shaft, under the same twist, the radius of the shaft being equal to that of the inscribed circle which touches the boundary at the point in question.

Also it is never greater than twice this quantity unless the boundary is concave.

(iv) At a sharp corner projecting inwards the stress will be very high.

(v) At a sharp corner projecting outwards the stress will be zero.

Example 1.—As an illustration of the method let us first take an equilateral triangle as the section of the rod, since we can compare the results with those obtained by exact analysis.

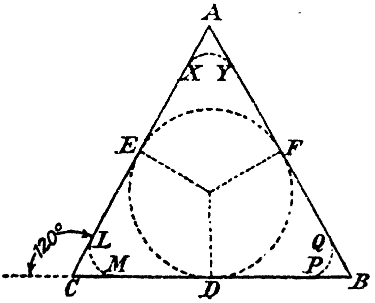

FIG. 371.

In Fig. 371 let ABC be an equilateral triangle, the length of each side being a.

The largest circle within the section, which touches the boundary at more than two points, is the inscribed circle of the triangle, i.e. the circle DEF, the radius of which is

$$b = \frac{a}{2}\tan 30° = 0{\cdot}2887a.$$

At each corner $a = 120°$, so that $a/180° = 0{\cdot}667$. From Fig. 368 we find $r/b = 0{\cdot}3$, so that $r = 0{\cdot}0866a$. With this radius we round off the corners A, B, C by the arcs XY, PQ, LM. Then, by measurement with a planimeter, or by calculation, we find

S_1 = area $LXYQPML$ $= 0{\cdot}4176a^2$.
P_1 = perimeter $LX \ldots ML = 2{\cdot}644a$.

Hence our first approximation gives

$$k = \frac{0{\cdot}835a^2}{2{\cdot}644a} = 0{\cdot}3159a.$$

The area of the triangle ABC is $S = 0{\cdot}433a^2$, and the perimeter is $P = 3a$. Therefore $h = \dfrac{0{\cdot}866a^2}{3a} = 0{\cdot}2887a$, so that $b/h = 1$. Thus we can take the above value of k as correct, so that

$$K = \tfrac{1}{2}Sk^2 = 0{\cdot}02160a^4.$$

The correct value of K is $a^4\sqrt{3}/80$ (p. 477) $= 0{\cdot}02165a^4$, so that in this instance the error is $-\tfrac{1}{4}$ per cent. to four significant figures.

Next let us apply formula (15) to calculate the maximum shearing stress. We have

$$\frac{b^2}{S} = 0{\cdot}192\,; \quad \frac{\pi^2 b^4}{S^2} = 0{\cdot}363\,; \quad 0{\cdot}15\frac{\pi^2 b^4}{S^2} = 0{\cdot}0545.$$

Also, at the point of contact of the inscribed circle $\dfrac{1}{\rho} = 0$. Hence

$$q = \frac{0{\cdot}578aC\theta}{1{\cdot}363} \times 1{\cdot}0545 = 0{\cdot}448aC\theta.$$

The correct value (p. 477) is

$$q = \frac{20T}{a^3} = \frac{0{\cdot}433a^4C\theta}{a^3} = 0{\cdot}433aC\theta.$$

Thus the formula (15) over-estimates the maximum shearing stress by only 3 per cent.

Example 2.—Calculate the torsion constant K for a thin tube of circular cross section, the wall of which has a narrow slit parallel to the axis.

In this case we use formula (14), p. 483; let a be the mean radius of the tube, and t the thickness. Then y is constant and equal to t; l is the perimeter of the tube and equals $2\pi a$; hence

$$\int_0^l y^3 ds = 2\pi a t^3.$$

Also $S = 2\pi r t$. Thus (14) gives

$$K = \frac{\tfrac{2}{3}\pi a t^3}{1 + \dfrac{4 \times 2\pi a t^3}{6\pi a t \times 4\pi^2 a^2}} = \frac{\tfrac{2}{3}\pi a t^3}{1 + \dfrac{t^2}{3\pi^2 a^2}} = \tfrac{2}{3}\pi a t^3,$$

approximately, since t/a is supposed small.

Example 3.—Calculate the torsion constant for a thin channel section having flanges of length a and thickness t_1, and web of depth b and thickness t_2, using formula (14), p. 483.

For each flange we have

$$l = a\,; \quad \int_0^l y^3 ds = a t^3_1\,; \quad S = a t_1\,;$$

whilst for the web we have

$$l = b\,; \quad \int_0^l y^3 ds = b t_2^3\,; \quad S = b t_2.$$

For the whole section, then,

$$l = 2a + b\,; \quad \int_0^l y^3 ds = 2a t_1^3 + b t_2^3\,; \quad S = 2a t_1 + b t_2.$$

$$\frac{4}{3Sl^2}\int_0^l y^3 ds = \frac{4(2a t_1^3 + b t_2^3)}{3(2a t_1 + b t_2)(2a + b)^2}, = n, \text{ say.}$$

Hence $$K = \frac{\tfrac{1}{3}(2a t_1^3 + b t_2^3)}{1 + n},$$

and usually we can take

$$K = \tfrac{1}{3}(2at_1^3 + bt_2^3),$$

when the thickness is very small compared with a or b.

Example 4.—Calculate the angle of twist and the maximum shearing stress in a beam whose section is shown in Fig. 372, subjected to a twisting moment T.

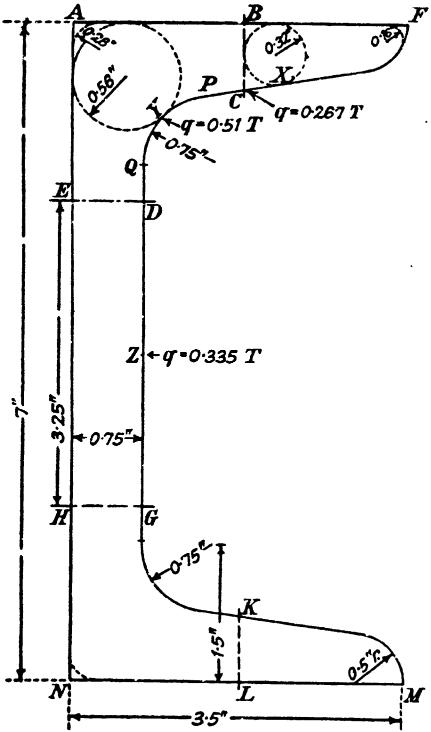

FIG. 372.

The first step is to divide the section into the five areas $ABCDE$, BCF, $HGKL$, KLM, $EDGH$. The division lines are found thus: the points P and Q mark the beginnings of the straight portions of the boundary, then PC is made equal to half the thickness of the flange at P and QD is made equal to half the thickness of the web at Q. The lines HG and KL are located in the same way.

We next round off any sharp corners: to do this we must first draw the largest possible inscribed circle in each portion. These circles are drawn by trial and are shown in the figure. The radii of the circles for rounding off the corners A and F are found as in Example 1, p. 484. For $ABCDE$ we have $b = 0{\cdot}56''$; at A, $\dfrac{a}{\pi} = 0{\cdot}5$ which gives $r/b = 0{\cdot}5$, so that $r = 0{\cdot}28''$ is the radius for the corner A; similarly $r = 0{\cdot}16''$ is the radius for the corner F.

The following table shows the remainder of the calculations for finding K:

Part.	S_1	P_1	S	P	$h = 2S/P$	b	$\dfrac{b}{h}$	λ	$\dfrac{2S_1}{P_1}$	K
	in.^{2}	ins.	in.^{2}	ins.	ins.	ins.			ins.	ins.^{4}
FBC	0·87	3·55*	0·91	3·59	0·506	0·32	0·63	0·845	0·502	0·097
MKL	—	—	—	—	—	—	—	—	—	0·097
$ABCDE$	2·26	5·51*	2·29	5·67	0·81	0·56	0·69	0·89	0·82	0·760
$HGKLN$	—	—	—	—	—	—	—	—	—	0·760
$EDGH$	2·43	6·48	2·43	6·48	0·75	0·375	0·5	0·667	0·75	0·455

Total $K = 2{\cdot}169$

---

* It must be remembered that only the portions of the boundary which belong to the whole section are included in estimating P and P_1; thus for the part $ABCDE$, $P = EA + AB + CPQD$.

We shall take $K = 2{\cdot}17$ ins.^{4}
Hence the angle of twist per unit length of rod is given by

$$\theta = \frac{T}{2{\cdot}17C} = 0{\cdot}46T/C, \text{ radians.}$$

We next calculate the shear stresses in terms of θ.
(i) *BCF.*
The mean shearing stress round the boundary is (p. 483)

$$2C\theta S/P = 0{\cdot}506C\theta.$$

The maximum shearing stress will occur at or near the point X, and is found from (15), p. 484. Putting $\rho = \infty$ we find

$$q = 0{\cdot}58C\theta = 0{\cdot}267T.$$

The shearing stress at F is zero.
(ii) *ABCDE.*
The maximum shearing stress will occur at Y.
Since the boundary is concave the radius of curvature is negative, and we must use formula (16), p. 484. We have

$$\rho = -0{\cdot}75'', \ b = 0{\cdot}56'', \ b/\rho = -0{\cdot}746.$$

$$a = 81^\circ, \ \frac{2a}{\pi} = 0{\cdot}9, \ \tanh\frac{2a}{\pi} = 0{\cdot}716.$$

$$\log_e\left(1 - \frac{b}{\rho}\right) = \log_e 1{\cdot}746 = 0{\cdot}557$$

$$0{\cdot}118 \log_e\left(1 - \frac{b}{\rho}\right) - 0{\cdot}238\frac{b}{\rho} = 0{\cdot}0657 + 0{\cdot}177 = 0{\cdot}2423.$$

Then

$$q = \frac{1{\cdot}12C\theta(1 + 0{\cdot}2423 \times 0{\cdot}716)}{1{\cdot}185} = 1{\cdot}11C\theta = 0{\cdot}51T.$$

This being greater than the value of the maximum shearing stress for *BCF*, and the stress at F being zero, we conclude that the maximum stress in the section *BCF* will occur at C not X.
(iii) *EDGH.*
The maximum stress will occur at Z and is given by (15), p. 484. We have $\rho = \infty$, $b = 0{\cdot}375$, $S = 2{\cdot}43$, from which we find

$$q = 0{\cdot}728C\theta = 0{\cdot}335T.$$

Thus the greatest stress anywhere on the boundary is $0{\cdot}51T$ at Y.

335. The Torsion of Hollow Shafts of any Section.—This problem is of much greater difficulty than any of those treated above, and, so far as the author is aware, no quick drawing-office method has yet been devised for hollow shafts having relatively thick walls. In any case the problem can be solved experimentally by the method of Griffiths and Taylor,* but this involves the use of special and costly apparatus. A better way of dealing with the problem is that given by L. Bairstow and A. J. S. Pippard,† who have devised a means of finding graphically the shearing stresses for any given boundaries. Their method is somewhat long and tedious but quite within the range of an intelligent draughtsman, and is the only way of dealing with the problem when it must be done accurately. In most practical cases, when there are no sharp re-entrant angles, the formulæ of § 333 will be sufficiently accurate, but these formulæ would give no indication of the maximum stress in a hollow shaft having a keyway with sharp corners (cf. p. 111).

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* Aeronautical Research Committee, *Reports and Memoranda*, No. 392.
† Institution of Civil Engineers, 1922.

Bairstow and Pippard's method can be applied equally well to solid shafts but is not so convenient as that given above, although it has the slight advantage of enabling the stress distribution all over the section to be found.

EXAMPLES XXX

1. Calculate the angle of twist and maximum shearing stress in a shaft of rectangular section 4″ × 1″, using the method of § 334, and compare the results with those given by § 332, when the applied torque is T.

2. Calculate the torsion constant for a section which is bounded by two arcs of 3″ and 4″ radius with a common chord of 4″, the arcs being on opposite sides of the chord.

3. Estimate the torsion constant for a channel section

$$3'' \times 1'' \times 1'' \times 0{\cdot}104''.$$

4. What is the maximum torque which can be applied to a shaft 2″ × 2″ square section if the shearing stress is not to exceed 20 tons/in.^{2}, and what will be the twist in a length of 5 ft. if $C = 5{,}500$ tons/in.^{2}?

5. A closely coiled helical spring is to deflect 1″ under a pull of 3 tons, and the shearing stress is not to exceed 10 tons/in.^{2} Taking $C = 5{,}500$ tons/in.^{2}, compare the weight of a spring made of round wire with one made of square wire.

CHAPTER XXXI

STRESSES IN FLAT PLATES DUE TO BENDING

336. Statement of the Problem and Assumptions.—When a flat plate of material is supported at its boundary and loaded by forces applied at right angles to its surface, flexure takes place just as it does in the case of a beam. But the flexure of a flat plate presents a more difficult problem than that of a beam since the curvature is not confined to taking place in parallel planes. The general theory of the bending of flat plates is difficult, and we shall here consider only an approximate theory analogous to the simple theory of flexure in beams. The cases which are covered by this theory include most of those which are of practical interest, and it will appear that even this theory involves us in sufficiently laborious algebra.

We assume that the plate is of uniform thickness and material, that the deflections of the middle surface are small* compared with the thickness of the plate. The plane midway between the faces of the plate is referred to as the middle-plane, or, when deformed, the middle-surface, of the plate; we assume that this surface is unextended. We also assume that elements of the plate originally straight and perpendicular to the middle-plane remain straight and become perpendicular to the middle surface when strained. We neglect normal stresses across planes parallel to the middle-surface.

The simplest case is that of a circular plate loaded by forces which are uniformly disposed round the axis, so that stresses and strains are the same for all radii. We shall therefore deal with this first.

SYMMETRICALLY LOADED CIRCULAR PLATE †

337. General Equations.—In Fig. 373 let OA be the axis of the plate, and BC a section of the strained middle surface by any meridian plane. Take the origin O at the point where the axis of the plate cuts the middle plane of the unstrained plate.

Let P and Q be points on the middle surface, at distances r and $r + \delta r$ from the axis. Then the normals at P and Q intersect the axis OA at points which ultimately coincide and become the centre of curvature of the middle-surface at P, when $\delta r \to O$. Let φ and $\varphi + \delta\varphi$ be the inclination of these normals to the axis OA.

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* The flexure of plates when the deflections are not small compared with the thickness is treated by J. Prescott in *Phil. Mag.*, Jan., 1922, or see the same writer's book, *Applied Elasticity*. For an account of some experimental work, see *Engineering*, 1927(i).

† For the general equations, when the plate is not circular, see Appendix, p. 563.

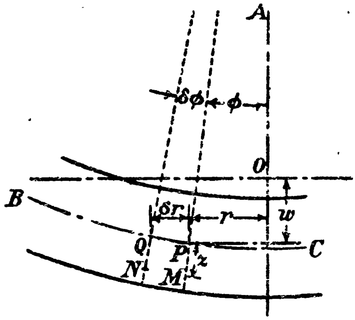

FIG. 373.

Let w = the deflection of P from its initial position.

Let M be a point on the normal through P and let $PM = z$, z being positive in the same direction as w.

Let p_r and p_θ = the radial and circumferential direct stresses at M.

Let e_r and e_θ be the corresponding strains.

We neglect shearing stresses in radial planes and direct stresses across the middle surface.

Before strain the distance of M from the axis was r; after strain it becomes $r + z\varphi$. Hence the circumferential strain is given by

$$e_\theta = \frac{2\pi(r + z\varphi) - 2\pi r}{2\pi r} = \frac{z\varphi}{r} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

If N be the point on the normal through Q corresponding with M on the normal through P, i.e. $QN = z$, the length of NM changes from δr to $\delta r + z\delta\varphi$. Hence the radial strain is given by

$$e_r = \text{Lt.} \frac{z\delta\varphi}{\delta r} = z\frac{d\varphi}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Also, since φ is a small angle, we can write

$$\varphi = -\frac{dw}{dr}.$$

Hence the strains become

$$e_\theta = -\frac{z}{r}\cdot\frac{dw}{dr} \text{ and } e_r = -z\frac{d^2w}{dr^2} \quad . \quad . \quad . \quad \text{(iii)}$$

The stress-strain equations are

$$\frac{p_r}{E} - \frac{p_\theta}{mE} = e_r = -z\frac{d^2w}{dr^2}$$

$$\frac{p_\theta}{E} - \frac{p_r}{mE} = e_\theta = -\frac{z}{r}\frac{dw}{dr}.$$

Solving these for p_r and p_θ we get,

$$p_r = -\frac{mEz}{m^2 - 1}\left(\frac{1}{r}\frac{dw}{dr} + m\frac{d^2w}{dr^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$p_\theta = -\frac{mEz}{m^2 - 1}\left(\frac{m}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We must now consider the stress-resultants and stress-couples, that is the resultant forces and couples on an element of the plate due to the stresses. Consider the element *ABCD* (Fig. 374), the dimensions of which are shown in the figure.

We shall first find the resultant force and couple acting on the element due to the stresses p_θ.

Let t = the thickness of the plate.

Let *LKM* be the intersection of the middle-surface with the element; let *O'N* be the radius bisecting *KM*, and *NT* the tangent at *N* to the arc *MK*.

Let *QRS* be an elementary layer of the plate, at a distance z from the middle-surface, and of thickness δz.

Let *P* be the middle point of *QR*, and *U* the middle point of *RS*.

We shall take the radial and circumferential stresses separately and estimate their total action on the element *ABC* . . . *H*. Then, by considering the equilibrium of the element as a whole, we shall deduce an expression for the shearing force over the face *ADEF*, and finally form a differential equation for finding the displacement.

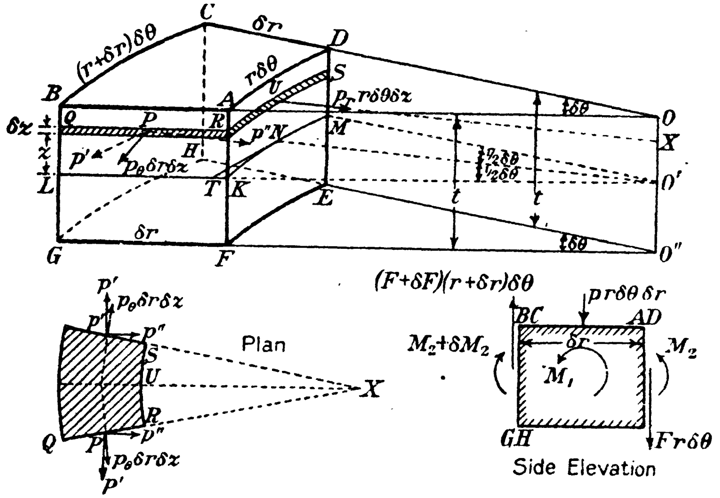
C
δr
(r+δr)δθ
D
rδθ
S
U
B
A
$p_r r\delta\theta\delta z$
δz
Q
P
R
p″
N
M
O
X
z
p′
H
$p_\theta\delta r\delta z$
½δθ
½δθ
L
T
K
E
t
t
O′
δθ
δθ
δr
G
F
O″
p′
$p_\theta\delta r\delta z$
p′
p″
Plan
S
U
X
R
Q
P
p″
$p_\theta\delta r\delta z$
p′
(F+δF)(r+δr)δθ
$pr\delta\theta\delta r$
BC
AD
$M_2+\delta M_2$
δr
M_2
M_1
GH
Frδθ
Side Elevation

FIG. 374.

Consider first the effect of the circumferential stresses p_θ. The total action over the strip *QR* is a force $p_\theta \,.\, \delta r \,.\, \delta z$ acting through *P* at right angles to *QR* and parallel to the middle surface of the element. This force can be resolved into components p' and p'', respectively perpendicular and parallel to the radius through *U*, as shown more clearly in the small plan view. The stress on the face *CDEH* will give equal components at *P'* as shown. The components p' on the two faces cancel so that there is no resultant force perpendicular to the radius.

Again, we see from (2) that p_θ, and therefore p'', is proportional to z, so that p'' changes sign with z and the integral of p'' over each of the faces *ABGF* and *DCHE* will vanish. Hence the hoop stresses p_θ produce no resultant radial force on the element. They do, however, produce a couple: we have

$$p'' = p_\theta\delta r.\delta z \,.\, \sin\frac{\delta\theta}{2} = \tfrac{1}{2}p_\theta\delta r.\delta z.\delta\theta,$$

since $\delta\theta$ is a small angle. The moment of this force about the line NT is $\frac{1}{2}p_\theta.\delta r.\delta z.\delta\theta.z$, and each face AG and CE will contribute a couple of moment $\frac{1}{2}\int_{-\frac{t}{2}}^{\frac{t}{2}} p_\theta z \,.\, \delta r \,.\, dz \,.\, \delta\theta.$

Let M_1 = the resultant couple about NT due to the hoop stresses, then, using (2), we have

$$M_1 = -\frac{mE}{m^2-1}\left(\frac{m}{r}\cdot\frac{dw}{dr} + \frac{d^2w}{dr^2}\right)\delta r \,.\, \delta\theta\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2 dz$$

or

$$M_1 = -\frac{mEt^3}{12(m^2-1)}\left(\frac{m}{r}\cdot\frac{dw}{dr} + \frac{d^2w}{dr^2}\right)\delta r \,.\, \delta\theta \quad . \quad . \quad . \quad (3)$$

This is a couple tending to rotate the element about a line parallel to NT, and represents the whole action of the hoop stresses.

Now consider the radial stresses on the face $ADEF$. The force on the strip RS is $p_r \,.\, r\delta\theta \,.\, \delta z$, acting along the radius through U. From (1) we see that p_r changes sign with z, so that the resultant force over the whole face $ADEF$ is zero. The resultant force parallel to NT is also zero, but there is a couple about NT, the moment of which is

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} p_r r\delta\theta \,.\, dz \,.\, z.$$

Denoting this by M_2 we have from (1)

$$M_2 = -\frac{mE}{m^2-1}\left(\frac{1}{r}\frac{dw}{dr} + m\frac{d^2w}{dr^2}\right)r\delta\theta \,.\, \int_{-\frac{t}{2}}^{\frac{t}{2}} z^2 \,.\, dz$$

or

$$M_2 = -\frac{mEt^3}{12(m^2-1)}\left(\frac{1}{r}\frac{dw}{dr} + m\frac{d^2w}{dr^2}\right)r\delta\theta \quad . \quad . \quad . \quad (4)$$

This represents the whole action of the radial stresses on the face $ADEF$. Similarly the radial stresses on the face $BCHG$ produce a couple

$$M_2 + \delta M_2 = M_2 + \frac{dM_2}{dr}\delta r,$$

about a line parallel to NT. From (4) we have

$$\delta M_2 = -\frac{mEt^3}{12(m^2-1)}\left(\frac{d^2w}{dr^2} + m\frac{d^2w}{dr^2} + mr\frac{d^3w}{dr^3}\right)\delta\theta \,.\, \delta r \quad . \quad (4\text{A})$$

Now let us consider the equilibrium of the whole element.

Let F = the shearing force on the face $ADEF$, per unit length of arc, as shown in Fig. 374. Then the total shearing force on $ADEF$ is $F \,.\, r\delta\theta$, and that on the face $BCHG$ is $(F + \delta F)(r + \delta r)\delta\theta$

Let p = the pressure per unit area on the face $ABCD$, where p may be a function of r but not of θ. Then taking moments about the line NT we have

$$(M_2 + \delta M_2) + (F + \delta F)(r + \delta r)\delta\theta \,.\, \delta r - M_2 - M_1 = 0.$$

This reduces to

$$\delta M_2 + Fr\delta\theta \,.\, \delta r - M_1 = 0.$$

Substituting for δM_2 from (4A) and M_1 from (3), we get

$$Fr\delta\theta \,.\, \delta r = -\frac{mEt^3}{12(m^2 - 1)}\left(\frac{m}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right)\delta r \,.\, \delta\theta$$
$$+ \frac{mEt^3}{12(m^2 - 1)}\left(\frac{d^2w}{dr^2} + m\frac{d^2w}{dr^2} + mr\frac{d^3w}{dr^3}\right)\delta r \,.\, \delta\theta$$

Hence

$$F = \frac{m^2Et^3}{12(m^2 - 1)}\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \quad . \quad . \quad . \quad (5)$$

Again, resolving vertically, we have

$$p \,.\, r \,.\, \delta\theta \,.\, \delta r - (F + \delta F)(r + \delta r)\delta\theta + Fr\delta\theta = 0$$

which reduces to

$$pr \,.\, \delta\theta \,.\, \delta r - r\delta F \,.\, \delta\theta - F\delta r \,.\, \delta\theta = 0.$$

Hence, in the limit, after dividing by $r\delta r \,.\, \delta\theta$, we get

$$\frac{dF}{dr} + \frac{F}{r} = p.$$

Substituting for F from (5) we get

$$\frac{m^2Et^3}{12(m^2 - 1)}\left(\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3} \,.\, \frac{dw}{dr}\right) = p \quad . \quad . \quad (6)$$

Let us, for brevity, write

$$\frac{\mathbf{m^2Et^3}}{\mathbf{12(m^2 - 1)}} = \mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then we can put (6) in the form

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = \frac{p}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is the general differential equation for the deflection of a circular plate when the load is distributed symmetrically about the axis. When w has been found the radial and hoop stresses are given by (1) and (2), and the shearing force by (5).

338. General Solution when the Load is Uniform.—When the load is uniformly distributed, p is constant and we have from (8)

$$\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]=\frac{p}{D}r$$

$$\therefore\ r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}=\frac{pr^2}{2D}+A_1$$

$$\therefore\ \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}=\frac{p}{2D}r+\frac{A_1}{r}$$

$$\therefore\ \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=\frac{p}{4D}r^2+A_1\log r+A_2$$

$$\therefore\ \frac{d}{dr}\left(r\frac{dw}{dr}\right)=\frac{p}{4D}r^3+A_1r\log r+A_2r$$

$$\therefore\ r\frac{dw}{dr}=\frac{pr^4}{16D}+\frac{A_1r^2}{2}(\log r-\tfrac{1}{2})+\tfrac{1}{2}A_2r^2+A_3$$

$$\therefore\ \frac{dw}{dr}=\frac{p}{16D}r^3+\tfrac{1}{2}A_1r(\log r-\tfrac{1}{2})+\tfrac{1}{2}A_2r+\frac{A^3}{r}\quad .\quad .\quad .\qquad (9)$$

$$\therefore\ w=\frac{pr^4}{64D}+\frac{A_1}{4}r^2(\log r-1)+\tfrac{1}{4}A_2r^2+A_3\log r+A_4\qquad (10)$$

There are four constants of integration to be determined, A_1, A_2, A_3, and A_4.

If the plate is continuous to the centre A_1 and A_3 must be zero since w does not become infinite at the centre.

It will be convenient here to obtain general expressions for the stress-couple M_2 and the shearing force F. From (9) we have

$$\frac{d^2w}{dr^2}=\frac{3pr^2}{16D}+\tfrac{1}{2}A_1(\log r+\tfrac{1}{2})+\tfrac{1}{2}A_2-\frac{A_3}{r^2}$$

$$\frac{d^3w}{dr^3}=\frac{3pr}{8D}+\frac{A_1}{2r}+\frac{2A_3}{r^3}$$

Substituting for $\frac{dw}{dr}$, $\frac{d^2w}{dr^2}$ and $\frac{d^3w}{dr^3}$ in (4) and (5), we find, after simplification,

$$\frac{M_2}{r\,.\,\delta\theta}=-\frac{(3m+1)}{m}\cdot\frac{pr^2}{16}-\frac{D}{m}\left[\frac{A_1}{2}\{(m+1)\log r+\tfrac{1}{2}(m-1)\}\right.$$
$$\left.+\frac{A_2}{2}(m+1)-\frac{A_3}{r^2}(m-1)\right]\qquad (11)$$

$$F=\frac{pr}{2}+\frac{A_1D}{r}\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\qquad (12)$$

where D is given by (7).

Similarly from (1) and (2), we have when $z = \frac{t}{2}$

$$p_r = -\frac{3(3m+1)pr^2}{8mt^2} - \frac{mEt}{2(m^2-1)}\left[\frac{A_1}{2}\{(m+1)\log r + \tfrac{1}{2}(m-1)\} + \frac{A_2}{2}(m+1) - \frac{A_3}{r^2}(m-1)\right] \quad (13)$$

$$p_\theta = -\frac{3(m+3)pr^2}{8mt^2} - \frac{mEt}{2(m^2-1)}\left[\frac{A_1}{2}\{(m+1)\log r - \tfrac{1}{2}(m-1)\} + \frac{A_2}{2}(m+1) + \frac{A_3}{r^2}(m-1)\right] \quad (14)$$

These expressions give the maximum values of the radial and hoop tensile stresses at any radius.

The constants of integration can be obtained in various cases from the following conditions:

If an edge is freely supported, w and M_2 vanish at that edge.

If an edge is encastré or clamped, w and $\frac{dw}{dr}$ vanish at that edge.

At a free edge, M_2 and F must vanish.

If the load has different uniform values $p_1, p_2 \ldots$, over different regions of the plate, M_2 and w must have the same values for the p_1 region and the p_2 region where they meet.

If only one edge is supported the value of F round that edge is determined by the total load on the plate, and thus A_1 is given by (12).

We shall now consider some particular cases.

339. Solid Circular Plate, Uniformly Loaded over the Whole Area: Edge Freely Supported.—Let a be the radius of the plate. Since it is continuous up to the centre we must have

$$A_1 = A_3 = 0.$$

When $r = a$ we must have $M_2 = 0$ and $w = 0$. The first condition, using (11), gives

$$-\frac{(3m+1)}{m}\frac{pa^2}{16} - \frac{DA_2}{2m}(m+1) = 0$$

whence

$$A_2 = -\frac{(3m+1)pa^2}{8(m+1)D}.$$

The condition $w = 0$ when $r = a$ gives, from (10),

$$A_4 = -\frac{pa^4}{64D} + \frac{(3m+1)pa^4}{32(m+1)D} = \frac{(5m+1)pa^4}{64(m+1)D}.$$

Inserting the values of the constants in (10), we get after some simplification,

$$w = \frac{3p(m^2-1)}{16m^2Et^3}(a^2 - r^2)\left(\frac{5m+1}{m+1}a^2 - r^2\right) \quad . \quad . \quad . \quad (15)$$

The stresses are then given by (13) and (14) :—

$$p_r = \frac{3(3m+1)p}{8mt^2}(a^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$p_\theta = \frac{3p}{8mt^2}\{(3m+1)a^2 - (m+3)r^2\} \quad . \quad . \quad . \quad (17)$$

These both have their greatest value when $r = 0$, giving

$$\mathbf{p_r\ (max.) = p_\theta\ (max.) = \frac{3(3m+1)pa^2}{8mt^2}} \quad . \quad . \quad . \quad (18)$$

If we take the strain energy as the criterion of failure and f be the elastic limit of the material in pure tension we must have

$$p_r^2 + p_\theta^2 - \frac{2}{m}p_r p_\theta = f^2$$

where p_r and p_θ have the values given by (18). This gives

$$\frac{9(3m+1)^2(m-1)}{32m^3t^4}p^2a^4 = f^2$$

From this we find that the pressure required to cause elastic failure is given by

$$\frac{p}{f} = \frac{4m}{3(3m+1)}\sqrt{\frac{2m}{m-1}}\cdot\frac{t^2}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

340. Solid Circular Plate, Uniformly Loaded over the Whole Area: Edge Clamped.—As before we must have $A_1 = A_3 = 0$. When $r = a$ we must also have $w = 0$ and $\frac{dw}{dr} = 0$. The latter condition gives, from (9),

$$A_2 = -\frac{pa^2}{8D}.$$

The condition $w = 0$ when $r = a$ gives, from (10),

$$A_4 = -\frac{pa^4}{64D} + \frac{pa^4}{32D} = \frac{pa^4}{64D}.$$

Hence we find $$w = \frac{pr^4}{64D} - \frac{pa^2r^2}{32D} + \frac{pa^4}{64D}$$

$$= \frac{p}{64D}(a^2 - r^2)^2$$

or, inserting the value of D,

$$w = \frac{3(m^2-1)p}{16m^2Et^3}(a^2 - r^2)^2.$$

From (13) and (14) the maximum stresses at radius r are found to be

$$p_r = \frac{3p}{8mt^2}\{(m+1)a^2 - (3m+1)r^2\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$p_\theta = \frac{3p}{8mt^2}\{(m+1)a^2 - (m+3)r^2\} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

As in the case of the plate with the edge freely supported, these stresses are greatest at the centre, and there

$$\mathrm{p_r}\ (\text{max.}) = \mathrm{p}_\theta\ (\text{max.}) = \frac{3(\mathrm{m}+1)\mathrm{pa}^2}{8\mathrm{mt}^2} \quad . \quad . \quad . \quad (22)$$

Comparing (18) with (22) it will be seen that the maximum stress when the edge is clamped is only about 0·4 times the maximum stress when the edge is free, if m be 10/3.

341. Annular Ring Freely Supported at the Outer Edge and Loaded Uniformly Round the Inner Edge (Fig. 375 shows a diametral section).—Let the radii of the outer and inner boundaries be a and b. Let there be a total load W distributed uniformly round the inner edge of the plate, the surface of the plate being free from load.

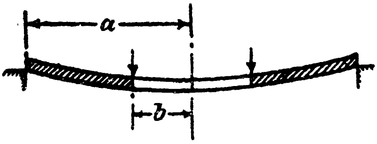

FIG. 375.

We must have $M_2 = 0$ when $r = a$ and $r = b$, hence, from (11), putting $p = 0$,

$$\frac{A_1}{2}\{(m+1)\log a + \tfrac{1}{2}(m-1)\} + \frac{A_2}{2}(m+1) = \frac{A_3}{a^2}(m-1)$$

and

$$\frac{A_1}{2}\{(m+1)\log b + \tfrac{1}{2}(m-1)\} + \frac{A_2}{2}(m+1) = \frac{A_3}{b^2}(m-1).$$

Hence, by subtraction,

$$(m-1)\left(\frac{1}{a^2} - \frac{1}{b^2}\right)A_3 = \frac{A_1}{2}(m+1)\log\frac{a}{b}$$

$$\therefore\ A_3 = -\frac{m+1}{m-1} \cdot \frac{a^2b^2}{2(a^2-b^2)}A_1 \log\frac{a}{b}. \quad . \quad . \quad (\alpha)$$

Also, multiplying the first of the above equations by a^2, the second by b^2, and subtracting, we get

$$(a^2-b^2)(m+1)A_2 = -a^2A_1\{(m+1)\log a + \tfrac{1}{2}(m-1)\}$$
$$+ b^2A_1\{(m+1)\log b + \tfrac{1}{2}(m-1)\}$$

Hence

$$A_2 = \left[\frac{b^2\log b - a^2\log a}{a^2-b^2} - \frac{m-1}{2(m+1)}\right]A_1 . \quad . \quad (\beta)$$

The shearing force per unit length of outer edge is $W/2\pi a$, hence from (12) we have, since $p = 0$,

$$A_1 = \frac{W}{2\pi D}$$

Thus the constants A_1, A_2 and A_3 are determined ; A_4 is found from the condition $w = 0$ when $r = a$.

Omitting the details of the algebra, we finally get *

$$w = \frac{W}{2\pi D}\left[\frac{1}{4}\left\{-\frac{3m+1}{2(m+1)} + \frac{a^2}{a^2-b^2}\log\frac{b}{a} + \log\frac{r}{b}\right\}r^2\right.$$
$$\left. + \frac{1}{2}\frac{m+1}{m-1}\cdot\frac{a^2b^2}{a^2-b^2}\log\frac{a}{b}\,.\,\log\frac{a}{r} + \frac{1}{4}\frac{a^2b^2}{a^2-b^2}\log\frac{a}{b} + \frac{(3m+1)}{8(m+1)}a^2\right] \quad (23)$$

From this we can calculate the stresses by means of (13) and (14).

342. Solid Plate Uniformly Loaded Round a Circle : Edge Freely Supported (see Fig. 376).—The conditions of this problem are the same as the last except that the plate is continuous to the centre. A total load W is uniformly distributed round the circumference of a circle of radius b, the radius of the plate being a. We must divide the plate into two regions : $a > r > b$ and $b > r > 0$. For the first region we must have

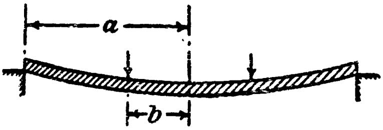

FIG. 376.

$$M_2 = 0,\ w = 0, \text{ when } r = a.$$
$$F = W/2\pi a, \text{ when } r = a.$$

For the region $r < b$ the constants A_1 and A_3 in (10) disappear, and $F = 0$ for all values of r. When $r = b$, M_2 and w and $\frac{dw}{dr}$ must have the same values whether calculated for the first or second region. Let us write

$$r < b,\quad w = \tfrac{1}{4}A_2r^2 + A_4 \quad . \quad . \quad . \quad . \quad (i)$$

$$r > b,\quad w' = \frac{A'_1}{4}r^2(\log r - 1) + \tfrac{1}{4}A'_2r^2 + A'_3\log r + A'_4 \quad (ii)$$

Then :

$w' = 0$, $r = a$ gives

$$\frac{A'_1}{4}a^2(\log a - 1) + \tfrac{1}{4}A'_2a^2 + A'_3\log a + A'_4 = 0 \quad . \quad (iii)$$

$M_2 = 0$, $r = a$, gives

$$\frac{A'_1}{2}\{(m+1)\log a + \tfrac{1}{2}(m-1)\} + \frac{A'_2}{2}(m+1) - \frac{A'_3}{a^2}(m-1) = 0 \quad (iv)$$

$F = \dfrac{W}{2\pi a}$ when $r = a$ gives

$$A'_1 = \frac{W}{2\pi D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (v)$$

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* Kelvin and Tait, *Natural Philosophy*, Pt. II, p. 198.

$w' = w$ when $r = b$ gives

$$\frac{A'_1}{4}b^2(\log b - 1) + \frac{A'_2}{4}b^2 + A'_3 \log b + A'_4 = \frac{A_2}{4}b^2 + A_4 \quad . \quad . \quad \text{(vi)}$$

The condition that M_2 calculated from (ii) $=$ M_2 calculated from (i), when $r = b$, gives

$$\frac{A'_1}{2}\{(m+1)\log b + \tfrac{1}{2}(m-1)\} + \frac{A'_2}{2}(m+1) - \frac{A'_3}{b^2}(m-1) = \frac{A_2}{2}(m+1) \quad \text{(vii)}$$

$\dfrac{dw'}{dr} = \dfrac{dw}{dr}$, when $r = b$ gives

$$\frac{A'_1}{2}b(\log b - \tfrac{1}{2}) + \frac{A'_2}{2}b + \frac{A'_3}{b} = \frac{A_2}{2}b \quad . \quad \text{(viii)}$$

The equations (iii)–(viii) determine the six constants of integration.

Considerations of space prevent us from giving the details of the algebra ; the steps for finding the constants are as follows : from (vii) and (viii) find A'_3 in terms of A'_1, then (iv), (iii), (viii), (vi), give A'_2, A'_4, A_2 and A_4, in that order, in terms of A'_1, which is known from (v). Finally we get

$$w = \frac{W}{8\pi D}\left[-(r^2+b^2)\log\frac{a}{b} + (r^2-b^2) + \frac{(3m+1)a^2-(m-1)b^2}{2(m+1)a^2}(a^2-r^2)\right] \quad (24)$$

$$w' = \frac{W}{8\pi D}\left[-(r^2+b^2)\log\frac{a}{r} + \frac{(3m+1)a^2-(m-1)b^2}{2(m+1)a^2}(a^2-r^2)\right] \quad (25)$$

343. Solid Plate with Load Concentrated at the Centre.*—If the load could be truly concentrated at a point at the centre of the plate the stresses at that point would be infinitely great. In practice the load will either be distributed round a very small circle or over the area of a very small circle. In the former case the above analysis will apply, and if we take the expression for w', making b infinitesimal we shall get an expression for the deflection except near the point of application of the load. Doing this we find, with the edge freely supported,

$$w = \frac{W}{8\pi D}\left[-r^2\log\frac{a}{r} + \tfrac{1}{2}\frac{3m+1}{m+1}(a^2-r^2)\right] \quad . \quad . \quad (26)$$

From this we have, by substitution in (1) and (2),

$$p_r = \frac{3W}{2\pi t^2}\left(1+\frac{1}{m}\right)\log\frac{a}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$p_\theta = \frac{3W}{2\pi t^2}\left[\left(1+\frac{1}{m}\right)\log\frac{a}{r} + \left(1-\frac{1}{m}\right)\right] \quad . \quad . \quad (28)$$

at any point on the surface outside the region of application of the load.

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* For circular plates of varying thickness see an article by G. D. Birkhoff, *Phil. Mag.*, May, 1922. See also Prescott's *Applied Elasticity*. For effect of shear stress on deflection, see *Engineering*, Vol. 123 (1927), p. 343 and Vol. 125 (1923), p. 31.

344. Rectangular Plate, Supported at the Edges.*—B. C. Laws (*loc. cit. infra*) gives the following formulæ for the maximum stress and deflection: the greatest principal stress occurs at the middle points of the longer sides and is given by

$$\frac{a^2}{2a^2 + 6b^2} \cdot \frac{pb^2}{t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

whilst the maximum deflection is given by

$$\frac{a^4}{a^4 + 2b^4} \cdot \frac{pb^4}{32Et^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where t is the thickness of the plate.

For general solution see p. 568 below.

EXAMPLES XXXI

1. In the case considered in § 342, if the edge of the plate be clamped, show that

$$r < b,\ w = \frac{W}{8\pi D}\left[-(r^2 + b^2)\log\frac{a}{b} + r^2 - b^2 + \frac{(a^2 + b^2)(a^2 - r^2)}{2a^2}\right]$$

$$r > b,\ w = \frac{W}{8\pi D}\left[-(r^2 + b^2)\log\frac{a}{r} + \frac{(a^2 + b^2)(a^2 - r^2)}{2a^2}\right]$$

Hence show that the deflection due to a concentrated load at the centre is given by

$$w = \frac{W}{8\pi D}\left[-r^2\log\frac{a}{r} + \tfrac{1}{2}(a^2 - r^2)\right].$$

2. Show that the displacement of a circular plate of radius a, clamped at the edge, and carrying a load W at a point O distant c from the centre is given by †

$$w = \frac{W}{8\pi D}\left[-R^2\log\frac{cR'}{aR} + \tfrac{1}{2}\left(\frac{c^2}{a^2}R'^2 - R^2\right)\right]$$

where R denotes the distance of any point in the plate from the point O, and R' is the distance of the same point from O' the inverse of O with respect to the circle.

3. A circular plate of radius a is freely supported at the edge and carries a total load W uniformly distributed over the area of a concentric circle of radius b.

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* The formulæ given here are not exact, although the complete solution has been worked out; the solution for a rectangular plate with clamped edges has not been worked out, but an approximate method of solution has been given by Ritz, *J. f. Math.* (*Crelle*), Bd. 135, 1909. See also *Edinburgh Proc. Royal Soc.*, 1912, for the application to the case of a square by C. G. Knott.

See also papers in *Trans. Inst. Naval Architects*: A. M. Robb, 1921 (or *Engineering*, April, 1921); G. H. Bryan in Vol. XXXV; I. G. Boobnoff in Vol. XLIV; J. Montgomerie in Vol. LXI; T. B. Abell, 1923; also *Engineering*, June 7, 1929.

Also paper by Hankey: *Der Spannungszustand in rechteckigen platten*, R. Oldenburg, Berlin, 1913; and B. C. Laws, *Proc. Inst. C.E.*, 1921-2, I.

† J. H. Michell, *Proc. London Math. Soc.*, 1902. See also Love's *Theory of Elasticity*, 3rd Ed., p. 495.

Show that

$r > b$,

$$w = \frac{W}{8\pi D}\left[\left(r^2 + \frac{b^2}{2}\right)\log\frac{r}{a} + \frac{2(3m+1)a^2 - (m-1)b^2}{4(m+1)} - \frac{(3m+1)r^2}{2(m+1)}\right]$$

$r < b$,

$$w = \frac{W}{8\pi D}\left[\left(r^2 + \frac{b^2}{2}\right)\log\frac{b}{a} + \frac{r^4 - b^4}{4b^2} - \frac{4ma^2 - (m-1)b^2}{4(m+1)a^2}(r^2 - b^2)\right.$$
$$\left. + \frac{2(3m+1)a^2 - (7m+1)b^2}{4(m+1)}\right]$$

Hence show that, at the centre, the maximum stresses are given by

$$p_r = p_\theta = \frac{3(m+1)W}{2\pi m t^2}\left[\frac{m}{m+1} + \log\frac{a}{b} - \frac{m-1}{m+1}\cdot\frac{b^2}{4a^2}\right].$$

4. If the plate in question (3) be clamped at the edge show that the stresses on the surface, at the centre, are given by

$$p_r = p_\theta = \frac{3(m+1)W}{2\pi m t^2}\left(\log\frac{a}{b} + \frac{b^2}{4a^2}\right);$$

whilst at the circumference

$$p_r = \frac{3W}{2\pi t^2}\left(1 - \frac{b^2}{2a^2}\right).$$

5. The end of a gas-engine piston may be regarded as a diaphragm 12" diameter and ⅜" thick clamped at its edge. If the piston is subjected to a pressure of 400 lbs./in.² determine the intensity of greatest radial and hoop stress. (Mech. Sc. Trip., B., 1915.)

CHAPTER XXXII

THE WHIRLING OF SHAFTS

345. Definition of Whirling Speed.—When a shaft is rotating in bearings, it is unlikely that its axis will be mathematically straight, for, apart from initial crookedness, the deadweight of the shaft must cause some deflection. Consequently, the geometrical axis of the shaft not coinciding with the axis of rotation, centrifugal forces will tend to make the shaft deflect further, until they are balanced by the restoring forces arising from the stiffness of the shaft. The speed which just gives balance between the two sets of forces, is called the *whirling speed* of the shaft.

346. Unloaded Shaft.*—Let ω be the angular velocity of the shaft, w the weight per unit length, and I the moment of inertia of the cross section, which is assumed to be uniform. Measuring x along the axis of rotation, let y be the deflection of the centre line. Then the centrifugal force, per unit length of shaft, is $\frac{wy\omega^2}{g}$. Hence, from § 96, the equation for the strained central line is

$$EI\frac{d^4y}{dx^4} = \frac{wy\omega^2}{g}$$

or

$$\frac{d^4y}{dx^4} - a^4y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$a^4 = \frac{w\omega^2}{gEI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The general solution of (1) is :—

$$y = A \sin ax + B \sinh ax + C \cos ax + D \cosh ax \quad . \quad (3)$$

If the shaft runs in bearings which are very short compared with their distance apart, the conditions approximate to free ends. As in the case of a beam carrying a steady load, this requires $y = 0$ and $d^2y/dx^2 = 0$ at each end, and, if the origin be taken at one of the bearings, we find $C = D = 0$. If l be the length of the shaft, the equation becomes

$$y = A\left(\sin ax - \frac{\sin al}{\sinh al}\sinh ax\right) \quad . \quad . \quad . \quad . \quad (4)$$

---

* For a graphical method of dealing with non-uniform heavy shafts see Appendix, p. 558.

We then find that the condition $d^2y/dx^2 = 0$ when $x = l$ requires $A \sin al = 0$. Hence, either $A = 0$, or $\sin al = 0$. In the former case there is no deflection. In the latter case $al = \pi, 2\pi \ldots.$ The constant A is then indeterminate and a state of instability exists. The lowest speed for which this can happen is given by

$$al = \pi$$

or

$$\frac{w\omega^2}{gEI} = \frac{\pi^4}{l^4}, \text{ from (2)}$$

which gives

$$\omega = \frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is the whirling speed for a long shaft running in very short bearings, the cross section of the shaft being uniform and of dimensions which are small compared with the distance between the bearings.

If the shaft runs in long bearings the conditions approximate to those of "fixed" ends. In this case we find that the critical speed is given by

$$\cos al \, . \cosh al = 1.$$

The smallest root of this is $al = 4{\cdot}73$ or approximately $\frac{3\pi}{2}$, which gives a critical speed

$$\omega = \frac{9\pi^2}{4l^2}\sqrt{\frac{gEI}{w}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

347. Single Concentrated Load on a Light Shaft.—If we have a very light shaft carrying a single load, the dimensions of which are small compared with the length of the shaft, we can find the whirling speed by equating the centrifugal force to the elastic restoring force. Treating the shaft as a light beam carrying a concentrated load W, we have (p. 226),

$$y_1 = \frac{Wa^2b^2}{3EIl},$$

where a and b are the lengths of the two parts into which W divides the length l, and y_1 denotes the deflection under the load, the ends being freely supported. In the case of the rotating shaft, W is replaced by the centrifugal force $\frac{Wy_1\omega^2}{g}$. Hence in the critical state we must have

$$y_1 = \frac{Wy_1\omega^2a^2b^2}{3gEIl}$$

or

$$\omega = \sqrt{\frac{3gEIl}{Wa^2b^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let us consider the effect of an initial deflection h at the point of attachment of the load. At speed ω the centrifugal force will be

$$\frac{W\omega^2}{g}(y_2 + h),$$

where y_2 is the *extra* deflection. For equilibrium we must have

$$\frac{W\omega^2}{g}(y_2 + h) = \text{restoring force for deflection } y_2, = \frac{3EIl}{a^2b^2}y_2$$

or

$$y_2\left(\frac{W\omega^2}{g} - \frac{3EIl}{a^2b^2}\right) = -\frac{W\omega^2h}{g}$$

$$\therefore\ y_2 = \frac{-W\omega^2h}{W\omega^2 - \frac{3gEIl}{a^2b^2}}.$$

When ω is less than the value given by (7) the shaft is quite stable; if ω is equal to this value the shaft is unstable and y tends to infinity. But, when ω is increased beyond the critical value, the shaft becomes stable again and the deflection approaches the value $-h$ as the speed is increased. The interpretation of this is that the axis of rotation and the c.g. of the weight approach one another as the speed is increased. The arrangement is perfectly safe provided the speed is not kept near the critical value, *en passant*, long enough for excessive deflection to develop. This is the principle of the flexible shaft used in a de Laval turbine.

If the shaft is running in long bearings so as to approximate to the condition of fixed ends, we have $y_1 = \frac{Wa^3b^3}{3EIl^3}$ instead of $\frac{Wa^2b^2}{3EIl}$, so that we get for the whirling speed

$$\omega = \sqrt{\frac{3gEIl^3}{Wa^3b^3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have postulated above that the dimensions of the load are small compared with the distance between the bearings. If we did not make this stipulation we should have to examine the effect of the continual changing of the plane of rotation. The general effect is to introduce a gyroscopic torque which stiffens the shaft and increases the whirling speed. This is important in propeller shafts for aeroplanes, the airscrew having considerable stiffening effect.

348. Single Concentrated Load on a Heavy Shaft.—If, in the last problem, the centrifugal forces on the shaft itself are not negligible, we obtain a solution thus :—

Let the load be at the centre of the shaft, which is of length l, and weight w per unit length. Taking the origin at one end, the general equation (3) on p. 502 still holds for either half of the shaft. If the bearings are short we must have $y = 0$ and $\frac{d^2y}{dx^2} = 0$ at $x = 0$, and therefore $C = D = 0$, and the equation reduces to

$$y = A \sin ax + B \sinh ax.$$

On account of symmetry we must have $\frac{dy}{dx} = 0$ when $x = \frac{l}{2}$,

$$\therefore\ 0 = A \cos\frac{al}{2} + B \cosh\frac{al}{2}.$$

Hence the equation for y becomes

$$y = A\left(\sin ax - \frac{\cos\frac{al}{2}}{\cosh\frac{al}{2}}\sinh ax\right).$$

The shearing force is given by $EI\frac{d^3y}{dx^3}$; at the centre the centrifugal force on the load W is supported by the shearing forces on each side of it, and these will be equal by symmetry. Hence, if y_0 denote the deflection at the centre, we must have

$$-\,2EI\left[\frac{d^3y}{dx^3}\right]_{x=\frac{l}{2}} = \frac{Wy_0\omega^2}{g}.$$

That is

$$2EIa^3A\left(\cos\frac{al}{2} + \frac{\cos\frac{al}{2}}{\cosh\frac{al}{2}}\cosh\frac{al}{2}\right)$$

$$= \frac{W\omega^2}{g}A\left(\sin\frac{al}{2} - \frac{\cos\frac{al}{2}}{\cosh\frac{al}{2}}\sinh\frac{al}{2}\right)$$

or

$$A\left[\frac{W\omega^2}{g}\left(\sin\frac{al}{2}\cosh\frac{al}{2} - \cos\frac{al}{2}\sinh\frac{al}{2}\right) - 4EIa^3 . \cos\frac{al}{2}\cosh\frac{al}{2}\right] = 0.$$

If $A = 0$ there is no deflection; therefore, the condition for whirling is that the expression inside the brackets vanishes, that is

$$\tan\frac{al}{2} - \tanh\frac{al}{2} = \frac{4EIa^3g}{W\omega^2}$$

or

$$\tan\frac{al}{2} - \tanh\frac{al}{2} = \frac{4w}{aW} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This is the condition for whirling. When the numerical values of w, W, and l are known, we must solve this equation for a by trial and error. The whirling speed is then given by (2).

349. Shaft Subjected to End Thrust.—If the rotating shaft be subjected to an end thrust P, the thrust will increase the deflecting action of the centrifugal forces.

Referring to Fig. 377, the bending moment at any point B due to the centrifugal forces is obtained by integrating the distributed load

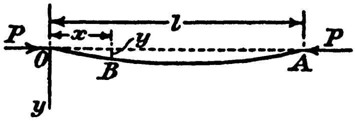

FIG. 377.

twice (§ 96, p. 117). If M_1 denote this bending moment we have

$$\frac{d^2M_1}{dx^2} = \frac{-wy\omega^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The bending moment due to the end thrust is Py. Hence we have

$$-EI\frac{d^2y}{dx^2} = Py + M_1.$$

If the section be uniform, differentiating this twice, we get

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} + \frac{d^2M_1}{dx^2} = 0$$

Substituting from (i) we get

$$\frac{d^4y}{dx^4} + \frac{P}{EI}\frac{d^2y}{dx^2} - \frac{w\omega^2}{gEI}y = 0 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The solution of this equation is

$$y = a_1e^{m_1x} + a_2e^{m_2x} + a_3e^{m_3x} + a_4e^{m_4x}$$

where m_1, m_2, m_3, m_4 are the roots of the equation

$$m^4 + \frac{P}{EI}m^2 - \frac{w\omega^2}{gEI} = 0.$$

The roots of this are

$$\pm\sqrt{\left\{-\frac{P}{2EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}}\right\}} \text{ and}$$

$$\pm\sqrt{\left\{-\frac{P}{2EI} - \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}}\right\}}$$

Thus there are two real roots, and two imaginary roots, and the solution of (ii) may be written

$$y = A\cos a_1x + B\sin a_1x + C\cosh a_2x + D\sinh a_2x \quad \text{(iii)}$$

where

$$a_1^2 = \frac{P}{2EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}} \text{ and}$$

$$a_2^2 = -\frac{P}{2EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}} \quad \text{(iv)}$$

If the bearings are short so as to offer no constraint to the direction of the axis, we must have $y = 0$ and $\dfrac{d^2y}{dx^2} = 0$ when $x = 0$ and when $x = l$. These conditions lead to the equation

$$(a_1^2 + a_2^2)B \sin a_1 l = 0.$$

From (iv) we see that $a_1^2 + a_2^2 \neq 0$, so that we must have, unless $B = 0$, in which case there is no deflection, $a_1 l = \pi, 2\pi \ldots\ldots$ Thus the lowest speed at which instability will occur is given by $a_1^2 = \pi^2/l^2$, or

$$\frac{\mathrm{P}}{2\mathrm{EI}} + \sqrt{\left(\frac{\mathrm{P}}{2\mathrm{EI}}\right)^2 + \frac{\mathrm{w}\omega^2}{\mathrm{gEI}}} = \frac{\pi^2}{l^2} \quad \ldots\ldots \quad (10)$$

The effect of eccentricity of end load, concentrated loads, etc., might be taken into account as in the case of struts.

350. Shaft Subjected to End Thrust and Torque.—If the shaft in the last problem be subjected to torque the problem becomes far more complicated; the analysis is long and tedious, but, fortunately, the results are comparatively simple. We shall give here only the outline of the analysis, and the results, obtained by R. V. Southwell.*

When the shaft is deflected, the applied torque gives component bending moments, as in the problem of the torsion of a beam (p. 375). Taking the axis of x along the unstrained axis of the shaft, whilst the axes of y and z are two axes perpendicular to this and revolving with the shaft, the component bending moments due to a torque T are (p. 375) $-T\dfrac{dz}{dx}$ about the axis of z and $T\dfrac{dy}{dx}$ about the axis of y. The component bending moments due to the thrust P are Py and Pz respectively. The distributed loads, per unit length, due to centrifugal forces, are $\dfrac{w\omega^2}{g}y$ and $\dfrac{\omega^2 w}{g}z$, respectively.

Considering flexure in the xy plane, let M_1 be the bending moment due to the lateral forces. Then we have

$$-EI\frac{d^2y}{dx^2} = M_1 + Py - T\frac{dz}{dx}$$

Differentiating this, we have, if the section be uniform,

$$-EI\frac{d^3y}{dx^3} - P\frac{dy}{dx} + T\frac{d^2z}{dx^2} = \frac{dM_1}{dx} = F_1$$

Differentiating again we get

$$-EI\frac{d^4y}{dx^4} - P\frac{d^2y}{dx^2} + T\frac{d^3z}{dx^3} = \frac{d^2M_1}{dx^2} = w_1 = -\frac{w\omega^2}{g}y$$

or

$$T\frac{d^3z}{dx^3} = EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} - \frac{w\omega^2}{g}y \quad \ldots \quad \text{(i)}$$

Similarly, considering flexure in the plane xz, we get

$$-T\frac{d^3y}{dx^3} = EI\frac{d^4z}{dx^4} + P\frac{d^2z}{dx^2} - \frac{w\omega^2}{g}z \quad \ldots \quad \text{(ii)}$$

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* *British Association Report,* 1921.

Now multiply* (ii) by i $(=\sqrt{-1})$, add to (i) and let

$$u = y + iz.$$

Then we get

$$EI\frac{d^4u}{dx^4} + iT\frac{d^3u}{dx^3} + P\frac{d^2u}{dx^2} - \frac{w\omega^2}{g}u = 0.$$

The solution of this is

$$u = A_1e^{i\lambda_1x} + A_2e^{i\lambda_2x} + A_3e^{i\lambda_1x} + A_4e^{i\lambda_4x},$$

where λ_1, λ_2, λ_3, λ_4 are the roots of

$$EI\lambda^4 + T\lambda^3 - P\lambda^2 - \frac{w\omega^2}{g} = 0.$$

If the bearings are long, so that the axis is held straight by them, the terminal conditions are

$$y = z = 0\,;\ \frac{dy}{dx} = \frac{dz}{dx} = 0,$$

or

$$u = 0 \text{ and } \frac{du}{dx} = 0,$$

when $x = 0$ or l. For brevity let $e^{i\lambda_1x} = a$, etc. Then these conditions give

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 &= 0 \\ \lambda_1A_1 + \lambda_2A_2 + \lambda_3A_3 + \lambda_4A_4 &= 0 \\ a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 &= 0 \\ a_1\lambda_1A_1 + a_2\lambda_2A_2 + a_3\lambda_3A_3 + a_4\lambda_4A_4 &= 0. \end{aligned}$$

Eliminating A_1, A_2, A_3, A_4 in the form of a determinant and expanding we get

$$\Sigma(a_1a_2 + a_3a_4)(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) = 0 \quad . \quad . \quad . \text{(iii)}$$

Dividing through by $\sqrt{a_1a_2a_3a_4}$, and noticing that

$$\sqrt{\frac{a_1a_2}{a_3a_4}} + \sqrt{\frac{a_3a_4}{a_1a_2}} = 2\cos(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\frac{l}{2},$$

(iii) becomes

$$\Sigma(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\cos(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\frac{l}{2} = 0 \quad . \quad . \quad (11)$$

This is the criterion for stability.

If the bearings are short, so that no constraint is offered to bending, the terminal conditions are

$$y = z = 0\,;$$

$EI\dfrac{d^2y}{dx^2} - T\dfrac{dz}{dx} = 0$, since M_1 and y vanish at the ends.

$EI\dfrac{d^2z}{dx^2} + T\dfrac{dy}{dx} = 0$, ,, M_2 ,, z ,, ,, ,,

---

* This elegant method of obtaining equations (11) and (12) from (i) and (ii) is due to H. A. Webb.

These conditions lead to the equation

$$\Sigma(\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2)\cos(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\frac{l}{2} = 0 \quad . \quad (12)$$

for the criterion of stability. From equations (11) and (12) Southwell obtains the diagrams given in Fig. 378, where

$$A = \frac{lT}{2EI},\ B = \frac{l^2P}{4EI},\ C = \frac{w\omega^2 l^4}{16EIg}.$$

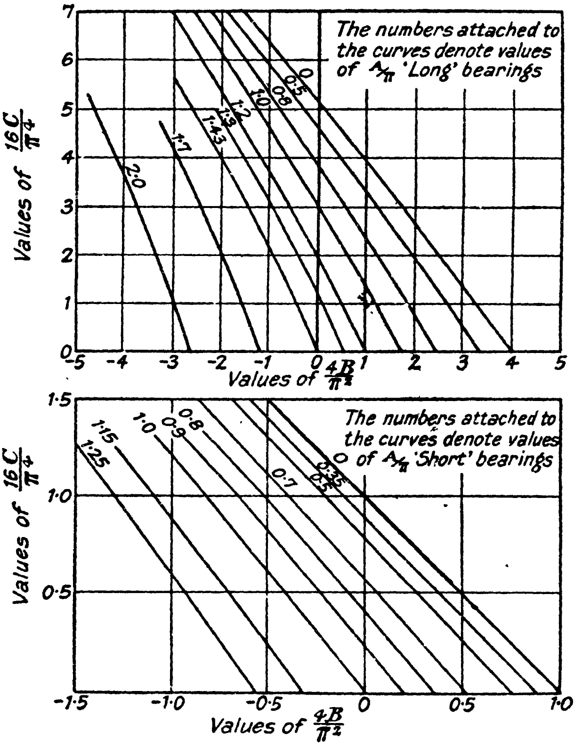

FIG. 378.

For a further treatment of this subject see the following :—

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C. Chree, in *Phil. Mag.*, 1904, 1905.

A. Morley, *Engineering*, July 30 and Aug. 13, 1909; Nov. 22 and 29, 1918.

W. Kerr, *Engineering*, Feb. and March, 1916, and correspondence arising from these articles.

A. Fage, *Engineering*, July 20, 1917.

R. V. Southwell, *Phil. Mag.*, March, 1921.

J. Morris, *Advisory Committee for Aeronautics*, R. and M., 551 (1918).

W. L. Cowley and H. Levy, ditto, R. and M., 485 (1918).

G. Greenhill, ditto, R. and M. 560.

J. Morrow, *Phil. Mag.*, 1905, 1906, 1907.

F. B. Pidduck, *Proc. London Math. Soc.*, 1920.

P. F. Ward, *Phil. Mag.*, 1913.

C. H. Lees, *Phil. Mag.*, 1919, vol. i.

H. A. Webb, *Engineering*, Nov. 2, 9, 16, 1917, on the Whirling of Shafts of non-uniform section, with graphical methods.

EXAMPLES XXXII

1. Show that the whirling speed of a long overhanging shaft, projecting from a long bearing, is given by $\cos al . \cosh al + 1 = 0$, and that the smallest root of this equation is 1·875, the notation being the same as on p. 502. Hence show that the critical speed is

$$\omega = 3{\cdot}52 \sqrt{gEI/wl^4}$$

2. Show that the whirling speed of a long shaft with a short bearing at one end, and a long bearing at the other, is given by $al = 3{\cdot}927$, or

$$\omega = 15{\cdot}4 \sqrt{gEI/wl^4}$$

3. A long shaft is carried by three short bearings, the lengths of the two spans being l_1 and l_2. Show that the whirling speed is given by

$$\frac{\sin a(l_1 + l_2)}{\sinh a(l_1 + l_2)} = \frac{\sin al_1 . \sin al_2}{\sinh al_1 . \sinh al_2}.$$

4. Find the whirling speed of a steel shaft, 6 ft. long between the bearings, which are very short. The diameter of the shaft is 1". Take $E = 30 \times 10^6$ lbs./in.^{2}, and the weight of steel = 0·283 lbs./in.^{3}

5. A light shaft 12" long carries, at its centre, a small flywheel weighing 5 lbs. The shaft is made of steel (as in Ex. 4) and is $\frac{3}{8}$" diameter. Calculate the whirling speed. If the c.g. of the wheel be 0·01" from the axis of the shaft, calculate the maximum stress when running at a speed equal to 0·95 of the critical speed. The bearings are short.

6. A steel shaft, 1" diameter and 5′ 6" long between its short bearings, is subjected to an end thrust of 250 lbs. Calculate the critical speed of the shaft.

7. A steel shaft, $\frac{3}{4}$" diameter, is 15" long between short bearings, and carries at its centre a pulley weighing 30 lbs. Find the whirling speed, allowing for the weight of the shaft.

8. A light shaft runs in two short bearings, one at one end, and one at a distance a from the other end. At the extremity of the overhanging end is a load of weight w. Show that the whirling speed is $\sqrt{3EIg/lwa^2}$, where l is the total length of the shaft. (Morris.)

9. A pulley is keyed to a light elastic shaft and the two vibrate violently when running through the critical speed at 1,000 revs. per minute. It is proposed to paint a red zone on the tachometer dial, as a warning. If the initial eccentricity of the pulley is 0·002 in., and the flexure of the shaft should not exceed 0·010 in., how far should the red zone extend above and below 1,000 revs. per minute? (R.N.E. College, Keyham, 1937.)

10. A thick circular disc is mounted at the centre of a shaft of length l, supported at each end in spherical bearings. The c.g. of the disc is on the axis of the shaft, but its plane is out of truth by an angle φ. Show that at speed ω, the angle between the axis of the disc and the axis of rotation is approximately $\varphi \Big/ \left\{1 - \dfrac{\omega^2 l}{12EI}(I_2 - I_1)\right\}$, where I_1 and I_2 are the moments of inertia of the disc about its axis and about a diameter respectively, and EI is the flexural stiffness of the shaft.

Examine the behaviour of the shaft according as I_1 is $>$ or $< I_2$.

CHAPTER XXXIII

TRANSVERSE OSCILLATIONS OF BEAMS DUE TO PULSATING AND TRAVELLING LOADS

PULSATING LOADS *

351. Introductory.—We shall consider first the deflections and stresses when a load, which is fixed on a beam, varies as a periodic function of time. Such a case, for instance, would arise from the out-of-balance forces of a reciprocating engine supported on girders; another example is the cross beam of a railway bridge due to the passage of a train. Before considering these problems, however, we must examine the free oscillations of a girder when it is given a small deflection and released, the only load being its own weight.

352. Free Oscillations of a Beam Simply Supported at Both Ends.—Let w be the weight, per unit length, of the beam and any permanent load it may carry; let y be the displacement of any point, on the axis of the beam, distant x from some fixed point on the axis, at time t. Then the "load" per unit length, due to inertia, is $\frac{-w}{g} \cdot \frac{\partial^2 y}{\partial t^2}$. Neglecting the gravitational forces (if the beam is not vertical) in comparison with the inertia forces, we have from (3), p. 218,

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\frac{w}{g} \cdot \frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the section of the beam be uniform this can be written

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2 \frac{\partial^2 y}{\partial t^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\lambda^2 = w/gEI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Assume that y is a harmonic function of the time, and take

$$y = y' \sin 2\pi nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

where y' is a function of x only. Substituting in (2) we get:

$$\frac{d^4 y'}{dx^4} - a^4 y' = 0, \text{ where } a^2 = 2\pi n\lambda \quad . \quad . \quad . \quad . \quad (ii)$$

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* Practically the whole of §§ 351–359 is due to C. E. Inglis, *Theory of Transverse Oscillations in Girders* (Proc. Inst. C.E., 1924). The matter is also dealt with briefly by Prof. S. P. Timoshenko (*Phil. Mag.*, May, 1922). Reference should be made to the work of Bridge Stress Committee, *Engineering*, 1929(i).

The general solution of this is

$$y' = A \sin ax + B \sinh ax + C \cos ax + D \cosh ax,$$

where A, B, C and D are constants of integration.

Putting this value of y' in (i) we get for the general solution of (2)

$$y = (A \sin ax + B \sinh ax + C \cos ax + D \cosh ax) \sin \frac{a^2 t}{\lambda} \quad . \quad (4)$$

Taking the origin at one end, y must vanish with x for all values of t; therefore $D = -C$, and we can write

$$y = \{A \sin ax + B \sinh ax + C(\cos ax - \cosh ax)\} \sin \frac{a^2 t}{\lambda} \quad . \quad . \quad . \quad (5)$$

At a freely supported end the bending moment, and therefore $\partial^2 y/\partial x^2$, must vanish, so that $C = 0$. A further condition to be satisfied is $y = 0$ when $x = l$, the length of the girder, which requires

$$A \sin al + B \sinh al = 0$$

Hence the equation for y becomes

$$y = A\left(\sin ax - \frac{\sin al}{\sinh al} \sinh ax\right) \sin 2\pi nt$$

Now we must have $\partial^2 y/\partial x^2 = 0$ when $x = l$, which requires $\sin al = 0$. Hence we have the following series of values for a, etc.

$$\left.\begin{array}{rlllll}
al = & \pi & 2\pi & 3\pi & . & . \; . \; . \; . \; . \\
a = & \frac{\pi}{l} & \frac{2\pi}{l} & \frac{3\pi}{l} & . & . \; . \; . \; . \\
n = \frac{a^2}{2\pi\lambda} = & \frac{\pi}{2\lambda l^2} & \frac{2^2\pi}{2\lambda l^2} & \frac{3^2\pi}{2\lambda l^2} & . & . \; . \; . \; . \\
\text{or} \quad n = & n_0 & 4n_0 & 9n_0 & . & . \; . \; . \; .
\end{array}\right\} \; . \; . \quad (6)$$

where $n_0 = \pi/2\lambda l^2$ is the lowest frequency.

Each value of n corresponds with a separate mode of oscillation, and the equation representing the fundamental mode is

$$y = A \sin \frac{\pi x}{l} . \sin 2\pi n_0 t,$$

and the complete expression is

$$y = A_1 \sin \frac{\pi x}{l} \sin 2\pi n_0 t + A_2 \sin \frac{2\pi x}{l} . \sin 8\pi n_0 t + \quad . \quad . \quad (7)$$

All the conditions are equally well met if we write $\cos 2\pi nt$ instead of $\sin 2\pi nt$, so that there is a second series of modes of oscillation expressed by the equation

$$y = B_1 \sin \frac{\pi x}{l} \cos 2\pi n_0 t + B_2 \sin \frac{2\pi x}{l} \cos 8\pi n_0 t + \quad . \quad . \quad (8)$$

In the series (7) we get $y = 0$ for all values of x when $t = 0$, but there is an initial velocity given by

$$\frac{\partial y}{\partial t} = 2\pi n_0\left(A_1 \sin\frac{\pi x}{l} + 4A_2 \sin\frac{2\pi x}{l} + 9A_3 \sin\frac{3\pi x}{l} + \ldots .\right)$$

If we know the initial values of $\frac{\partial y}{\partial t}$ all along the beam we can find the values of A_1, $A_2 \ldots .$

On the other hand, with the series (8), $\frac{\partial y}{\partial t}$ vanishes with t, but there is an initial deflection given by

$$y = B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \ldots .$$

The constants B_1, $B_2 \ldots .$ are determined if we know the initial shape into which the axis of the beam is bent in order to start the oscillations.

The frequency of the fundamental oscillation, in terms of the constants of the beam, is given by

$$n_0 = \frac{\pi}{2}\sqrt{\frac{gEI}{wl^4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

In the above analysis we have neglected the effects of shear and of rotary inertia. The corrections to be applied on account of these have been found by Timoschenko.*

Example.—Find the fundamental frequency of a standard 24″ × 7½″ × 100 lbs./ft. rolled steel joist 15 ft. long, the value of I being 2,654 ins.^{4}, and $E = 30 \times 10^6$ lbs./in.^{2}

We have

$$n_0 = \frac{\pi}{2}\sqrt{\frac{32{\cdot}2 \text{ ft./sec.}^2 \times 30 \times 10^6 \text{ lbs./in.}^2 \times 2{,}654 \text{ ins.}^4}{100 \text{ lbs./ft.} \times 15^4 \text{ . ft.}^4}}$$

$$= \frac{\pi}{2}\sqrt{\frac{32{\cdot}2 \times 30 \times 10^4 \times 2{,}654}{50{,}625}\left(\frac{\text{ins.}}{\text{ft.}}\right)^2 \frac{1}{\text{sec.}^2}}$$

$$= \frac{\pi}{2}\sqrt{\frac{32{\cdot}2 \times 30 \times 2{,}654}{5{\cdot}0625 \times 144}} \text{ per sec.} = 93 \text{ per sec.}$$

353. Pulsating Sinusoidal Load on Freely Supported Beam.—

In order to deal with the effects of pulsating loads we shall have to make use of the methods of harmonic analysis which we applied to static deflections in Chapter XIV. As an easy example, and one which gives results that we shall make use of later, we shall take the case of a beam, freely supported at each end, carrying a load which at any time is distributed along the beam in a sinusoidal manner and which varies harmonically with the time. Such a load distribution will be given by the equation

$$w = w_0 \sin\frac{k\pi x}{l} \sin 2\pi n t,$$

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* *Phil. Mag.*, Jan., 1922.

The vibrations of rotating cantilevers, such as airscrew blades, are dealt with in *Aero. Research Com., Reports and Memoranda*, 426, 451, 486; vibrations of beams of varying sections in *R. and M.*, 566; beams under thrust in *R. and M.*, 453.

where k is any integer and n may have any real value. The general equation for the deflection is then, instead of (2),

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2 \frac{\partial^2 y}{\partial t^2} = \frac{w_0}{EI} \sin \frac{k\pi x}{l} \sin 2\pi nt \quad . \quad . \quad . \quad . \quad (10)$$

We can find a particular integral of this by assuming

$$y = y_0 \sin \frac{k\pi x}{l} \sin 2\pi nt,$$

and substituting in (10) to find the value of y_0. We find

$$y_0 = \frac{w_0 l^4}{\pi^4 EI\left(k^4 - \dfrac{n^2}{(\pi/2\lambda l^2)^2}\right)} = \frac{w_0 l^4}{\pi^4 EI\left(k^4 - \dfrac{n^2}{n_0^2}\right)}$$

where n_0 is the fundamental frequency of free oscillation of the girder (p. 512). Hence a particular integral of (10) is

$$y = \frac{w_0 l^4}{\pi^4 EI\left(k^4 - \dfrac{n^2}{n_0^2}\right)} \sin \frac{k\pi x}{l} \sin 2\pi nt \quad . \quad . \quad . \quad . \quad (11)$$

The complementary function is obtained by making the right hand zero, so that it is given by (7) on p. 512. Hence the complete solution of (10) is

$$y = A_1 \sin \frac{\pi x}{l} \sin 2\pi n_0 t + \ldots + A_k \sin \frac{k\pi x}{l} \sin 2k^2 \pi n_0 t + \quad . \quad . \quad . \quad .$$

$$+ \frac{w_0 l^4}{\pi^1 EI\left(k^4 - \dfrac{n^2}{n_0^2}\right)} \sin \frac{k\pi x}{l} \sin 2\pi nt.$$

This satisfies the conditions that y and $\dfrac{\partial^2 y}{\partial x^2}$ vanish when $x = 0$ or l.

We also require that $\dfrac{\partial y}{\partial t} = 0$, when $t = 0$, for all values of x. Hence we see that all the A's must vanish except A_k, which is given by

$$A_k = -\frac{n w_0 l^4}{k^2 n_0 \pi^4 EI\left(k^4 - \dfrac{n^2}{n_0^2}\right)}$$

Thus the final expression for y is *

$$y = \frac{w_0 l^4 \sin \dfrac{k\pi x}{l}}{\pi^4 EI\left(k^4 - \dfrac{n^2}{n_0^2}\right)} \left(\sin 2\pi nt - \frac{n}{k^2 n_0} \sin 2k^2 \pi n_0 t\right) \quad . \quad . \quad (12)$$

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* If the applied load be $w_0 \sin \dfrac{k\pi x}{l} \cos 2\pi nt$, the particular integral is given by (11) with $\cos 2\pi nt$ written instead of $\sin 2\pi nt$, and the complementary function is zero since this particular integral satisfies the condition $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$.

The corresponding bending moment is given by

$$M = -EI\frac{\partial^2 y}{\partial x^2} = \frac{k^2 w_0 l^2 \sin\frac{k\pi x}{l}}{\pi^2\left(k^4 - \frac{n^2}{n_0{}^2}\right)}\left(\sin 2\pi n t - \frac{n}{k^2 n_0}\sin 2k^2\pi n_0 t\right) \quad . \quad (13)$$

If $n = k^2 n_0$, the above expressions for y become indeterminate, and we must take another form for the particular integral of (10). It can easily be verified by substitution that when $n = k^2 n_0 = \frac{k^2\pi}{2\lambda l^2}$,

$$y = \frac{w_0 l^3 x}{4k^3\pi^3 EI}\cos\frac{k\pi x}{l} \,.\, \sin 2\pi n t$$

and

$$y = -\frac{w_0 n_0 l^4 t}{k^2\pi^3 EI}\sin\frac{k\pi x}{l} \,.\, \cos 2\pi n t$$

both satisfy the equation (10). Since y must vanish when $x = l$, the first of these solutions is not admissible. We therefore take as a complete solution of (10)

$$y = A_k \sin\frac{k\pi x}{l} \,.\, \sin 2k^2\pi n_0 t - \frac{w_0 n_0 l^4 t}{k^2\pi^3 EI}\sin\frac{k\pi x}{l} \,.\, \cos 2\pi n t.$$

The condition $\frac{\partial y}{\partial t} = 0$ when $t = 0$ gives

$$A_k = \frac{w_0 l^4}{2k^4\pi^4 EI}$$

Hence the solution becomes

$$y = \frac{w_0 l^4}{2k^4\pi^4 EI}(\sin 2k^2\pi n_0 t - 2k^2\pi n_0 t \,.\, \cos 2\pi n t)\sin\frac{k\pi x}{l}$$

or

$$y = \frac{w_0 l^4}{2k^4\pi^4 EI}(\sin 2\pi n t - 2\pi n t \,.\, \cos 2\pi n t)\sin\frac{k\pi x}{l} \quad . \quad . \quad . \quad . \quad (14)$$

The second term within the brackets shows that the amplitude of the oscillation increases uniformly with the time, and may become dangerous if the alternating load endures for a sufficient length of time. This is a case of synchronism.

Returning to the general case and taking $k = 1$, i.e. the load follows the law $w = w_0 \sin(\pi x/l)$, the greatest value that y can have is

$$\frac{w_0 l^4}{\pi^4 EI} \,.\, \frac{\sin 2\pi n t - \frac{n}{n_0}\sin 2\pi n_0 t}{1 - \frac{n^2}{n_0{}^2}} \quad \text{(from (12))} \quad . \quad . \quad . \quad (15)$$

The worst that can happen is that $2\pi n_0 t = 2\pi n t + \pi$, for then the second term will have the same sign as the first, and both will have their maximum value (unity) at the same time. In this case the maximum value of this expression is $\frac{n_0}{n_0 - n}$. Comparing (15) above with (31) on p. 242, we see then that the greatest magnification of deflection and bending moment which the alternations of load can produce is $\frac{n_0}{n_0 - n}$.

354. Alternating Load Uniformly Distributed on Freely Supported Beam.—We shall next consider the case when the load is uniformly distributed along the beam, but alternates with a frequency n alternations per second.

Let $w = w_0 \sin 2\pi nt$.

The equation for the deflections becomes

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2 \frac{\partial^2 y}{\partial t^2} = \frac{w_0}{EI} \sin 2\pi nt \quad . \quad . \quad . \quad . \quad . \quad (16)$$

To obtain a particular integral, assume

$$y = y_0 \sin 2\pi nt,$$

where y_0 is a function of x only, and substitute in (16). We get

$$\frac{d^4 y_0}{dx^4} - 4\pi^2 n^2 \lambda^2 y_0 = \frac{w_0}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If n_0 be the frequency of the fundamental free oscillation, we have (p. 512) $n_0 = \pi/2\lambda l^2$, so that $4\pi^2 n^2 \lambda^2 = \frac{\pi^4}{l^4} \cdot \frac{n^2}{n_0^2}$. Hence we can write (17) in the form

$$\frac{d^4 y_0}{dx^4} - \frac{\pi^4 n^2}{l^4 n_0^2} y_0 = \frac{w_0}{EI}.$$

A particular integral of this is $y_0 = -\frac{w_0 l^4}{\pi^4 EI} \frac{n_0^2}{n^2}$, and the complete solution is

$$y_0 = \frac{w_0 l^4 n_0^2}{\pi^4 EI n^2}\left[-1 + A \sin \frac{\pi}{l}\sqrt{\frac{n}{n_0}}\, x + B \sinh \frac{\pi}{l}\sqrt{\frac{n}{n_0}} \,.\, x \right.$$
$$\left. + C \cos \frac{\pi}{l}\sqrt{\frac{n}{n_0}} \,.\, x + D \cosh \frac{\pi}{l}\sqrt{\frac{n}{n_0}}\, x\right] \quad . \quad (18)$$

where A, B, C, D are constants to be determined. We must have $y_0 = 0$ when $x = 0$; therefore

$$C + D = 1.$$

We must also have $\frac{d^2 y_0}{dx^2} = 0$ when $x = 0$; therefore

$$-C + D = 0$$

Hence $C = D = \frac{1}{2}$.

Again, we require $y_0 = 0$ and $\frac{d^2 y_0}{dx^2} = 0$ when $x = l$. These two conditions give

$$A \sin \pi \sqrt{\frac{n}{n_0}} + B \sinh \pi \sqrt{\frac{n}{n_0}} = 1 - \tfrac{1}{2}\cos \pi \sqrt{\frac{n}{n_0}} - \tfrac{1}{2}\cosh \pi \sqrt{\frac{n}{n_0}}$$

$$-A \sin \pi \sqrt{\frac{n}{n_0}} + B \sinh \pi \sqrt{\frac{n}{n_0}} = + \tfrac{1}{2} \cos \pi \sqrt{\frac{n}{n_0}} - \tfrac{1}{2}\cosh \pi \sqrt{\frac{\pi}{n_0}}$$

From these two equations we get

$$A = \tfrac{1}{2} \tan \frac{\pi}{2} \sqrt{\frac{n}{n_0}}, \text{ and } B = - \tfrac{1}{2} \tanh \frac{\pi}{2} \sqrt{\frac{n}{n_0}}.$$

Substituting for A, B, C, D in (19) and simplifying we find

$$y_0 = \frac{w_0 l^4 n_0^2}{\pi^4 E I n^2} \left[\frac{\cos\left(1 - \frac{2x}{l}\right)\frac{\pi}{2}\sqrt{\frac{n}{n_0}}}{2 \cos \frac{\pi}{2}\sqrt{\frac{n}{n_0}}} + \frac{\cosh\left(1 - \frac{2x}{l}\right)\frac{\pi}{2}\sqrt{\frac{n}{n_0}}}{2 \cosh \frac{\pi}{2}\sqrt{\frac{n}{n_0}}} - 1 \right]$$

Multiplying this by sin $2\pi nt$ we get a particular integral of (16), whilst the complementary function is given by (7) on p. 512. Thus the complete solution of (16) may be expressed in the form

$$y = \frac{w_0 l^4 n_0^2}{\pi^4 E I n^2} \left[\frac{\cos\left(1 - \frac{2x}{l}\right)\frac{\pi}{2}\sqrt{\frac{n}{n_0}}}{2 \cos \frac{\pi}{2}\sqrt{\frac{n}{n_0}}} + \frac{\cosh\left(1 - \frac{2x}{l}\right)\frac{\pi}{2}\sqrt{\frac{n}{n_0}}}{2 \cosh \frac{\pi}{2}\sqrt{\frac{n}{n_0}}} - 1 \right] \sin 2\pi nt$$

$$+ A_1 \sin \frac{\pi x}{l} \sin 2\pi n_0 t + A_2 \sin \frac{2\pi x}{l} . \sin 2.2^2 \pi n_0 t + \;.\;.\;. \quad (19)$$

In this the first line represents the forced oscillation and the second the free oscillations. The constants A_1, A_2 . . . are determined from the condition $\frac{\partial y}{\partial t} = 0$ when $t = 0$. In order to find these constants we must express the forced oscillation as a series of sines of multiples of $\pi x/l$. This is readily done by making use of § 181 and § 352. The uniformly distributed load w_0 may be expressed in the form (p. 243)

$$w_0 = \frac{4w_0}{\pi}\left(\sin \frac{\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \frac{1}{5} \sin \frac{5\pi x}{l} + \;.\;.\;.\;.\;.\right)$$

Thus the pulsating load w_0 sin $2\pi nt$ is equivalent to a series of loads

$$\frac{4w_0}{\pi} \sin \frac{\pi x}{l} \sin 2\pi nt + \frac{4w_0}{3\pi} \sin \frac{3\pi x}{l} \sin 2\pi nt + \;.\;.\;.\;.\;.$$

Then, applying (13) of p. 515 to these loads, we shall get as another form of the complete solution of (16)

$$y = \frac{4w_0 l^4}{\pi^5 EI}\left[\left\{\frac{\sin 2\pi nt - \dfrac{n}{n_0}\sin 2\pi n_0 t}{1 - \dfrac{n^2}{n_0{}^2}}\right\}\sin\frac{\pi x}{l}\right.$$
$$\left. + \left\{\frac{\sin 2\pi nt - \dfrac{n}{3^2 n_0}\sin 2.3^2.\pi n_0 t}{3\left(3^4 - \dfrac{n^2}{n_0{}^2}\right)}\right\}\sin\frac{3\pi x}{l} + \ . \ . \ \right] \quad . \ . \ (20)$$

In most cases the series is very rapidly convergent and it is sufficient to retain only the first harmonic, that is, the terms involving sin $(\pi x/l)$ only.

355. Single Pulsating Load on Freely Supported Beam.—Let the beam be acted on by a transverse load

$$W = W_0 \sin 2\pi nt,$$

acting at a distance $x = a$ from one end, which is taken as origin.

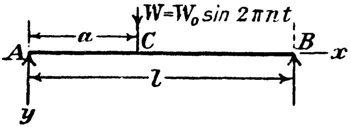

FIG. 379.

For either part of the beam, AC or CB (Fig. 379) the equation to be solved is

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2 \frac{\partial^2 y}{\partial t^2} = 0,$$

but we must have a different solution for the two parts on account of the discontinuity of shear at C. The solution

$$y = (A \sin ax + B \sinh ax) \sin 2\pi nt,$$

where $a^2 = 2\pi n\lambda = \dfrac{\pi^2}{l^2} \cdot \dfrac{n}{n_0}$, n_0 being the fundamental frequency of free oscillation of the beam, satisfies the conditions $y = 0$ and $\dfrac{\partial^2 y}{\partial x^2} = 0$ when $x = 0$ and so can be used for the part AC. The case of a stationary load (p. 225) suggests that we should take for the whole beam

$$y = [A \sin ax + B \sinh ax + \{A' \sin a(x - a)$$
$$+ B' \sinh a(x - a)\}] \sin 2\pi nt \quad . \ . \ . \ . \ . \ (21)$$

the terms in { } being rejected when $x < a$.

The conditions to be satisfied are:

$$y = 0 \text{ and } \frac{\partial^2 y}{\partial x^2} = 0 \text{ when } x = l.$$

y, $\frac{\partial y}{\partial x}$ and $\frac{\partial^2 y}{\partial x^2}$ must have the same values at C whether they are calculated for AC or CB, i.e. whether the $\{\ \}$ are retained or rejected when $x = a$.

The shear, $EI\frac{\partial^3 y}{\partial x^3}$, must be discontinuous at C by an amount $W_0 \sin 2\pi nt$.

The conditions that y and $\frac{\partial^2 y}{\partial x^2}$ should have the same values each side of C are satisfied.

From (21) we have

$$EI\frac{\partial^3 y}{\partial x^3} = EIa^3[-A\cos ax + B\cosh ax$$
$$+ \{-A'\cos a(x-a) + B'\cosh a(x-a)\}]\sin 2\pi nt.$$

Hence, just to the left of C, we have

$$EI\frac{\partial^3 y}{\partial x^3} = EIa^3(-A\cos aa + B\cosh aa)\sin 2\pi nt,$$

whilst, just to the right of C,

$$EI\frac{\partial^3 y}{dx^3} = EIa^3(-A\cos aa + B\cosh aa - A' + B')\sin 2\pi nt.$$

From these two equations we see that the discontinuity of shear at C requires

$$a^3EI(-A' + B') = W_0 \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

Again, from (21),

$$\frac{\partial y}{\partial x} = a[A\cos ax + B\cosh ax + \{A'\cos a(x-a)$$
$$+ B'\cosh a(x-a)\}]\sin 2\pi nt.$$

Hence, for the continuity of $\frac{\partial y}{\partial x}$ at C, we must have

$$A' + B' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

From (i) and (ii) we get

$$A' = -\frac{W_0}{2a^3EI},\ B' = \frac{W_0}{2a^3EI}.$$

Then the conditions $y = 0$ and $\frac{\partial^2 y}{\partial x^2} = 0$, when $x = l$, give

$$A\sin al + B\sinh al = \frac{W_0}{2a^3EI}\{\sin a(l-a) - \sinh a(l-a)\}$$

$$-A\sin al + B\sinh al = -\frac{W_0}{2a^3EI}\{\sin a(l-a) + \sinh a(l-a)\}.$$

Hence

$$A = \frac{W_0}{2a^3EI} \cdot \frac{\sin a(l-a)}{\sin al}$$

$$B = \frac{-W_0}{2a^3EI} \cdot \frac{\sinh a(l-a)}{\sinh al}.$$

The complete equation for y is therefore

$$y = \frac{W_0}{2a^3EI}\left[\frac{\sin a(l-a)}{\sin al}\sin ax - \frac{\sinh a(l-a)}{\sinh al}\sinh ax - \{\sin a(x-a) - \sinh a(x-a)\}\right]\sin 2\pi nt \quad (22)$$

This is the equation for the forced oscillations, to which must be added the deflections for free oscillations. Instead of the finite form for y which we have just found, it will be more convenient for our immediate purpose to express y as a series of sines. To this end we use the formula (38), § 182, and express W_0 as a series of harmonically distributed loads :

$$W_0 = \frac{2W_0}{l}\left(\sin\frac{\pi a}{l}\sin\frac{\pi x}{l} + \frac{\sin 2\pi a}{l}\sin\frac{2\pi x}{l} + \dots\dots\right)$$

The equation for y is then

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2\frac{\partial^2 y}{\partial t^2} = \frac{2W_0}{l}\sum_{k=1}^{k=\infty}\left(\sin\frac{k\pi a}{l}\sin\frac{k\pi x}{l}\right)\sin 2\pi nt.$$

The complete integral is given by applying (12), p. 514.

$$y = \frac{2W_0 l^3}{\pi^4 EI}\sum_{k=1}^{k=\infty}\frac{\sin 2\pi nt - \dfrac{n}{k^2 n_0}\sin 2k^2\pi n_0 t}{k^4 - \dfrac{n^2}{n_0^2}}\sin\frac{k\pi a}{l}\sin\frac{k\pi x}{l}. \quad (23)$$

In nearly all practical cases this series converges so rapidly that it is sufficient to consider only the terms $k = 1$, it being only when n is large enough to embrace higher harmonics of the free oscillations that this is not the case.

356. Freely Supported Beam Subjected to a Load which Varies Uniformly with the Time.—Any distribution of load along a beam can be expressed in the form

$$\sum_1^\infty w_k \sin\frac{k\pi x}{l},$$

so let us take, as a typical load of the type we are now going to consider, $wt\sin\frac{k\pi x}{l}$.

The equation for the oscillations is

$$\frac{\partial^4 y}{\partial x^4} + \lambda^2\frac{\partial^2 y}{\partial t^2} = \frac{wt}{EI}\sin\frac{k\pi x}{l} \quad \dots\dots \quad (24)$$

Try as a solution

$$y = t \, . f(x),$$

and substitute in the equation : we get

$$\frac{d^4f}{dx^4} = \frac{w}{EI}\sin\frac{k\pi x}{l},$$

Hence a solution of our equation is (cf. p. 242)

$$y = t\left(\frac{l^4w}{k^4\pi^4EI}\sin\frac{k\pi x}{l} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D\right)$$

The conditions $y = 0$ and $\dfrac{\partial^2 y}{\partial x^2} = 0$ when $x = 0$ require that $B = D = 0$, whilst the conditions that the same quantities vanish when $x = l$ demand that $A = C = 0$. Hence we have for the particular integral

$$y = \frac{wl^4t}{k^4\pi^4EI}\sin\frac{k\pi x}{l}$$

whilst the complete solution is

$$y = \frac{wl^4t}{k^4\pi^4EI}\sin\frac{k\pi x}{l} + \left(A_1\sin\frac{\pi x}{l}\sin 2\pi n_0 t \right.$$
$$\left. + A_2\sin\frac{2\pi x}{l}\sin 2.2^2\pi n_0 t + \ldots\ldots\right)$$

The condition that $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$ gives $A_1 = A_2 = A_{k-1} = A_{k+1}$ $\ldots = 0$, and

$$A_k = -\frac{wl^4}{2k^6\pi^5n_0EI}.$$

Thus the final solution of the equation (24) is

$$y = \frac{wl^4}{k^4\pi^4EI}\left(t - \frac{\sin 2k^2\pi n_0 t}{2k^2\pi n_0}\right)\sin\frac{k\pi x}{l} \quad \ldots \ldots \quad (25)$$

TRAVELLING LOADS

357. Concentrated Load Advancing over Freely Supported Girder.—Let a load of constant magnitude W advance along a girder of length l at a constant velocity v, the ends of the girder being freely supported (Fig. 380). Then, at any time t, after the load comes on to the beam, it will have advanced a distance $a = vt$. Using the result

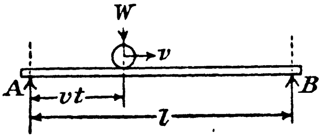

FIG. 380.

of § 182, p. 244, we can resolve the load W into a series of distributed loads by the equation

$$W = \frac{2W}{l}\left(\sin\frac{\pi vt}{l}\sin\frac{\pi x}{l} + \sin\frac{2\pi vt}{l}\sin\frac{2\pi x}{l} + \ldots\ldots\right)$$

Hence, from (12), p. 514, the deflection is given by

$$y = \frac{2Wl^3}{\pi^4 EI}\sum_{k=1}^{k=\infty}\frac{\sin\dfrac{k\pi vt}{l} - \dfrac{v}{2k^2 n_0 l}\,.\sin 2k^2\pi n_0 t}{k^4 - \dfrac{k^2v^2}{4n_0{}^2l^2}}\,.\sin\frac{k\pi x}{l},$$

or, since $n_0 = \dfrac{\pi}{2\lambda l^2}$,

$$y = \frac{2Wl^3}{EI}\sum_{k=1}^{k=\infty}\frac{\sin\dfrac{k\pi vt}{l} - \dfrac{\lambda vl}{k^2\pi}\sin 2k^2\pi n_0 t}{k^2\pi^2(k^2\pi^2 - \lambda^2v^2l^2)}\,.\sin\frac{k\pi x}{l} \quad . \; . \; . \quad (26)$$

This is a complete solution of the problem, satisfying all the required conditions.* The series comprising the first terms in the numerators represents the forced oscillation. In practice λvl will nearly always be small in comparison with π^2, so that the deflection due to the forced oscillation is very closely given by

$$\frac{2Wl^3}{\pi^4 EI}\left(\sin\frac{\pi a}{l}\sin\frac{\pi x}{l} + \frac{1}{2^4}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + \ldots\ldots\right),$$

where a is the distance that the load has advanced along the beam. It will be seen that this is identical with the static deflection given by (37), p. 244. The additional deflection represented by the free oscillation is at the same time given by

$$-\frac{2Wl^3}{\pi^4 EI}\left(\frac{\lambda vl}{\pi}\right)\left(\sin 2\pi n_0 t\,.\sin\frac{\pi x}{l} + \frac{1}{2^5}\sin 8\pi n_0 t\,.\sin\frac{2\pi x}{l} + \ldots\ldots\right) \text{ or}$$

$$-\frac{2Wl^3}{\pi^4 EI}\left(\frac{v}{2ln_0}\right)\left(\sin 2\pi n_0 t\,.\sin\frac{\pi x}{l} + \frac{1}{2^5}\sin 8\pi n_0 t\,.\sin\frac{2\pi x}{l} + \ldots\ldots\right). \quad (27)$$

The greatest positive value of this is almost exactly $\dfrac{2Wl^3}{\pi^4 EI}\left(\dfrac{v}{2ln_0}\right)$ at the centre, whilst the static deflection at the centre, to the same order of approximation, is $\dfrac{2Wl^3}{\pi^4 EI}$. Hence we have

$$\frac{\textbf{max. dynamic deflection}}{\textbf{max. static deflection}} = \mathbf{1} + \frac{\mathbf{v}}{\mathbf{2ln_0}}.$$

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* Prof. Inglis (*loc. cit.*, p. 511) obtains the forced oscillation in a finite form :—

$$y = \frac{W}{\lambda^3 v^3 EI}\left[\sin\lambda v(l - vt)\frac{\sin\lambda vx}{\sin\lambda vl} - \lambda v(l - vt)\frac{x}{l} - \{\sin\lambda v(x - vt) - \lambda v(x - vt)\}\right]$$

where the terms in { } are rejected where $x < vt$, but later makes use of the form given above.

358. Uniformly Distributed Load Advancing over Freely Supported Girder.—A load, which is uniform and equal to w per unit length advances across a girder with constant speed v, the ends of the girder being freely supported. At time t from when the load comes on to the beam, the head of the load will have travelled a distance $a = vt$ (Fig. 381). The load on the beam at any time can be expressed in the form *

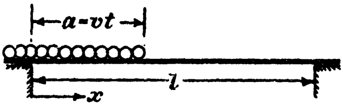

FIG. 381.

$$\frac{2w}{\pi}\sum_{k=1}^{k=\infty}\frac{1}{k}\sin\frac{k\pi x}{l} - \frac{2w}{\pi}\sum_{k=1}^{k=\infty}\frac{1}{k}\sin\frac{k\pi x}{l}\cdot\cos\frac{k\pi vt}{l}.$$

Hence, from 31, p. 242, and (12), p. 514, the deflection is given by

$$y = \frac{2wl^4}{\pi^5 EI}\sum_{k=1}^{k=\infty}\frac{1}{k^5}\sin\frac{k\pi x}{l} - \frac{2wl^4}{\pi^5 EI}\sum_{k=1}^{k=\infty}\frac{\sin\frac{k\pi x}{l}\cos\frac{k\pi vt}{l}}{k\left(k^4 - \frac{k^2v^2}{4l^2n_0^2}\right)}$$

$$= \frac{2wl^4}{\pi^5 EI}\sum_{k=1}^{k=\infty}\frac{(-1)^{k+1}}{k^5}\left\{1 - \frac{\cos\frac{k\pi vt}{l}}{1 - \frac{\lambda^2v^2l^2}{k^2\pi^2}}\right\}\sin\frac{k\pi x}{l} \quad . \quad . \quad . \quad (28)$$

This satisfies all the conditions of the problem, including the condition of zero initial velocity; in this case, there is no free oscillation and the above expression represents the forced oscillation.† It has already been remarked that $\lambda^2v^2l^2$ is usually very small compared with π^2, so that the deflection is given with considerable accuracy by the more simple expression

$$y = \frac{4wl^4}{\pi^5 EI}\sum_{k=1}^{k=\infty}\frac{(-1)^{k+1}}{k^5}\sin^2\frac{k\pi vt}{2l}\,.\sin\frac{k\pi x}{l} \quad . \quad . \quad . \quad (29)$$

Since $vt = a$, we see that the deflection is equal to the statical deflection under a stationary load extending over the same amount of the beam.

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* See Ex. 17, p. 243.

† In addition to the above harmonic form Prof. Inglis finds the deflection due to the forced oscillation in finite form:

$$y = \frac{w}{\lambda^4v^4EI}\left[\frac{l-x}{l} - \frac{\cos\lambda v^2t.\ \sin\lambda v(l-x)}{\sin\lambda lv} + \frac{\lambda^4v^4(l-x}{360l}\left((2lx-x^2)(4l^2+6lx-3x^2+\frac{60}{\lambda^2v^2})\right.\right.$$
$$\left.\left. - \frac{180t^2}{\lambda^2}\right) - 2\left\{\sin^2\frac{\lambda v(vt-x)}{2} - \frac{\lambda^2v^2}{4}(vt-x)^2\right\}\right]$$

where the { } terms are omitted when $x > vt$.

359. Single Pulsating Load Advancing over Freely Supported Girder.—Let a concentrated load of magnitude $W = W_0 \cos 2\pi nt$ travel across a freely supported beam of span l at a uniform speed v, so that, after a time t, it has advanced a distance $a = vt$.

Then by (38), p. 244, we can write

$$W = \frac{2W_0}{l}\left(\sum_{k=1}^{k=\infty}\sin\frac{k\pi vt}{l}\sin\frac{k\pi x}{l}\right)\cos 2\pi nt$$

$$= \frac{W_0}{l}\sum_{k=1}^{k=\infty}\left\{\sin\frac{k\pi x}{l}\sin\left(\frac{\pi vk}{l}+2\pi n\right)t + \sin\frac{k\pi x}{l}\sin\left(\frac{k\pi v}{l}-2\pi n\right)t\right\}$$

Thus the given load is equivalent to these two infinite series of loads, each of which can be treated by the formula (12) on p. 514. Applying this we get *

$$y = \frac{W_0 l^3}{EI}\sum_{k=1}^{k=\infty}\frac{\sin\left(\frac{k\pi v}{l}+2\pi n\right)t - \left(\frac{kv}{2l}+n\right)\frac{2\lambda l^2}{\pi k^2}\sin 2k^2\pi n_0 t}{\pi^2\left\{\pi^2 k^4 - \left(n+\frac{kv}{2l}\right)^2 4\lambda^2 l^4\right\}}\cdot\sin\frac{k\pi x}{l}$$

$$+\frac{W_0 l^3}{EI}\sum_{k=1}^{k=\infty}\frac{\sin\left(\frac{k\pi v}{l}-2\pi n\right)t - \left(\frac{kv}{2l}-n\right)\frac{2\lambda l^2}{\pi k^2}\sin 2k^2\pi n_0 t}{\pi^2\left\{\pi^2 k^4 - \left(\frac{kv}{2l}-n\right)^2 4\lambda^2 l^4\right\}}\cdot\sin\frac{k\pi x}{l}\quad . \quad . \quad (30)$$

It is usually sufficient to consider only the terms which occur when $k = 1$.

The effects of the pulsating load on the girder will be greatest when the frequency of pulsation is equal to the natural frequency of the beam, i.e. when $n = n_0 = \frac{\pi}{2\lambda l^2}$. Consider the case when this happens and the load has just reached the end of the girder, i.e. when $t = l/v$. Making these substitutions and taking $k = 1$, we get from (30)

$$y = \frac{W_0 l^3}{\pi^4 EI}\left[-\frac{\sin 2\pi n_0 t + \left(1+\frac{\lambda l v}{\pi}\right)\sin 2\pi n_0 t}{1-\left(1+\frac{\lambda l v}{\pi}\right)^2} + \frac{\sin 2\pi n_0 t + \left(1-\frac{\lambda l v}{\pi}\right)\sin 2\pi n_0 t}{1-\left(1-\frac{\lambda l v}{\pi}\right)^2}\right]\sin\frac{\pi x}{l}$$

If we remember that, in practice, $\frac{\lambda l v}{\pi}$ is usually of the order 0·1, and accordingly neglect $\lambda^2 l^2 v^2/\pi^2$ this reduces to

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* For the expression of the forced oscillation in finite form see Prof. Inglis' paper. The series given above was deduced by Timoshenko by a different method.

$$y = \frac{W_0 l^3}{\pi^4 EI}\left[-\frac{\sin 2\pi n_0 t + \left(1 + \frac{\lambda l v}{\pi}\right)\sin 2\pi n_0 t}{-\frac{2\lambda l v}{\pi}} + \frac{\sin 2\pi n_0 t + \left(1 - \frac{\lambda l v}{\pi}\right)\sin 2\pi n_0 t}{\frac{2\lambda l v}{\pi}}\right]\sin\frac{\pi x}{l}$$

$$= \frac{2 W_0 l^3}{\pi^4 EI}\cdot\frac{\pi}{\lambda l v}\sin 2\pi n_0 t \,.\, \sin\frac{\pi x}{l}$$

or

$$y = \left(\frac{2W_0 l^3}{\pi^4 EI}\right)\left(\frac{2n_0 l}{v}\right)\sin 2\pi n_0 t \,.\, \sin\frac{\pi x}{l} \quad . \; . \; . \; . \; . \quad (31)$$

From this we see that, at the centre of the girder, the greatest value of the deflection, as the load reaches the end, is $\frac{2W_0 l^3}{\pi^4 EI}(2N)$, where N is the total number of complete oscillations made by the load in passing over the girder. The factor $2W_0 l^3/\pi^4 EI$ is approximately the central deflection due to a static load W_0, so that the worst effect of the load is to magnify this $2N$ times.

EXAMPLES XXXIII

1. A girder of length l is freely supported at both ends, and has a weight W attached to it at a distance a from each end, the distance between the weights being $2b = l - 2a$. Show that the frequencies of the free oscillations are

$$\frac{\alpha_1^2 l^2}{\pi^2}n_0,\ \frac{\alpha_2^2 l^2}{\pi^2}n_0,\ \frac{\alpha_3^2 l^2}{\pi^2}n_0, \ldots\ldots$$

where $\alpha_1, \alpha_2, \alpha_3 \ldots.$ are the roots of the equation

$$\frac{\cos \alpha b \,.\, \sin \alpha a}{\cos(\alpha l/2)} - \frac{\cosh \alpha b \,.\, \sinh \alpha a}{\cosh(\alpha l/2)} = \frac{gEI\alpha^3}{2\pi^2 n^2 W} = \frac{2w}{\alpha W},$$

w being the weight per unit length of the girder and n_0 being the natural fundamental frequency of the girder without the weights W attached. (Inglis.)

2. In the last example the weights are subjected to forces $P \sin 2\pi n t$. Show that, so far as the first harmonic is concerned, the oscillations of the girder are given by

$$y = \frac{4l^3 P \sin\frac{\pi a}{l}\sin\frac{\pi x}{l}}{\pi^4 EI\left(1 - \frac{n^2}{n_0^2}\right)(1 - \gamma)}\left(\sin 2\pi n t - \frac{n}{n_1}\sin 2\pi n_1 t\right)^{*},$$

---

* When $W = 0$, i.e. when the girder is simply subjected to the pulsating forces P, the oscillations are given by putting $\gamma = 0$ and $n_1 = n_0$ in this equation.

where n_1 is the fundamental frequency of the loaded girder, and

$$\gamma = \frac{2\pi^2 n^2 W}{gEIa^3}\left(\frac{\cos ab.\sin aa}{\cos(al/2)} - \frac{\cosh ab.\sinh aa}{\cosh(al/2)}\right)$$

(Inglis.)

3. A 24″ × 7½″ × 100 lbs./ft. standard rolled steel girder having $I = 2{,}654$ ins.^{4} and $E = 30 \times 10^6$ lbs./in.^{2}, and 15 ft. long is freely supported at each end. It is acted on by a pulsating load at the centre, making 90 complete oscillations per second. Show that, when $t = 0{\cdot}082$ sec., the central deflection is about 22 times, and bending moment about 23 times, the deflection and bending moment due to a constant load of the same maximum magnitude.

4.* Four equal axle-loads, at intervals of 12 ft., cross a bridge of 60 ft. span at 120 ft./sec. The natural frequency of the bridge is 10 per sec. Show that the dynamical deflection is about 8 per cent. greater than the maximum statical deflection. Examine the effect of reducing the axle spacing to 6 ft. (Inglis.)

5. A load which alternates 6 times per sec. moves across a span of 60 ft. at a speed of 120 ft./sec. The natural frequency of the bridge is 9 per sec. Show that the maximum deflection at the centre is $5{\cdot}15Wl^3/\pi^4EI$, where W is the maximum value of the load.

6. In the case of a pulsating single load crossing a girder whose natural fundamental frequency of oscillation is n_0, show that the greatest addition to the central deflection which can possibly occur is

$$\frac{2W_0l^3}{\pi^4EI}\left[\frac{1}{1-\frac{n}{n_0}\left(1+\frac{1}{2N}\right)} + \frac{1}{1+\frac{n}{n_0}\left(1-\frac{1}{2N}\right)}\right],$$

if the following conditions are simultaneously fulfilled (i) the fluctuating load has its maximum value at the same time as the load reaches the centre of the girder, (ii) the natural frequency is such that at this moment the free oscillation is making its greatest contribution to the deflection, (iii) N is a whole number. The notation is the same as in § 359. (Inglis.)

7. The load on a beam rises uniformly from zero to a maximum value W in time T, and then decreases uniformly to zero in further time T. If y_0 be the maximum static deflection, show that the dynamical deflection is given very closely by

$$y = y_0 - \frac{y_0}{2\pi n_0 T}[\sin 2\pi n_0 t - 2\sin 2\pi n_0\{t - T\}],$$

where the last term is omitted when $t < T$. (Inglis.)

8. On account of the balance weights on the driving wheels, the passage of a single driving axle of a locomotive over a bridge subjects the cross girders to forces which are equivalent to a central force which increases according to the law $W = R\sqrt{2}\sin\frac{\pi a}{l}.\frac{t}{T}.\cos 2\pi N(T - t)$. In this formula R is the maximum additional rail-pressure due to the balance weights, T is the time required to pass from one cross girder to the next, N is the speed of rotation of the wheels, and a is the distance of the rail-bearers from the ends of the cross girder. From the time $t = T$ to $t = 2T$ the force falls off in the same manner. Show that the oscillations of the cross girder are given by

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[*] Details of a Locomotive and Bridge, useful for arithmetical investigations, will be found in *Engineering*, Sept. 8, 1922.

$$y = \frac{l^3 R\sqrt{2}.\sin\frac{\pi a}{l}\sin\frac{\pi x}{l}}{\pi^4 n_0 TEI\left(1-\frac{N^2}{n_0^2}\right)^2}\left[2\pi n_0 t\left(1-\frac{N^2}{n_0^2}\right)\cos 2\pi N(T-t)\right.$$

$$-\frac{2N}{n_0}\sin 2\pi N(T-t)-\left(1+\frac{N^2}{n_0^2}\right)\cos 2\pi NT\sin 2\pi n_0 t$$

$$\left.-\frac{2N}{n_0}\sin 2\pi NT.\cos 2\pi n_0 t+\left\{2\left(1+\frac{N^2}{n_0^2}\right)\sin 2\pi n_0(t-T)\right\}\right]$$

where the term in { } is rejected when $t < T$, and n_0 is the natural fundamental frequency of the girder.

In the particular case when R is equal to P, $2P$ being the steady axle-load when the engine is stationary, $N = 5$, $T = 0{\cdot}125$ sec., $n_0 = 40$, show that the deflection is increased 74 per cent. at the time $t = 0{\cdot}25$ sec. (Inglis.)

CHAPTER XXXIV

ALTERNATING STRESSES AND FATIGUE

360. Introductory.—In many branches of engineering practice, members, the strengths of which have to be calculated, are subjected to loads which vary in a periodic manner with respect to time. This applies to most parts of moving machinery: for instance, the piston-rod and connecting-rod of a reciprocating double-acting engine are subjected to loads which alternate between tensions and approximately equal compressions during each revolution of the crankshaft, whilst the latter is itself subjected to a torque which varies in a cyclic manner, and so on for all the moving parts. Again, consider a railway bridge: if a train be at rest on the bridge, all the members will be subjected to certain definite stresses on account of the deadweight of the train and the bridge itself. If, however, the train travel over the bridge, the latter will be set oscillating according to the principles set forth in Chapter XXXIII. The effect of this will be that the strains will be reduced for some of the members and increased for others during one-half of the oscillation, and the effects will be reversed in the other half. Thus every part of the bridge is subjected to stresses which oscillate between certain limits with the frequency of oscillation of the bridge: a member which is free from stress in the static condition of the bridge will probably be subjected to stresses which alternate between equal tensile and compressive values; another which in the static condition is stressed up to, say, 5 tons per sq. in. tension, may, when the bridge is oscillating, carry a stress which varies between 4 tons per sq. in. and 6 tons per sq. in., both of the same kind; other members may have stresses which fluctuate between small tensile values and large compressive values, or vice versa. These two instances should suffice to show that the occurrence of fluctuating stresses is common in practice, and no doubt the reader will be able to think of many more examples.

In all such instances, experience has taught us that fracture will frequently occur when the greatest stress reached in a cycle is far less than the ultimate strength of the material found by a static test, if the cycle is repeated sufficiently often. It is therefore of the greatest importance to the practical engineer to know what are the limits of stress which can be allowed when the load is a periodic function of the time.

We shall distinguish between two kinds of fluctuating stress: when the stress oscillates between equal or nearly equal tensile and compressive values we shall use the term *alternating* stresses; when the stresses fluctuate between zero and some definite maximum value we shall call them *repeated* stresses; the adjective "fluctuating" will be applied in a general way to all stresses which vary periodically.

Before discussing the stress limits for fluctuating stresses it will be well to pause to examine one or two phenomena which occur in connection with static tests of metals.

361. Raising the Yield Point by Stress.—When a specimen of wrought iron, mild steel, or other ductile metal, is loaded beyond the yield point, both the limit of proportionality and the yield point are found to be higher on a subsequent re-loading. This is shown clearly[*] in Fig. 382, obtained by Ewing. The test, on a piece of soft iron wire, was carried to the point *a*, well beyond the limit of proportionality and yield point, after which the load was removed and the specimen reloaded. It was found that a new yield point *b* was reached, slightly above the previous maximum load. Similar effects are shown in Fig. 383, which refers to a specimen of mild steel tested by Unwin,[†] the load being removed three times.

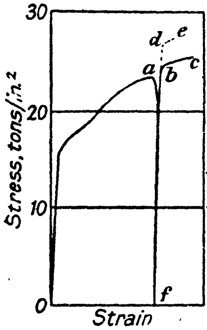

FIG. 382.

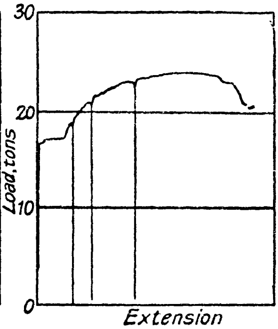

FIG. 383.

362. Effects of Time on Recovery of Elasticity.—In the tests referred to in Figs. 382 and 383 the reloading followed almost immediately on the unloading. In such cases it will be found, on reloading, that the material is very imperfectly elastic: the limit of proportionality has either disappeared altogether or been considerably reduced. If, however, an interval of rest of several hours be allowed before the second loading, the metal will be found to have recovered its elasticity either completely, or to a great extent, according to the length of the rest.[‡] At the same time the yield point is raised, as shown at *d* in Fig. 382. This gradual recovery with time is shown very clearly in Fig. 384, plotted from observations by J. Muir.[§]

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[*] Ewing's *Strength of Materials* (1899), p. 34. See also his paper *Proc. R.S.*, 1880.

[†] *The Testing of Materials of Construction* (Longmans, 1910).

[‡] See Ewing, *Proc. R. Soc.*, 1895, and Bauschinger, *Dingler's Journal*, Bd. 224, s. 5; *Civilingenieur*, 1881; *Mitth. aus dem Mech.-Tech. Lab. in München*, Heft. XIII and XXV.

[§] *Phil. Trans.*, 1900; see also *Proc. R.S.*, Vol. 67, 1900.

Curve (1) shows the result of the first test, the yield point being 27 tons/in.^{2}, whilst curve (2) refers to a reloading shortly after the first loading. It will be seen that there is no obvious limit of proportionality. Curves 3, 4, 5, 6 were taken after rests of 4, 23, 70 and 147 hours respectively, and show the gradual recovery of elasticity. In the last loading the metal was elastic very nearly up to the previous yield point, whilst the new yield point is not well marked. The several curves are drawn from different origins for the sake of clearness, in Figs. 384 and 385.

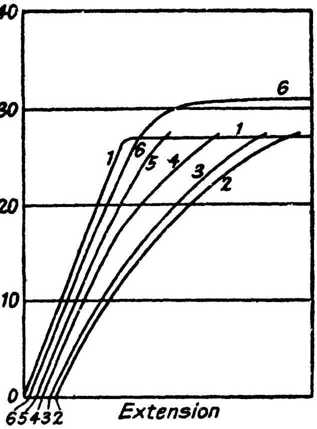

FIG. 384.

363. Recovery of Elasticity with Moderate Heat.

—The restoration of the elastic properties of metals such as mild steel, after being stressed beyond the yield point, can be considerably hastened by moderate heating: a few minutes' immersion in boiling water will effect a considerable improvement. This is illustrated in Fig. 385, plotted from J. Muir's observations. The curve (1) is the first test beyond the yield point; curve (2) is the first reloading, twenty minutes later, and shows the complete disappearance of the limit of proportionality. Curve (3) shows the effect of four minutes' immersion in boiling water.

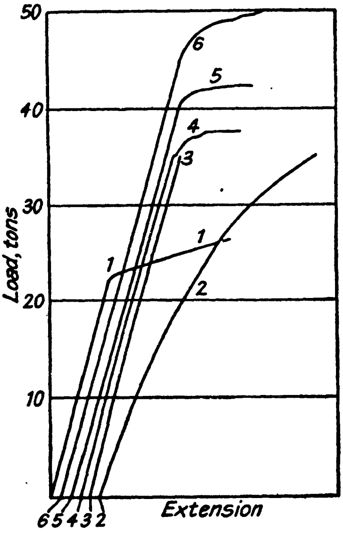

FIG. 385.

If a specimen of mild steel be taken just up to the first yield point, then immersed for a few minutes in boiling water, stressed up to the new yield point, again immersed in boiling water, and so on in a series of steps, until fracture occurs, the metal can be made to show a considerably greater ultimate strength than it had in its initial state. For instance, one of Muir's specimens of steel having, normally, a yield point of 27 tons/in.^{2} and an ultimate strength of 39 tons/in.^{2}, with an elongation of 20 per cent. on 8″, was made to reach a yield point 47 tons/in.^{2} with an ultimate strength of 49·5 tons/in.^{2} and a total elongation of only 12 per cent., after four immersions.

364. Primitive and Natural Elastic Limits.—It was observed by Bauschinger that the elastic limits of a material vary in position when the specimen is subjected to cyclic variations of stress. The following table shows some of the results of his experiments :

TENSILE STRESSES VARYING FROM 0 TO UPPER LIMIT

Material.	Elastic limit (tons/in.²).		Max. Load (tons/in.²).	Millions of repetitions before fracture.	Tensile strength. (tons/in.²).	
	Original.	Acquired by repetition of loads.			Original.	By repetition of loads.
Wrought Iron Plate	6·84	12·3	7·1	5·17	25·2	23·6
	6·84	14·4	13·1	5·18	25·2	24·5
	6·84	16·4	16·4	2·28	25·2	—
M.S. Plate .	15·6	18·0	16·0	3·55	28·5	—
	15·6	19·4	16·0	6·68	28·5	—
	15·6	19·9	23·0	0·16	28·5	—
	15·6	20·0	26·2	0·34	28·5	—
	15·6	12·3	26·2	0·11	28·5	—
	15·6	11·5	26·2	0·04	28·5	—
Bar Iron . .	11·8	21·4	13·2	9·11	26·6	28·2
	11·8	10·8	19·7	0·64	26·6	—
	14·8	16·3	13·8	16·48	26·7	27·1
	14·8	18·6	17·2	9·31	26·7	26·6
	14·8	11·9	19·7	0·67	26·7	—

Similar results have been obtained by Stanton and Bairstow.* The figures for three tests on Swedish Bessemer steel are given below (tons/in.² units) :—

	Original E.L. in Tension.	Acquired after repetition of loads.		Load varied between	
		Tension.	Compression.	Tension.	Compression.
A	27·7	17·3	—	18·3	13·1
B	25·0	16·4	—	16·9	12·0
C	25·0	—	12·6	16·9	12·0

In general, it seems that, with metals such as steel, the elastic limit, found from the first loading of a specimen, is fictitious, its value depending on the processes of manufacture. This elastic limit is called the *primitive* elastic limit. Regarding tensile stresses as positive, and compressive stresses as negative, the elastic limits in tension and compression can both be increased algebraically, or both decreased, by the application of fluctuating stresses. The algebraic difference between the elastic limits is called the *elastic range.* By the application of a few reversals of stress, through a range less than the primitive elastic range, the material acquires a new elastic range between equal tension and compression, less than the primitive range, and through which the material remains perfectly elastic for alternating stresses repeated indefinitely. The stress limits in this case are called the *natural* elastic limits. Similar limits can be found when the tensile and compressive stresses are unequal.

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* *Phil. Trans. R. Soc.*, Ser. A, Vol. 210.

365. Hysteresis.—The limits of proportionality are defined as the stresses between which the stress is directly proportional to the strain, and within this range Hooke's Law is sensibly true. At the same time it is doubtful whether every stress, however small, does not leave some slight permanent set. Ewing has shown,* by experiments on long wires, that, even with stresses considerably below what is usually considered the elastic limit, there is a lagging of strain with regard to stress. If a specimen be loaded and unloaded very carefully, and the strain measured accurately, it is found that, at any intermediate value of the load, the strain during unloading is greater than that during loading (see Fig. 386). This lagging of the strain behind the stress is called hysteresis. In Ewing's experiments the greatest difference between the strains for loading and unloading was about one three-hundredth of the greatest strain, so that, between what are called the elastic limits, we can safely take Hooke's law as true.

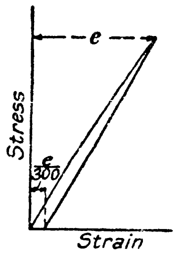

FIG. 386.

If, after unloading the tensile load, a gradually increasing compressive load be applied up to the same value and then taken off, the diagram will take the form shown in Fig. 387, and after a few reversals of stress the metal will take up the cyclic state, the same loop (Fig. 388) being traced

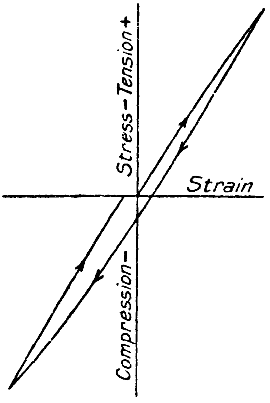

FIG. 387.

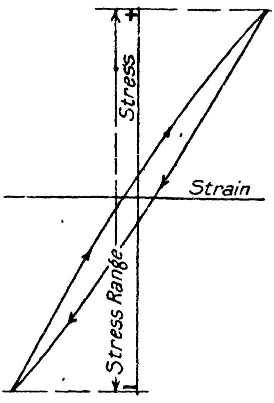

FIG. 388.

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* *British Assoc. Report,* 1899.

out for each stress cycle. This loop is called the hysteresis loop for that particular material and stress range.

As the stress range increases, the width and area of the hysteresis loop increases, as shown in Fig. 389, which gives the results of one of Bairstow's * experiments on a specimen of axle steel, the gauge length being 0·5″.

The straight line (1) is the graph for the earlier stress repetitions over a range ± 14·1 tons/in.^{2}, which was between the primitive and natural elastic ranges. For this range the material at first appears to be perfectly elastic, but as the repetitions were increased the hysteresis loop developed until it had acquired the shape (2) after 18,750 alternations. When the stress range was increased to ± 15 tons/in.^{2}, the curve (3) was obtained after 23,260 repetitions. With a range of ± 20·2 tons/in.^{2} and 29,280 repetitions, the large loop was obtained. Bairstow expresses the view that for a slightly smaller stress range, 13 tons/in.^{2}, say, the steel would have remained perfectly elastic.

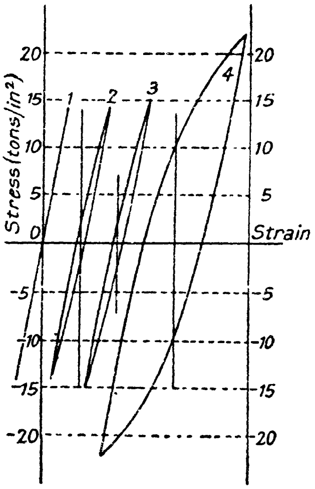

FIG. 389.

The probability of this being strictly true seems doubtful, except for very slow loading and unloading, for with rapid alternations of stress there is a rise of temperature, and consequently a loss of heat to the surroundings, a process which must be irreversible. Even with slow loading and unloading there is probably *some* hysteresis with all materials for all loads, however small. Thus Ewing † says: "In the case of iron, there is indirect evidence that all alteration of stress whatsoever affects the molecular structure in a way not consistent with the notion of perfect elasticity. When the state of stress in iron is varied, however slowly and however little, the magnetic and thermo-electric qualities of the metal are found to change in an essentially irreversible manner. Every variation leaves its mark on the quality of the piece; the actual quality at any time is a function of all the states of stress in which the piece has previously been placed. It can scarcely be doubted that sufficiently refined methods of experiment would detect a similar want of reversibility in the mechanical effects of stress, even when alterations of stress take place very slowly."

366. Fatigue Range.—We are now in a position to proceed with the discussion of the strength of metals under fluctuating stresses. The information which the practical engineer requires is: What are the limits to the stress which a piece can withstand, with an indefinite number of repetitions, without rupture ? These limits are called the fatigue limits, and their algebraic difference is the fatigue range. If the smaller stress is fixed, the maximum stress is fixed by the fatigue range, thus

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* *Loc. cit.*, p. 531.

† Ewing, *Strength of Materials* (1899), p. 55; see also *Phil. Trans.*, 1855, 1886; *B. Ass.*, 1889.

$$p_{max} - p_{min} = \text{fatigue range},$$

where p_{max} and p_{min} are the maximum and minimum values of the fluctuating stress.

When the stress is alternating, so that the maximum and minimum stresses are numerically equal, the value of either is half the fatigue range and is called the *fatigue limit.*

For alternating stresses the fatigue limit appears to be equal to the stress at which the hysteresis loop seems to begin. But when the maximum and minimum stresses are unequal there is early hysteresis which ceases as the metal settles down into a new condition of stability. Even with alternating stresses some materials show early hysteresis, which disappears later, when they have considerable internal stresses ; for instance, hardened but untempered steel, certain alloy castings, etc.

A considerable amount of research work has been done in connection with the strength of metals under fluctuating stresses, by Wöhler,* Reynolds and Smith,† Bauschinger (*loc. cit.* above), Bairstow and Stanton (*loc. cit.* above), Rogers,‡ Arnold,§ Eden,|| B. Hopkinson,¶ Gough,** Haigh,†† Mason,‡‡ Fulton,§§ Scoble,§§ Millington, Thompson,|||| and others. We shall only be able to give a very few samples of their results, and for the rest the reader is referred to the sources mentioned in the footnote below. Reference should also be made to C. F. Jenkin's *Materials of Construction Used in Aircraft* (H.M. Stationery Office), which is full of useful information concerning the mechanical properties of materials.

Fig. 390 and the Tables on p. 535 will afford examples of fatique ranges.

It will be noticed that it is the *range* of stress, rather than the maximum stress, which determines the number of repetitions required to bring about failure, and that the effect of sharp corners, at the junction between the wide and narrow parts of the specimen, considerably reduces the endurance for a given stress.

Curves showing the fatigue ranges can be conveniently plotted by taking the minimum stresses as abscissæ, and the corresponding ranges as ordinates (see Figs. 390 and 398).

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* See *Engineering*, Vol. XI, 1871 ; also Unwin's *Testing of Materials of Construction.*

† *Phil. Trans. R. Soc.*, 1902 ; *Journal of Iron and Steel Institute*, No. 2, 1910.

‡ *Journ. Iron and Steel Inst.*, No. 1, 1905.

§ *Brit. Ass.*, 1904 ; *Inst. Mech. Eng.*, 1904 ; *Proc. Inst. Naval Architects*, 1908.

|| *Proc. Inst. Mech. Eng.*, 1911.

¶ *Proc. R. Soc.*, A., 1912.

** *Engineer*, Aug. 12, 1921.

†† *B.A. Report*, 1915; *Jour. Inst. Metals*, No. 2, 1917; *Trans. Faraday Soc.*, 1928, and *Brit. Ass.*, 1923, 1924.

‡‡ *Brit. Ass. Report*, 1913, 1921, 1923.

§§ *Brit. Ass. Report*, 1919.

|||| *Journ. Inst. of Metals*, 1924.

(WÖHLER) REPEATED TENSILE STRESSES

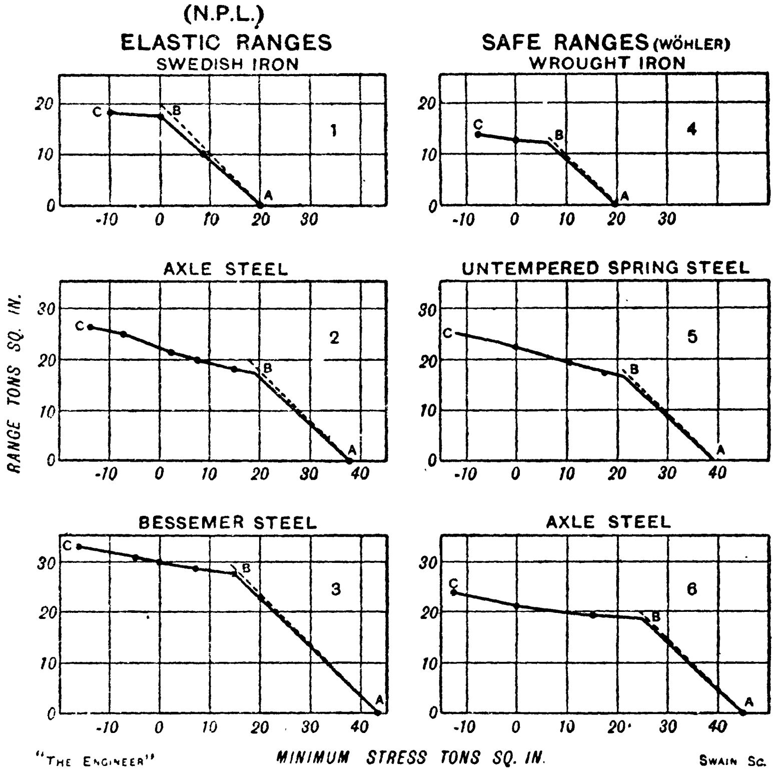
Material.	Applied Stress Tons/in.^{2}		Range of Stress Tons/in.^{2}	Number of repetitions before fracture.
	Max.	Min.		
Iron axle, Phœnix Iron Company. Ultimate static strength 21·1 tons/in.^{2} with 17·8 % elongation	21·01	0	21·01	106,910
	19·10	0	19·10	340,853
	17·19	0	17·19	409,481
	15·28	0	15·28	10,141,645
	21·01	9·55	11·46	2,373,424
	21·01	11·46	9·55	not broken after four million repetitions
Krupp's cast steel axles (square shoulders joining narrow part to wide part of test piece)	23·89	0	23·89	23,546
	21·97	0	21·97	35,486
	19·10	0	19·10	75,343
	15·29	0	15·29	274,970
	14·34	0	14·34	not broken after 1·1 million repetitions
Ditto, but with well-rounded shoulders	23·87	0	23·87	473,766
	22·92	0	22·92	not broken after 13·6 million
	21·95	0	21·95	not broken after 13·2 million
	38·2	23·87	14·33	not broken after 1·8 million
	38·2	16·7	21·5	not broken after 12 million

WÖHLER'S EXPERIMENTS ON BARS SUBJECTED TO ALTERNATING TRANSVERSE LOADS (ROTATING BARS); EQUAL AND OPPOSITE LIMITS

Material.	Stress Range Tons/in.^{2}	Number of cycles before fracture.
Phœnix iron for axles	+ 15·3 to − 15·3	56,430
	13·4 „ − 13·4	183 145
	11·5 „ − 11·5	909,840
	9·6 „ − 9·6	4,917,992
	8·6 „ − 8·6	19,186,791
	7·6 „ − 7·6	not broken after 132,250,000
Krupp's cast steel axles	+ 20·1 to − 20·1	55,100
	16·3 „ − 16·3	797,525
	15·3 „ − 15·3	642,675
	15·3 „ − 15·3	1,665,580
	15·3 „ − 15·3	3,114,160
	14·3 „ − 14·3	4,163,375
	14·3 „ − 14·3	45,050,640
Copper	+ 7·64 to − 7·64	30,875
	6·69 „ − 6·69	480,700
	5·97 „ − 5·97	798,000
	5·73 „ − 5·73	2,834,000
	4·78 „ − 4·78	19,327,000

Experiments were made at the N.P.L. on many varieties of steel, after undergoing various heat-treatments, and the results show that, in nearly all cases, the fatigue limit varied from 0·45 to 0·51 of the ultimate strength, but there was no clear relation between the fatigue limit and the elastic limit or yield point.*

Fig. 390 shows how the fatigue ranges were found to vary with the lower stress in six cases.



"THE ENGINEER" SWAIN SC.

FIG. 390.

367. Theory of Fatigue, Hysteresis, etc.—A great deal has been written on this subject, but we do not propose to go into the matter at great length; it is really outside the scope of this book and belongs more properly to the science of metallurgy, or even molecular physics. If we succeed in giving the reader some sort of mental picture of the behaviour of metals under stress we shall have achieved our object. To this end we shall follow the theory put forward by Prof. Jenkin,† as it affords a comparatively simple, possible explanation of the phenomena which occur, and is the most likely to provide a moderately clear idea of the type of changes which apparently go on when metals are strained.

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* Jenkin, *loc. cit.*, p. 534, where detailed results are given.

† *Proc. R. Soc.*, A., Vol. 103, 1923; *Aeronautical Journal*, March, 1923.

Those who wish to go more deeply into the subject must refer to the writings of Ewing, Bairstow, Haigh, and others.

Microscopic examination shows that metals are made up of crystalline grains, which are crystals of irregular shape. In each grain the elementary pieces composing them have a definite orientation, which varies among the several grains. Ewing and Rosenhain have shown * when a metal yields and takes up permanent set, slipping occurs between different parts of the crystalline grains. Thus, if *AC* and *CB* be two neighbouring grains, subjected to stress in the direction of the arrows (Fig. 391), they will develop *slip lines p, q, r, s,* denoting surfaces along which slipping takes place (Fig. 392). These surfaces are usually called *cleavage planes ;* in any sample of metal they will all have widely different orientations. On account of this varying orientation of the cleavage planes the shearing stresses will vary in intensity from crystal to crystal, so that slipping will occur first on those cleavage planes which carry the greatest stress. Before slipping begins it is prevented by adhesion across the cleavage planes ; after it has begun it is resisted by forces having the nature of solid friction.† In general the crystals, or at least some of them, will be in a state of stress before the application of any external load is begun. This may arise from such causes as the unequal contraction of a casting during cooling, cold work done on the material during manufacture, and so on. Thus the load at which any particular crystal will develop slip will depend on (i) its orientation, (ii) its initial condition with regard to stress.

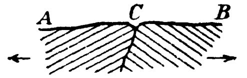

FIG. 391.

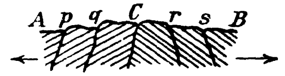

FIG. 392.

(Derived from Mellor's 'Crystallization of Iron & Steel')

If slip occurs in any crystal under alternating stresses, and the two parts slide backwards and forwards on each other, wear and tear will take place ; in due course a crack will be formed and so a "fatigue failure" is started. Sometimes this result is prevented by a sudden seizing, or "healing," of the sliding parts. If this healing process goes on there is a race between wear-and-tear and healing ; if wear-and-tear wins failure of the piece will occur.

Let us look at the matter a little more closely. Consider an individual crystal : up to a point it behaves elastically and its stress-strain diagram will be a straight line *OA* (Fig. 393). Then the adhesion breaks down and the stress will fall to a value corresponding with the friction force, after which it will remain constant, following the straight line BC. If the crystal be initially in a state of tension the graph will be as shown in Fig. 394 ; if in a state of compression it will be as shown in Fig. 395.

Now suppose we have a test-piece consisting only of three crystals which we designate 1, 2 and 3. Let us imagine that 1 and 3 are equally strong as regards adhesion and friction, but 2 is stronger than 1 or 3; imagine also that 1 is initially in a state of tension, 2 free from stress, and 3 in a state of compression which balances the tension in 1. The stress-strain diagram is drawn separately, and numbered, for each crystal, in Fig. 396 (*a*); that for the whole piece is obtained by combining them. At first all the crystals behave elastically, and we get the straight line *OP*, and *P* will be the limit of proportionality for the specimen, the crystal 1 then slipping owing to the adhesion breaking down. The load taken by that crystal then remains constant and equal to the force of friction on the cleavage plane. Crystal 3 is the next to slip, and finally 2, when the graph reaches the point *Y*, which is the yield point for the specimen. In reality, on account of the large number of crystals which go to make up a specimen, we do not get a jagged graph like this, but, rather, a smooth curve, as shown by the dotted line, which will be seen to resemble the stress-strain diagram for a material having a low limit of proportionality and ill-defined yield point.

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* *Proc. R. Soc.*, 1899. † Ewing, *Proc. R. Soc.*, Vol. 200.

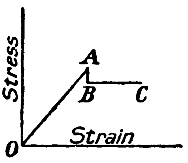

FIG. 393.

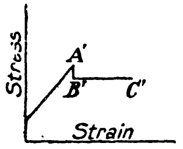

FIG. 394.

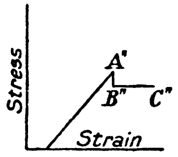

FIG. 395.

Fig. 396 (*b*) shows how the graph is altered by a rearrangement of the initial stresses. In this case 2 is initially in tension, whilst 1 and 3 are in compression. We get a much higher limit of proportionality and a more clearly marked yield point.

Underneath Figs. 396 (*a*) and (*b*) are shown the stress-strain diagrams for air-hardened 100-ton steel, untempered and tempered, and it will be seen that there is a resemblance between (*a*) and (*c*), and also between (*b*) and (*d*). The figures which we have traced out above should show how it is that any treatment, such as tempering, annealing, quenching, etc., which can redistribute the initial stresses in the crystals, will modify the stress-strain diagram and the position of the limit of proportionality.

A similar diagram may be drawn to explain the drop of stress which occurs at yield with materials like wrought-iron and mild steel, but it will be necessary to suppose a test-piece made up of about eight or ten crystals, most of which have nearly equal strengths. When this is the case, as soon as slip occurs in one crystal the extra load thrown on the others will cause the next crystal to slip, and so on.* We can also draw the compression curves by a similar process of reasoning. If each crystal is taken as equally strong in compression and tension it will be found that, on account of initial stresses, the limits of proportionality in tension and compression are unequal, but the modulus of elasticity will be the same in both cases. The compression curve for the conditions of Fig. 396 (*b*) is shown by *OP′Y′* in the lower part of Fig. 397.

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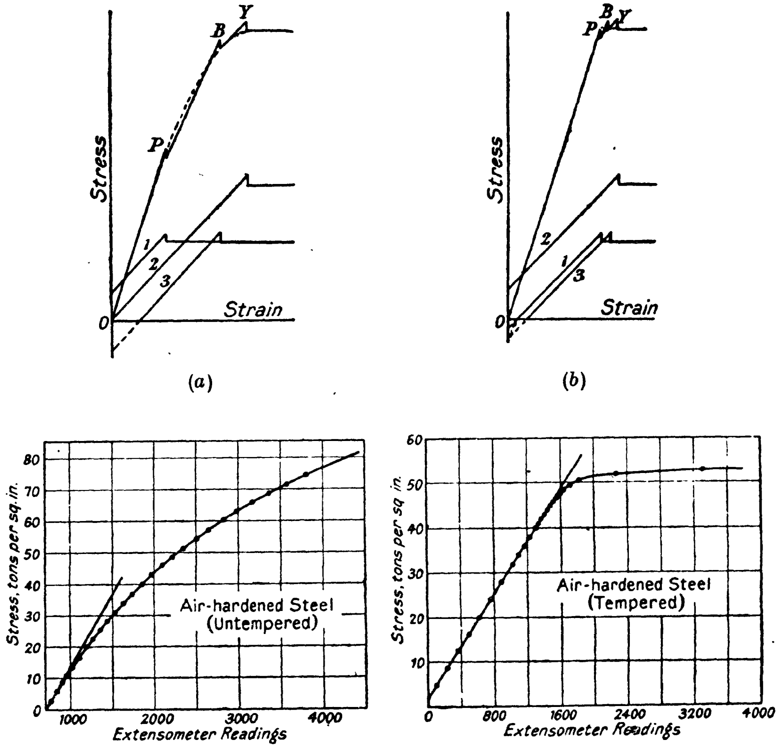
* Prof. Jenkin has devised an ingenious and delightfully simple model by which all these points can be illustrated. It is described in the papers referred to on p. 536 and can be made easily by anyone.

The hysteresis loop resulting from overstrain can be explained by a similar argument, and is shown dotted in Fig. 397 for the initial conditions taken for Fig. 396 (*b*), but here we neglect the drop of load which occurs when adhesion breaks down. Suppose, when all three crystals have slipped we keep the load on until the point H is reached, and then begin to reduce it. We are now, in effect, beginning a compression test with all the crystals in an initial state of tension, and, as we reduce the tension, the crystals will compress elastically until the thrust is great enough to overcome the friction of crystals 1 and 3. This happens at *K*. Crystals 1 and 3 then begin slipping in the direction of the thrust; the increase of load is then taken by 2 until that slips at *L*. If the load is now kept on until *M* is reached and then decreased, the graph will follow the dotted lines through *N* back to *H*.



(*a*) (*b*)

(*c*) (*d*)

FIG. 396.

Next suppose that our three-crystal specimen is subjected to an alternating load of sufficient amplitude to cause all the crystals to slip to and fro. As each new cycle is commenced there will be new " initial " stresses, with the result that, if we gradually decrease the range of the load, first one, and then another, and finally the last, of the crystals will cease slipping, having arrived at such a state that it is free from stress when the load is zero. The whole piece will then behave elastically; the range of stress is now the fatigue range, the limits being the natural elastic limits. It will be a profitable exercise if the reader, by a succession of applications of the above principles, will try to discover the fatigue range for the hypothetic case of Fig. 397.

If, instead of applying equal positive and negative loads, we apply an unsymmetrical fluctuating load, we shall arrive again at an elastic range in which none of the crystals slip, and this range will always be equal to the elastic range of the crystal which offers least resistance to slipping. Hence, as long as the lower stress is negative, the elastic range, and therefore the fatigue range, will be constant. But this cannot be the case when the value of the lower stress exceeds a certain value, since we clearly must not have the large stress limit greater than the yield point of the material. From these considerations, then, we should expect the graph of the fatigue range, plotted against the lower stress limit, to take the form shown in Fig. 398 (a), AB being at a height corresponding with the fatigue range for alternating stresses, and $OC = OD =$ the yield point. If we take into consideration the fact that the adhesion stress is greater than the friction stress, the diagram is modified in the

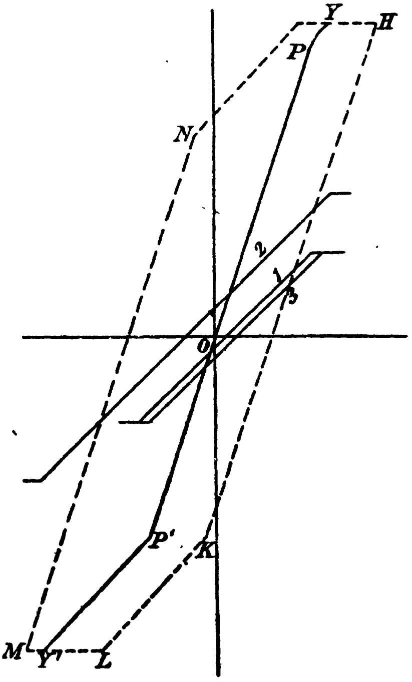


FIG. 397.

manner shown in Fig. 398 (*b*). These diagrams should be compared with Fig. 390.

The above must only be regarded as the outline of one of several

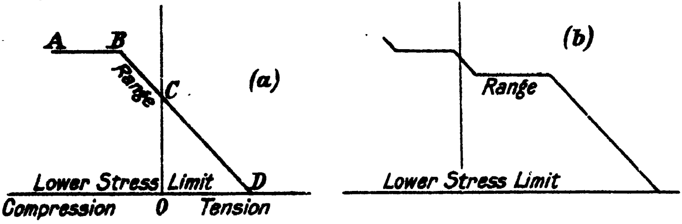


FIG. 398.

theories which have been put forward, any of which may ultimately be found to be the truth, although possibly the real explanation is to be found in some elemental truth as yet undiscovered. All we have tried to do here is to give the student a jumping-off ground for further reading.

368. Failure under Combined Cyclic Stresses.*—This differs from the case of failure under constant loads in that it is the *range* of stress which is the decisive factor, so that we have no reason to suppose that any of the theories enumerated in Chapter VI will be applicable. Gough and Clenshaw† give the following formula, which they have found gives very good agreement with experiment, so far as they have gone at present with ductile steels:

$$\frac{p^2}{p_1^2}+\frac{q^2}{q_1^2}=1$$

where $2p_1$ = fatigue range under bending only.
$2q_1$ = " " " torsion "
$2p$ = safe range of direct stress due to bending.
$2q$ = " " shear " " torsion.

When p_1 and q_1 have been determined by experiment, this formula will give the safe value of q for a given p, or *vice versa*.

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* See *Strength of Metals under Combined Alternating Stresses*, Gough and Pollard, Inst. Mech. Eng., 1935. Also *Fatigue of Metals*, Gough (Benn).

† *Inst. Marine Engineers*, Nov., 1935. For fatigue in welded joints, and under the influence of corrosion, see paper by Gough and Clenshaw cited above. This is altogether a valuable summary on the testing of materials.

APPENDIX I

1. PRINCIPAL STRESSES IN THREE-DIMENSIONS

369. Principal Stresses in Three-Dimensional System.—For any given stress system there are three planes across which the resultant stress is normal to the plane, i.e. there is no shearing stress on these planes. These planes are the principal planes of stress and the stresses are the principal stresses. We shall now proceed to prove this statement and show how the planes can be found.

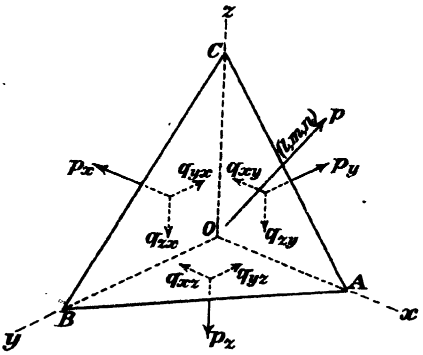

FIG. 399.

Let *OABC* be a tetrahedral element containing the point at which the stresses are under consideration, the stress components on the faces *OAB*, *OBC*, *OCA*, parallel to the three edges *OA*, *OB*, *OC*, being known.

Suppose that *ABC* is a plane on which the resultant stress is p normal to *ABC*.

Let S = the area of *ABC*, and let l, m, n be the direction cosines of the normal to this plane referred to *OA*, *OB*, *OC* as axes of x, y, z.

The stresses on the three faces *OAB*, *OBC*, *OCA* are shown in Fig. 399, and the edges *OA*, *OB*, *OC* are mutually perpendicular.

The areas of *OBC*, *OCA*, *OAB* are respectively lS, mS, nS.

The components of pS parallel to the three axes are $l.pS$, $m.pS$, $n.pS$.

Consider the equilibrium of the element and resolve in the directions Ox, Oy, Oz:

$$\begin{aligned} l.pS &= p_x.lS + q_{xy}.mS + q_{xz}.nS \\ m.pS &= p_y.mS + q_{yz}.nS + q_{yx}.lS \\ n.pS &= p_z.nS + q_{zx}.lS + q_{zy}.mS. \end{aligned}$$

Dividing through by S and remembering that

$$q_{yx} = q_{xy}; \quad q_{zy} = q_{yz}; \quad q_{xz} = q_{zx},$$

these equations become

$$\left.\begin{aligned} -l(p - p_x) + m.q_{xy} + n.q_{zx} &= 0 \\ lq_{xy} + m(p - p_y) + n.q_{yz} &= 0 \\ l.q_{zx} + mq_{yz} - n(p - p_z) &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (i)$$

Eliminating l, m, n we have:

$$\begin{vmatrix} p - p_x & -q_{xy} & -q_{zx} \\ -q_{xy} & p - p_y & -q_{yz} \\ -q_{zx} & -q_{yz} & p - p_z \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

This is a cubic equation for p, and it can be shown that the three roots are always real. If we take one of the roots of this equation and substitute in equations (i) we obtain three equations which, with the relation

$$l^2 + m^2 + n^2 = 1,$$

enable us to find the corresponding values of l, m, n.

Thus there are three planes which satisfy the required conditions.

Equation (ii) when expanded is:

$$\begin{aligned} p^3 - (p_x + p_y + p_z)p^2 + (p_xp_y + p_yp_z + p_zp_x - q_{xy}^2 - q_{yz}^2 - q_{zx}^2)p \\ + (p_xq_{yz}^2 + p_yq_{zx}^2 + p_zq_{xy}^2 - 2\, q_{xy}q_{yz}q_{zx} - p_xp_yp_z) = 0 \end{aligned}$$

and from this the principal stresses are found. Now the equation for finding the semi-axes of the conicoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzn + 2hy = 1$$

is

$$\begin{vmatrix} a - \dfrac{1}{r^2} & h & g \\ h & b - \dfrac{1}{r^2} & f \\ g & f & c - \dfrac{1}{r^2} \end{vmatrix} = 0$$

where r denotes the length of any of the semi-axes. We see that this is of exactly the same form as equation (iii).

It follows that if we construct the conicoid whose equation is

$$p_xx^2 + p_yy^2 + p_zz^2 + 2q_{yz}yz + 2q_{zx}zx + 2q_{xy}xy = 1,$$

the reciprocals of the squares of the semi-axes will be the values of the three principal stresses. Also, since we know that the semi-axes of a conicoid are mutually perpendicular, we see that the lines of action of the three principal stresses are also mutually perpendicular.

*Maximum Shear Stress.** Let p_1, p_2, p_3 be the principal stresses, and take axes Ox, Oy, Oz in the directions of these (Fig. 400). Now, instead of considering this system of stresses, consider that formed by subtracting from each the mean of the greatest and least. Let the order of magnitude of the principal stresses be $p_1 > p_2 > p_3$. Then to derive the second system we shall subtract from each $\frac{1}{2}(p_1 + p_3)$. This leaves us with $\frac{1}{2}(p_1 - p_3)$, $p_2 - \frac{1}{2}(p_1 + p_3)$, $-\frac{1}{2}(p_1 - p_3)$, which we shall denote by R, T, $-R$ and they will be in this order of magnitude.

---

* The demonstration given here is due to Boussinesq (see Todhunter and Pearson's *History of Electricity*, Vol. II., pt. ii., p. 219) and is instructive.

The first stress system will only differ from the second by the superposition of a stress uniform in all directions, so that the directions of the maximum and minimum stresses will be the same for both systems.

Now, let ABC be any plane cutting the axes at A, B, C, and let the area of ABC be S.

Let the normal to this plane make angles α, β, γ with the axes.

Let F be the resultant stress on ABC, the direction of F making angles α', β', γ' with the axes.

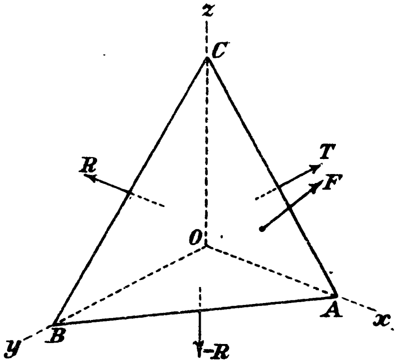

FIG. 400.

The total force acting on $ABC = FS$, and its components in the directions x, y, z are

$$FS \cos \alpha'; \quad FS \cos \beta'; \quad FS \cos \gamma'.$$

The total forces acting on the other faces of the tetrahedron are

$$R \times S \cos \alpha; \quad T \times S \cos \beta; \quad -R \times S \cos \gamma.$$

These components must be equal in pairs. Hence we must have

$$\cos \alpha' = \frac{R}{F} \cos \alpha; \quad \cos \beta' = \frac{T}{F} \cos \beta; \quad \cos \gamma' = -\frac{R}{F} \cos \gamma \quad . \quad \text{(i)}$$

Also, $FS =$ the resultant of the three forces $RS \cos \alpha$, $TS \cos \beta$, and $-RS \cos \gamma$. Therefore

$$\begin{aligned} F &= \sqrt{R^2 \cos^2 \alpha + T^2 \cos^2 \beta + R^2 \cos^2 \gamma} \\ &= \sqrt{R^2 - (R^2 - T^2) \cos^2 \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)} \end{aligned}$$

since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Now, since $R > T$, $R^2 - T^2$ is positive, and therefore we see from (ii) that F is a maximum when $\beta = \frac{\pi}{2}$ and is then equal to R.

Thus the greatest resultant stress is across a plane whose normal is perpendicular to Oy, i.e. perpendicular to the mean stress T. Hence the greatest stress has its direction in the plane xz containing R and $-R$.

If θ is the angle between F and the normal to the plane CA (Fig. 401), $\theta = a - a'$.

$$\therefore \cos\theta = \cos a \cos a' + \sin a \sin a'.$$

Since $F = R$ we have from (i) $\cos a' = \cos a$, and $\cos\gamma' = -\cos\gamma$, i.e. $\sin a' = -\sin a$, hence

$$\cos\theta = \cos^2 a - \sin^2 a = \cos 2a.$$

The shear component of F is $F \sin\theta = F \sin 2a$, which is a maximum when $a = \frac{\pi}{4}$, and is then equal to F. Thus the greatest shear stress is on a plane whose normal bisects the angle between the directions of the greatest and least principal stresses, and is equal to R, i.e. $\frac{1}{2}(p_1 - p_3)$, i.e. half the difference between the (algebraically) greatest and least principle stresses, for *both* the R_1 T_1 $-R$ and p_1 p_2 p_3 systems.

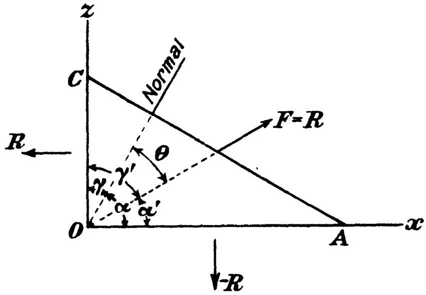

FIG. 401.

2. ANTICLASTIC CURVATURE

370. Change of Cross Section in Uniform Bending, neglecting the Radial Action between Filaments.

In Fig. 402, O is the centre of area of the strained cross-section, Oy and Oz are the principal axes, and P is any point in the section. The section is lettered to correspond with Fig. 168, so that positive values of y correspond with extensions of the longitudinal filaments. PY and PZ are the co-ordinates (y, z) of P.

All longitudinal fibres which after strain are above Oz suffer lateral contraction, and those which are below suffer lateral expansion.

We have seen that for all fibres which are at the same height y above Oz *after strain* the longitudinal strain is

$$e = \frac{y}{R},$$

where R is the radius into which the neutral filament is bent. Therefore the lateral strain of these filaments is given by

$$e' = -\frac{y}{mR}.$$

Let P_0Y_0 represent the *curved* line in the unstrained section which becomes the *straight* line PY in the strained section, and let the length of the arc P_0Y_0 be l. Then we have

$$\frac{l-z}{l} = \frac{y}{mR}$$

or

$$\frac{y}{mR} + \frac{z}{l} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

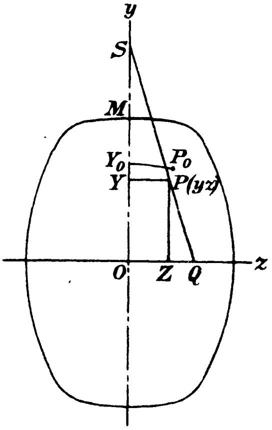

FIG. 402.

Since the distortion is small, let us regard P_0Y_0 as the arc of a circle of large radius ρ. Then

$$\frac{z_0}{\rho} = \sin\frac{l}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where z_0 is the abscissa of P_0. Since l/ρ is small we can write with sufficient accuracy:

$$\sin\frac{l}{\rho} = \frac{l}{\rho} - \frac{l^3}{6\rho^3}.$$

Hence from (ii)

$$\frac{z_0}{\rho} = \frac{l}{\rho} - \frac{l^3}{6\rho^3}$$

$$\therefore \frac{l-z_0}{l} = \frac{l^2}{6\rho^2},$$

which shows that we can neglect the difference between l and z_0. In this case, (i) becomes

$$\frac{y}{mR} + \frac{z}{z_0} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Hence the line represented by $z = z_0$ in the unstrained section becomes the line given by (iii) in the strained section. This line cuts Oy at a point S where $OS = mR$, that is at a fixed point. Thus all straight lines which were originally parallel to Oy are strained into straight lines passing through the point S.

Now we have assumed throughout that the longitudinal filaments have no action on one another. If this is so, it follows that the shapes of the sections of the filaments are not changed by strain, and that curves on the unstrained section which cut orthogonally will be transformed into curves which cut orthogonally in the strained section. Therefore, if the lines originally parallel to Oy are strained into straight lines radiating from S, the lines which were parallel to Oz will be strained into arcs of circles having S as centre; and the radius of the circle which passes through O is

$$R' = mR. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We must next consider how any vertical distance Q on the axis Oy is altered by the strain.

Let y_0 and $y_0 + dy_0$ be the heights, above O, of two neighbouring points on Oy in the unstrained section, and let them become y and $y + dy$ after strain. Then the distance between them is changed from dy_0 to dy, so that we must have

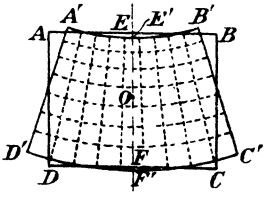

FIG. 403.

$$\frac{dy_0 - dy}{dy_0} = \frac{y}{mR} = \frac{y}{R'}$$

since y/mR is the lateral strain at height y.

$$\therefore \frac{dy}{dy_0} + \frac{y}{R'} = 1.$$

The solution of this equation, subject to the condition that $y = 0$ when $y_0 = 0$, is

$$y = R'\left(1 - e^{-\frac{y_0}{R}}\right) = y_0 - \frac{y_0^2}{2R'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

approximately, when y_0/R' is small.

We now have sufficient information to draw the strained cross section. In the case of a section which was originally rectangular, such as $ABCD$ in Fig. 403, the strained section takes the form shown by $A'B'C'D'$. The circular arc through O has a radius mR; the heights of E' and F' are given by (2).

371. Effect of the Radial Forces on a Beam of Rectangular Section.—Consider the element PQ (Fig. 404) of a filament bent to a radius $R + y$, R being the radius of the neutral filament. Let $ds = (R + y)d\theta$ be the length of PQ, and let dS be the area of the cross section of the filament. Let p be the normal stress on the ends of PQ due to a bending movement M. Then, as in § 118,

$$p = \frac{Ey}{R}$$

and the total force on each end of PQ is

$$pdS = \frac{Ey}{R}\,dS.$$

The resultant of these two forces is a radial force

$$\frac{Ey}{R}\,dS\,.\,d\theta.$$

Thus the force changes sign with y, acting inwards towards the axis of bending when y is positive, and outwards when y is negative.

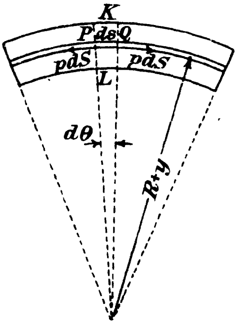

Fig. 404.

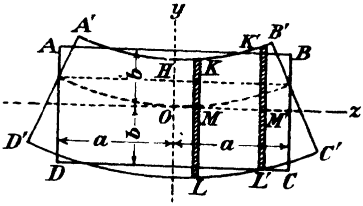

Fig. 405.

Considering any vertical strip (Fig. 405) parallel to the plane of bending, such as KL, the part KM will be pulled radially inwards with a force

$$\frac{Ed\theta}{R}\int_M^K y\,.\,dS = \frac{Ed\theta\,.\,dz}{R}\int_M^K ydy = \frac{E\,.\,d\theta\,.\,dz}{R}\,.\,\frac{MK^2}{2}.$$

Similarly the part ML will be pushed radially outwards with a force

$$\frac{E\,.\,d\theta\,.\,dz}{R}\,.\,\frac{ML^2}{2}.$$

If $ML > MK$ the resultant is an outward push, but if $ML < MK$ it is an inward pull.

Let us now concentrate our attention on a rectangular section of width $2a$ and depth $2b$, as shown in Fig. 405. On KL the nett result of the radial forces will be an upward push, whilst on $K'L'$ it will be downward pull. Hence the total effect of these radial forces on the right-hand half of the section will be a couple acting in a clockwise direction, and there will be a similar anticlockwise couple for the left-hand half of the section. The effect of these couples will evidently be to diminish the transverse curvature of the section.

Now the magnitude of these couples depends upon the difference in height between *KM* and *ML*. The radius of the arc through *O* is $R' = mR$, so that the height *OH* (Fig. 405) is $a^2/2R'$. When this is small compared with b the couple due to the radial forces must be small. Therefore, in this case, the results we have obtained above in § 118 and § 370 will be good approximations to the truth. The criterion, then, is that we should have b large compared with $a^2/2R'$, i.e. R' large compared with $a^2/2b$, or R large compared with $a^2/2bm$.

It is, however, quite possible for R' to be *small* compared with $a^2/2b$ in the case of thin flat beams, i.e. when a/b is large. For instance, consider the case of a strip of metal 3″ wide by 0·22″ thick, bent to a mean radius of 10″. If $m = 10/3$, we have $R' = 33\frac{1}{3}''$, and $a^2/2b = 112{\cdot}5$, which is more than three times as large as R'. In such cases the distortion of the section will not be given by the above results. This is shown by the following considerations:

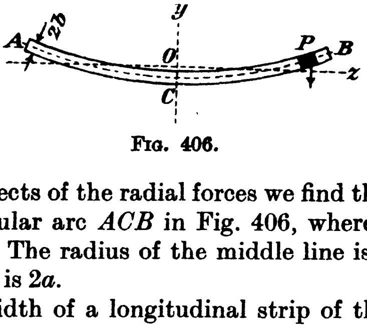

FIG. 406.

If we omit the effects of the radial forces we find that the cross section is bent into the circular arc *ACB* in Fig. 406, where *O* is the centre of gravity of this arc. The radius of the middle line is $R' = mR$, and the length of the arc *AB* is $2a$.

Let ds' be the width of a longitudinal strip of the beam at *P*, then the area of the cross section of this filament is $2b \,.\, ds'$. The radial force at *P* will therefore be $\frac{Ey}{R} 2b \,.\, ds' \,.\, d\theta$, where $d\theta$ has the same meaning as in Fig. 404. The moment of this about *O* will be $\frac{Ey}{R} 2b \,.\, ds' \,.\, d\theta \,.\, z$. Hence the total bending moment at *O* will be given by

$$M' = \int_C^B \frac{Ey}{R} \,.\, 2b \,.\, ds' \,.\, d\theta \,.\, z$$

$$= \frac{2bd\theta E}{R} \int_C^B yz \,.\, ds'.$$

If we neglect the difference between s' and z, this becomes

$$M' = \frac{2b \,.\, d\theta \,.\, E}{R} \int_0^a yz \,.\, dz. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

To evaluate the integral we must express y in terms of z. The arc ACB is approximately represented by the equation

$$y = \frac{z^2}{2R'} - OC \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

Now, if 2ϕ is the angle subtended by the arc ACB at its centre $\phi = a/R'$, and the distance of O from the centre of the arc is $R'\dfrac{\sin\phi}{\phi}$ since O is the centroid of ACB. Hence

$$OC = R' - R'\frac{\sin\phi}{\phi} = R'\frac{\phi^2}{6} = \frac{a^2}{6R'}$$

since ϕ is a small angle.

Therefore (ii) becomes

$$y = \frac{z^2}{2R'} - \frac{a^2}{6R'}.$$

Substituting this value of y in (i) gives

$$M' = \frac{2b \,.\, d\theta \,.\, E}{R}\int_0^a \left(\frac{z^3}{2R'} - \frac{a^2 z}{6R'}\right)dz = \frac{a^4 bEd\theta}{12RR'},$$

and M' is such as to decrease the curvature of ACB. Since $1/R$ is proportional to the bending moment producing R, and since R' is proportional to R, we see that M' is proportional to the square of the applied bending moment.

In order to obtain a rough estimate of the effect of this bending moment let us apply the formula for simple uniform bending. The moment of inertia of a section of the strip of width $Rd\theta$ is $\frac{2}{3}Rb^3d\theta$. Hence, if $\frac{1}{\rho}$ denote the change of curvature of ACB, we have

$$\frac{1}{\rho} = \frac{a^4 bEd\theta}{12RR' \,.\, E \,.\, \frac{2}{3}Rb^3 \,.\, d\theta} = \frac{a^4}{8b^2R^2R'} \quad . \quad . \quad . \quad . \qquad (3)$$

representing a decrease of curvature. This will be greater than $\frac{1}{R'}$ if $a^4 > 8b^2R^2$, that is if $R < \frac{a^2}{2\sqrt{2b}}$. Thus, when R is less than about $a^2/3b$ we may expect the curvature of ACB to be nearly or quite neutralized. In this case a different investigation of the problem is necessary.

372. Simple Bending of a Thin Flat Beam.—Experiment shows that when a beam, such as we have just considered, is bent there is no appreciable curvature of the transverse filaments. We shall, therefore, investigate the flexure of a thin rectangular beam on the assumption that the cross sections remain rectangular and parallel to the axis of bending.

In Fig. 407 $ABCD$ is a section of the beam when bent; $AB = 2a$ and $BC = 2b$; Oz is the line in which the neutral unstrained *longitudinal* filaments cut $ABCD$.

Consider a small rectangular element of area at *P*. Since this is rectangular before and after strain it must be acted on only by normal stresses. If we neglect the radial forces due to the curvature of the longitudinal filaments, the rectangle at *P* will be acted on only by tensile stresses p_z. The longitudinal stress will be denoted by p_x. Let e_x and e_z denote the corresponding strains. Then

$$e_x = \frac{p_x}{E} - \frac{p_z}{mE}$$

$$e_z = \frac{p_z}{E} - \frac{p_x}{mE},$$

whence we have

$$\frac{m^2-1}{m^2} p_x = Ee_x + \frac{Ee_z}{m} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\frac{m^2-1}{m^2} p_z = \frac{Ee_x}{m} + Ee_z \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where $e_x = \frac{y}{R}$, just as before in § 118.

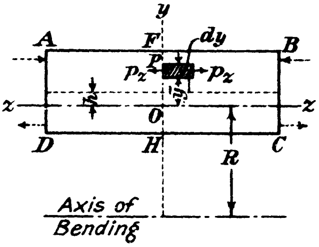

FIG. 407.

Now, let h = the height of the c.g. of *ABCD* above *Oz*.

Then $OF = b + h$, and $OH = b - h$, and I, the moment of inertia of *ABCD* about *Oz*, is $4ab\,(h^2 + \frac{1}{3}b^2)$.

Since the beam is bent by terminal couples only the resultant force normal to *ABCD* must be zero. Therefore

$$\int_{h-b}^{h+b} p_x \, . \, 2a \, . \, dy = 0.$$

Substituting for p_x from (i) and inserting the value of e_x this gives

$$\int_{h-b}^{h+b} \left(\frac{y}{R} + \frac{e_z}{m}\right) dy = 0.$$

Evaluating the integral and simplifying, this leads to

$$e_z = -\frac{mh}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then the bending moment about Oz is given by

$$
\begin{aligned}
M &= \int_{h-b}^{h+b} 2ap_x \cdot y\,dy \\
&= \frac{2am^2}{m^2-1}\int_{h-b}^{h+b}\left(\frac{Ey}{R}+\frac{Ee_z}{m}\right)y\,dy \quad . \quad . \quad . \quad . \quad [by\ (i)] \\
&= \frac{2am^2E}{m^2-1}\int_{h-b}^{h+b}\left(\frac{y}{R}-\frac{h}{R}\right)y\,dy \quad . \quad . \quad . \quad . \quad [by\ (iii)] \\
&= \frac{2am^2E}{m^2-1}\left\{\frac{6h^2b+2b^3}{3R}-\frac{2h^2b}{R}\right\} \\
&= \frac{4ab^3m^2E}{3(m^2-1)R}.
\end{aligned}
$$

When R is large compared with b we may neglect h^2 in comparison with $b^2/3$, and the moment of inertia is $\frac{4}{3}ab^3$ where a and b are initial dimensions. We can then write

$$M = \frac{m^2}{m^2-1}\cdot\frac{EI}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where I refers to the unstrained section.

We have here neglected the radial forces, and this is justified since they are proportional to $\frac{p_x}{R+y}$, that is to $1/R^2$ approximately, and b/R is small.

But the tensions corresponding with p_z do not exist at the edges AD and BC in reality, so that a correction ought to be made for this. We can neutralize them by applying forces indicated by the dotted arrows, which will tend to change the section of the beam. When this happens radial forces will be called into play which will tend to check this change. We may expect then that the distortion near the edges will be extremely slight, and this is confirmed by experiment.

373. Change in the Type of Bending.—The conclusion arrived at in § 371 was that, if R is less than about $a^2/3b$, the anticlastic curvature would be nearly or quite realized, but (§ 372) there may be a small distortion of the cross section near the edges, and the radius of the bent neutral filament is given by

$$\frac{M}{I} = \frac{m^2}{m^2-1}\cdot\frac{E}{R}.$$

If, however (§ 371), R is large compared with $a^2/2bm$, there will be anticlastic curvature of radius mR, R being given by

$$\frac{M}{I} = \frac{E}{R}.$$

Thus a thin flat blade is slightly less stiff for small curvatures than for large ones. If we apply a continuously increasing bending moment to such a blade, at first anticlastic curvature will be present, but a *critical* stage will be reached when the *type* of bending will change, the blade becoming flat except near the edges—another of those very frequent instances where continuous quantitative change brings about a discontinuity in quality.

3. DEFLECTION OF BEAMS

374. Alternative Graphical Treatment of Deflections.—It has been shown in § 159 that the relation between the deflection y of the axis of a beam, the bending moment M, and the moment of inertia I, is

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}.$$

It has also been shown in § 96 that the relation between the bending moment, M, and the load per unit length, w, is

$$\frac{d^2M}{dx^2} = -w.$$

These two equations show that the relationship has the same nature in both cases, and that we may deduce y from a curve of M/EI in the same way as we can deduce M from a curve of w. In fact the method given in §§ 172-174 to obtain the deflections is exactly the same as the method described in §§ 99 and 103 to derive the bending moment diagram from the load curve.

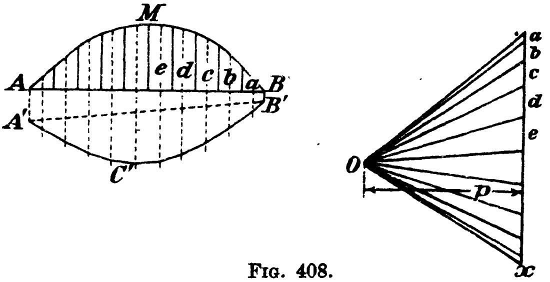

FIG. 408.

Similarly we can employ the method of § 104: the bending moment diagram is treated as a load curve, and the corresponding funicular polygon gives the deflection curve if redrawn on a horizontal base.

In Fig. 408 the curve *AMB* represents the bending moment diagram for the beam AB. This must be divided into a number of vertical strips as shown. Each strip is then imagined to be replaced by a vertical concentrated load proportional to its area and acting at the centroid of the strip. For practical purposes the centroid may be taken as being on the middle ordinate of the strip. We therefore draw vertical lines through the centre of each strip as shown, and the area of the strips may be taken as represented by the heights of their middle ordinates.

Next, on a vertical line, mark off lengths proportional to the heights of the middle ordinates of the strips; this is readily done with proportional compasses. In order to get the deflection curve the right way up the pole O must be taken on the left of the vertical line if the top division of this corresponds with the vertical strip nearest B, to the right if it corresponds with the vertical strip nearest A.

A pole O is then chosen and the funicular polygon $A'C'B'$ drawn in the usual way. Usually the line $A'B'$ will not be horizontal, and the deflections will be given by the *vertical* depth of the funicular polygon measured from this line. If desired the curve can be redrawn on a horizontal base.

SCALES:

Let the scales of the B.M. diagram be

$$1'' = s \text{ ins.}$$
$$1'' = m \text{ lbs. ins.}$$

Let the number vertical strips be n, and the length of the base AB be l inches on the diagram. Then the width of each strip will be $\frac{l}{n}$ inches, representing $\frac{ls}{n}$ inches on the real beam.

If $h =$ the height of one of the strips in inches, the area of the strip will be $\frac{hl}{n}$ square inches, representing $\frac{hlsm}{n}$ lbs. ins.^{2}.

Let the divisions on the vertical line ax be drawn $\frac{1}{k}$ times the heights of the corresponding strips, so that 1″ vertical on this line represents $\frac{khlsm}{n}$ lbs. in.^{2}.

Then the scale of the funicular polygon will be $1'' = \frac{khls^2m}{n}\,p$ lbs. in.^{3}, when p is the horizontal distance of the pole O from the vertical line ax. That is to say, vertical ordinates of the funicular polygon represent EIy to this scale, and therefore represent y to a scale

$$1'' = \frac{khls^2m}{nEI} \cdot p,$$

if E is in lbs./in.^{2}, and I ins.^{4}.

If I is not constant the starting point must be a curve of M/I instead of a curve of M.

4. FRAMES WITH STIFF JOINTS

375. Application of Strain Energy Principle.—A generalization of the theorem of least work (§ 45), is as follows: If the strain energy of an elastic solid structure be expressed as a function of any unknown forces or couples, the values which these forces and couples must have after strain are such as to make the strain energy a minimum, due regard being paid to the fundamental equations which exist between them.

This principle is frequently of use in solving problems such as those of § 257, and we shall illustrate its use by applying it to the following example.

In Fig. 409 $ABCD$ is a framework of rods of uniform section rigidly joined together at B and C, and pin-jointed to a rigid bed at A and D.

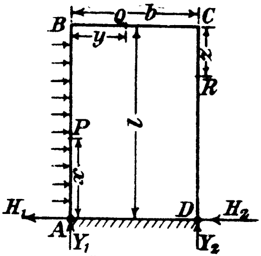

FIG. 409.

AB is subjected to a horizontal distributed load w per unit length. It is required to find the horizontal reactions at A and D, neglecting the lengthening or shortening of the rods, and the effects of shear.

Let H_1 and H_2 be the horizontal reactions at A and D.

Let $AB = DC = l$; $BC = b$.

Then evidently

$$-Y_1 = Y_2 = \frac{wl^2}{2b}$$

and

$$H_1 + H_2 = wl.$$

Let us chose H_1 as our unknown, and write

$$H_2 = wl - H_1. \qquad \text{(i)}$$

Since we are neglecting the extensions and contractions of the rods, the strain energy which we have to consider is that due to bending only.

At a point P in AB, distant x from A, the bending moment is

$$M_1 = H_1 x - \frac{wx^2}{2}, \qquad \text{(ii)}$$

and the strain energy of AB is (§ 136)

$$U_1 = \frac{1}{2E_1I_1}\int_o^l M_1^2 dx \qquad \text{(iii)}$$

where E_1 and I_1 are Young's Modulus and the sectional moment of inertia of AB.

At a point Q in BC, distant y from B, the bending moment is

$$\begin{aligned} M_2 &= H_1 l - \frac{wl^2}{2} + Y_1 y \\ &= H_1 l - \frac{wl^2}{2} - \frac{wl^2 y}{2b}, \qquad \text{(iv)} \end{aligned}$$

and the strain energy of BC is

$$U_2 = \frac{1}{2E_2I_2}\int_0^b M_2^2 dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v})$$

E_2 and I_2 referring to BC.

At R in CD, distant z from C, the bending moment is

$$\begin{aligned} M_3 &= -H_2(l-z). \\ &= -(wl - H_1)(l-z), \text{ from (i)} \\ &= -wl^2 + lH_1 + wlz - H_1z, \quad . \quad . \quad . \quad (\text{vi}) \end{aligned}$$

and the strain energy of CD is

$$U_3 = \frac{1}{2E_3I_3}\int_0^l M_3 dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vii})$$

The total strain energy to be considered is

$$U = U_1 + U_2 + U_3,$$

and we must have

$$\frac{\partial U}{\partial H_1} = \frac{\partial U_1}{\partial H_1} + \frac{\partial U_2}{\partial H_1} + \frac{\partial U_3}{\partial H_1} = 0.$$

Substituting the values of U_1, U_2 and U_3 from (iii), (v) and (vii) we have

$$\frac{1}{E_1I_1}\int_0^l M_1\frac{\partial M_1}{\partial H_1}dx + \frac{1}{E_2I_2}\int_0^b M_2\frac{\partial M_2}{\partial H_1}dy + \frac{1}{E_3I_3}\int_0^b M_3\frac{\partial M_3}{\partial H_1}dz = 0 \quad . \quad (\text{viii})$$

From (ii), (iv) and (vi) we have

$$\frac{\partial M_1}{\partial H_1} = x, \frac{\partial M_2}{\partial H_1} = l, \frac{\partial M_3}{\partial H_1} = l - z \quad . \quad . \quad . \quad . \quad . \quad (\text{ix})$$

Hence (viii) becomes

$$\frac{1}{E_1I_1}\int_0^l\left(H_1x^2 - \frac{wx^3}{2}\right)dx + \frac{1}{E_2I_2}\int_0^b l^2\left((H_1 - \frac{wl}{2} - \frac{wly}{2b}\right)dy$$
$$+ \frac{1}{E_3I_3}\int_0^l(l-z)\,(lH_1 - wl^2 + wlz - H_1z)dz = 0.$$

Performing the integrations we have

$$l^3E_1I_1(\tfrac{1}{3}H_1 - \tfrac{1}{8}wl) + bl^2E_2I_2(H_1 - \tfrac{3}{4}wl) + l^3E_3I_3(\tfrac{1}{3}H_1 - \tfrac{1}{3}wl) = 0,$$

whence

$$H_1 = \frac{\frac{l}{8}E_1I_1 + \frac{3b}{4}E_2I_2 + \frac{l}{3}E_3I_3}{\frac{l}{3}E_1I_1 + bE_2I_2 + \frac{l}{3}E_3I_3}\,.\,wl.$$

Hence

$$H_2 = wl - H_1 = \frac{\frac{5l}{24}E_1I_1 + \frac{b}{4}E_2I_2}{\frac{l}{3}E_1I_1 + bE_2I_2 + \frac{l}{3}E_3I_3}\,.\,wl.$$

Other problems dealing with frameworks of rods rigidly jointed together may be dealt with in a similar manner, and this treatment is frequently less laborious than the more direct treatment adopted in § 257.

5. THE WHIRLING OF SHAFTS OF NON-UNIFORM SECTION*

376. General Equation.—When we wish to calculate the whirling speed of turbo-alternators and other turbine machinery, we have to deal with " shafts " which vary in section throughout their length, and for which the dimensions of cross sections are not small compared with the length of the shaft. In this case the equation to be solved is, from (3), § 159, and § 346,

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = \frac{w\omega^2}{g}y \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Whether this equation can be solved or not depends on the manner in which I varies along the shaft, and the number of cases which have been solved is very small. In practice I is not generally a mathematical function of x, so that exact solutions cannot be expected. We can, however, obtain the value in any given case by methods to be explained below. We know that in a long shaft the deflection due to shearing forces is small compared with the deflection due to bending moments. In the present case, if we neglect the change in the deflection curve due to shear, there will be no change in the relative positions of cross sections as the shaft revolves. In that case the shearing forces can do no work, and there can be no transfer of energy from one transverse element of the shaft to its neighbours. The consequence of this argument is that the total energy in each transverse slice of shaft is independent of the time.

Let u be the displacement of any point on the axis at any time t, so that u is a function of both x and t. Then the strain energy in a slice of the shaft, of length dx is $M^2dx/2EI = \frac{1}{2}EI\left(\frac{\partial^2u}{\partial x^2}\right)^2 dx$. The kinetic energy of the same element is given by $\frac{wdx}{2g}\left(\frac{\partial u}{\partial t}\right)^2 dx$. Thus the above hypothesis states that

$$\frac{\partial}{\partial t}\left[EI\left(\frac{\partial^2u}{\partial x^2}\right)^2 + \frac{w}{g}\left(\frac{\partial u}{\partial t}\right)^2\right] dx = 0.$$

Now let $u = y \sin \omega t$, where y is the amplitude of vibration of the point on the central line distant x from the origin. Then

$$\frac{\partial^2u}{\partial x^2} = \frac{\partial^2y}{dx^2}\sin \omega t, \text{ and } \frac{\partial u}{\partial t} = \omega y \cos \omega t.$$

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* See an article by H. A. Webb, *Engineering*, Nov. 2, 9, 16, 1917.

Hence we get

$$\frac{d}{dt}\left[EI\left(\frac{d^2y}{dx^2}\right)^2 \sin^2 \omega t + \frac{w\omega^2y^2}{g} \cos^2 \omega t\right] = 0,$$

or

$$2EI\left(\frac{d^2y}{dx^2}\right)^2 \omega \sin \omega t \,.\, \cos \omega t - 2\,\frac{w\omega^3y^2}{g} \cos \omega t \,.\, \sin \omega t = 0,$$

or

$$EI\left(\frac{d^2y}{dx^2}\right)^2 = \frac{w}{g}\omega^2y^2.$$

$$\therefore \sqrt{EI}\,.\,\frac{d^2y}{dx'} = \pm\, \omega y \sqrt{\frac{w}{g}}.$$

Now with the fundamental mode of vibration, y and $\frac{d^2y}{dx^2}$ have opposite signs, so that we take

$$\frac{d^2y}{dx^2} + \sqrt{\frac{w\omega^2}{gEI}}\,.\,y = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

H. A. Webb (*loc. cit.*) has shown that this is correct when Iw is constant, in particular in the case of a cylindrical shaft. He has also solved (i) in four particular cases which are soluble, the range including a case where the maximum value of Iw is 50,000 times the minimum value, so that the likely practical range may be said to be covered. By comparing the results with those obtained from (ii), he obtained the following empirical correction, which, when applied to the whirling speed found from (ii), may be taken to give the correct answer within 3 per cent.

377. Correction to be Applied.—Let e = the distance between the c.g. of the shaft and the middle point of the line joining the bearings.

Let R = the average *numerical** value of $\frac{d}{dx}\sqrt{Iw}$, found graphically; let S = the average value of Iw along the shaft. Then the correction is: subtract $\left[3500\left(\frac{e}{l}\right)^3 + \frac{5}{9}\frac{l^2R^2}{S^2}\right]$ per cent. of the whirling speed found from equation (ii).

378. Graphical Method.†—Any practical case can be treated with all the accuracy that is required by using equation (ii), instead of (i), and employing the graphical process described on pp. 561-563.

The equation to be solved is

$$\frac{d^2y}{dx^2} = -\,\omega\sqrt{\frac{w}{gEI}}\,y. \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

---

* That is $\frac{d}{dx}\sqrt{Iw}$ is to be considered positive always.

† H. A. Webb (*loc. cit.*).

Let y_o be the deflection, and ϕ the slope, at the centre of the axis AC (Fig. 410).

Integrating (i) from A to B we have

$$\frac{dy}{dx} = \omega \int_x^{\frac{l}{2}} y \sqrt{\frac{w}{gEI}} + \phi,$$

and, from B to C,

$$\frac{dy}{dx} = \omega \int_{\frac{l}{2}}^{x} y \sqrt{\frac{w}{gEI}} + \phi.$$

Integrating again we get

$$y = \phi x + \omega \int_o^x \left[\int_x^{\frac{l}{2}} y \sqrt{\frac{w}{gEI}}\, dx \right] dx, \quad \text{from } A \text{ to } B,$$

$$y = \phi(x - l) + \omega \int_x^l \left[\int_{\frac{l}{2}}^{x} y \sqrt{\frac{w}{gEI}}\, dx \right] dx, \text{ from } B \text{ to } C.$$

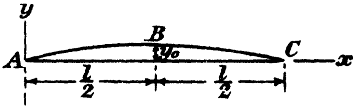

FIG. 410.

Therefore, since $y = y_o$, when $x = l/2$, we have

$$y_o = \frac{l\phi}{2} + \omega \int_o^x \int_x^{\frac{l}{2}} y \sqrt{\frac{w}{gEI}}\, dx \,.\, dx.$$

$$y_o = -\frac{l\phi}{2} + \omega \int_x^l \int_{\frac{l}{2}}^{x} y \sqrt{\frac{w}{gEI}}\, dx \,.\, dx.$$

Hence

$$2y_o = \omega \int_o^x \int_x^{\frac{l}{2}} y \sqrt{\frac{w}{gEI}}\, dx \,.\, dx + \omega \int_x^l \int_{\frac{l}{2}}^{x} y \sqrt{\frac{w}{gEI}}\, dx \,.\, dx \,. \text{(ii)}$$

Now assume $y = y_o \sin \frac{\pi x}{l}$, then (ii) gives

$$2y_o = \omega \int_o^x \int_x^{\frac{l}{2}} y_o \sqrt{\frac{w}{gEI}} \sin \frac{\pi x}{l}\, dx \,.\, dx + \omega \int_x^l \int_{\frac{l}{2}}^{x} y_o \sqrt{\frac{w}{gEI}} \sin \frac{\pi x}{l}\, dx \,.\, dx.$$

Hence

$$\frac{2\sqrt{gE}}{\omega} = \int_o^x \int_x^{\frac{l}{2}} \sqrt{\frac{w}{I}} \sin \frac{\pi x}{l} \,.\, dx \,.\, dx + \int_x^l \int_{\frac{l}{2}}^{x} \sqrt{\frac{w}{I}} \sin \frac{\pi x}{l}\, dx \,.\, dx \text{ (iii)}$$

The integrations are easily performed graphically. The following example, taken from Webb's article, should make the process clear.

Example of Graphical Method.—Find the whirling speed of a hollow truncated cone rotating about its axis, as shown in Fig. 411.

Let D_0, I_0, w_0 denote the diameter, moment of inertia, and weight per unit length, at the large end.

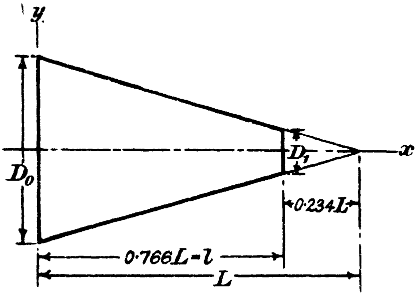

FIG. 411.

Then, at a distance x from that end,

$$I = \left(\frac{L-x}{L}\right)^3 I_0 \text{ and } w = \frac{L-x}{L} w_0.$$

Hence

$$\sqrt{\frac{I}{w}} = \sqrt{\frac{I_0}{w_0}}\left(1-\frac{x}{L}\right) = \sqrt{\frac{I_0}{w_0}}.\left(1-0\cdot766\frac{x}{l}\right).$$

$$\therefore \sqrt{\frac{I}{w}}\sqrt{\frac{w_0}{I_0}} = 1-0\cdot766\frac{x}{l}.$$

We now proceed with the calculation in a table as shown on page 562.

The second line is filled in by calculation and the third from the tables. The fourth line is found by dividing the figures of the third line by those of the second, and we then plot the curve of $\sqrt{\frac{I_0}{w_0}}\sqrt{\frac{w}{I}}\sin\frac{\pi x}{l}$, as shown in Fig. 412. This curve is then integrated to obtain the figures for the next two rows, which are then plotted. We then integrate these last curves and get

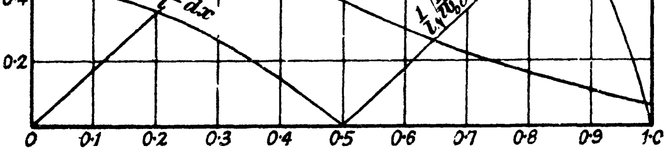

FIG. 412.

$$\frac{1}{l^2}\sqrt{\frac{I_0}{w_0}}\int_0^{\frac{1}{2}l}\int_x^{\frac{1}{2}l}\sqrt{\frac{w}{I}}\sin\frac{\pi x}{l}\,dx\,.\,dx = 0\cdot1417$$

$$\frac{1}{l^2}\sqrt{\frac{I_0}{w_0}}\int_{\frac{1}{2}l}^{l}\int_{\frac{1}{2}l}^{x}\sqrt{\frac{w}{I}}\sin\frac{\pi x}{l}\,.\,dx\,.\,dx = 0\cdot2032.$$

x/l	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$\sqrt{\frac{I}{w}}\sqrt{\frac{w_o}{I_o}}$	1	·9234	·8468	·7702	·6936	·6170	·5404	·4638	·3872	·3106	·234
$\sin\frac{\pi x}{l}$	0	·3090	·5878	·8090	·9511	1·0	·9511	·8090	·5878	·3090	0
$\sqrt{\frac{I_o}{w_o}}\sqrt{\frac{w}{I}}\sin\frac{\pi x}{l}$	0	·3347	·6940	1·050	1·371	1·621	1·760	1·745	1·518	·9950	0
$\frac{1}{l}\sqrt{\frac{I_o}{w_o}}\int_x^{\frac{1}{2}l}\sqrt{\frac{w}{I}}\sin\frac{\pi x}{l}\,.\,dx$	0·4270	·4105	·3592	·2718	·1504	0	—	—	—	—	—
$\frac{1}{l}\sqrt{\frac{I_o}{w_o}}\int_{\frac{1}{2}l}^{x}\sqrt{\frac{w}{I}}\sin\frac{\pi x}{l}\,.\,dx$	—	—	—	—	—	0	·1702	·3469	·5121	·6406	·6954
$\sqrt{\frac{I\,w}{I_o\,w_o}}=\left(1-0{\cdot}766\,\frac{x}{l}\right)^2$	1	·8526	·7171	·5932	·4810	·3807	·2920	·2151	·1500	·0965	·0548

Hence from (iii),

$$\frac{2}{w}\sqrt{gE} = 0{\cdot}3449\, l^2 \sqrt{\frac{w_o}{I_o}}$$

$$\therefore \omega = \frac{5{\cdot}8}{l^2}\sqrt{\frac{gEI_o}{w_o}}.$$

We must now apply the correction given in § 376. We find

$$e = 0{\cdot}104\, l.$$

$$\therefore\ 3500\left(\frac{e}{l}\right)^3 = 3{\cdot}94$$

$$R = \frac{1}{l}\left[\sqrt{I_o w_o} - \sqrt{I_1 w_1}\right] = 0{\cdot}9452\,\frac{\sqrt{I_o w_o}}{l},$$

$$S = \text{average value of} \sqrt{Iw} = 0{\cdot}4296\sqrt{I_o w_o},$$

from the curves of Fig. 412. Hence

$$3500\left(\frac{e}{l}\right)^3 + \frac{5}{9}\,\frac{l^2 R^2}{S^2} = 6{\cdot}63.$$

Hence we must subtract 6·63 per cent. from the value of ω found above, giving

$$\omega = \frac{5{\cdot}41}{l^2}\sqrt{\frac{gEI_o}{w_o}},\ \text{radians per second.}$$

This is 2½ per cent. above the correct value.

6. FLEXURE OF FLAT PLATES OF ANY SHAPE

379. General Equations for Flexure of Flat Plate.—We shall obtain the equations for the bending of a flat plate by a load applied normal to the surface, with the same assumptions as we made for the circular plate in Chapter XXXI.

We take any two convenient lines, which are mutually at right angles and in the middle-plane, as axes of x and y, and measure z in the direction of load.

Let ϕ and ψ indicate the angles between the axis of z and the radii of curvature at (x, y, z) in planes parallel to the planes xz and yz; let u, v, w be the displacements of the point (x, y) on the middle-plane. Then it is easy to show, exactly as on p. 490, that

$$u = z\phi = -z\frac{\partial w}{\partial x},$$

$$v = z\psi = -z\frac{\partial w}{\partial y}.$$

Hence the component strains are

$$e_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2};\quad e_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2};$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\,\frac{\partial^2 w}{\partial x\,.\,\partial y}.$$

The stresses are then given by

$$p_x = \frac{mE}{m^2-1}(e_y + me_x) = -\frac{mEz}{m^2-1}\left(m\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad . \quad . \quad (1)$$

$$p_y = \frac{mE}{m^2-1}(e_x + me_y) = -\frac{mEz}{m^2-1}\left(\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial y^2}\right) \quad . \quad . \quad (2)$$

$$q = Ce_{xy} = \frac{mE}{2(m+1)}e_{xy} = -\frac{mEz}{m+1}\cdot\frac{\partial^2 w}{\partial x\,.\,\partial y} \quad . \quad . \quad . \quad . \quad (3)$$

The directions of these stresses are shown in Fig. 413.

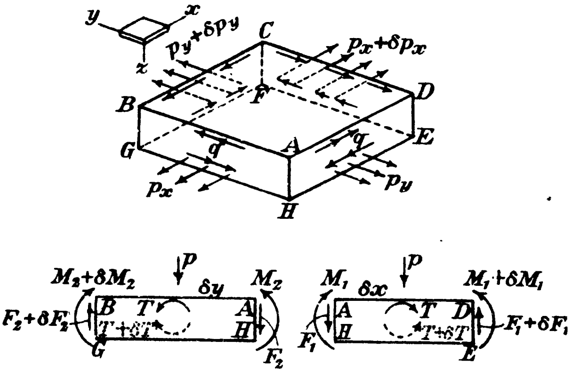
y
x
z
$p_y+\delta p_y$
C
$p_x+\delta p_x$
B
F
D
G
A
q
q
E
p_x
p_y
H
$M_2+\delta M_2$
P
δy
M_2
$F_2+\delta F_2$
B
T
A
T+δT
H
G
F_2
M_1
δx
P
$M_1+\delta M_1$
A
T
D
H
T+δT
F_1
E
$F_1+\delta F_1$

FIG. 413.

As before, let t be the thickness of the plate. The resultant action over the face $ABGH$, on account of p_x, is a force, per unit length,

$$P_1 = \int_{-\frac{t}{2}}^{\frac{t}{2}} p_x dz = 0,$$

and a couple, per unit length,

$$M_1 = \int_{-\frac{t}{2}}^{\frac{t}{2}} p_x\,.\,zdz = -\frac{mEt^3}{12(m^2-1)}\left(m\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad . \quad . \quad . \quad (4)$$

from (1).

Similarly the resultant action of p_y over the face $ADEH$ reduces to a couple, per unit length,

$$M_2 = \int_{-\frac{t}{2}}^{\frac{t}{2}} p_y dz\,.\,dz = -\frac{mEt^3}{12(m^2-1)}\left(\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial y^2}\right). \quad . \quad . \quad (5)$$

The stresses q, on the faces $ABGH$ and $ADEH$, give rise to stress-couples given by

$$T = \int_{-\frac{t}{2}}^{\frac{t}{2}} q\,.\,zdz = -\frac{mEt^3}{12(m+1)}\cdot\frac{\partial^2 w}{\partial x\,.\,\partial y} \quad . \quad . \quad . \quad . \quad (6)$$

The directions of the stress-couples are shown in Fig. 413.

Let F_1 and F_2 be the shearing forces on the edges AD and AB of the element, per unit length. Then we can find F_1 and F_2 from the conditions of equilibrium.

Let p = the pressure per unit area, in the direction of the axis of z.

The forces and couples, per unit length, are shown in Fig. 413. Taking moments about AB, we have

$$(M_1 + \delta M_1)\delta y - M_1\delta y + (F_1 + \delta F_1)\delta y \times \delta x - T\delta x + (T + \delta T)\delta x - p \,.\, \delta x \,.\, \delta y \,.\, \frac{\delta x}{2} = 0,$$

which gives

$$F_1 = -\frac{\partial T}{\partial y} - \frac{\partial M_1}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again, taking moments about AD, we get

$$M_2 \,.\, \delta x - (M_2 + \delta M_2)\delta x + T\delta y - (T + \delta T)\delta y - (F_2 + \delta F_2)\delta x \times \delta y + p\delta x \,.\, \delta y \,.\, \frac{\delta y}{2} = 0,$$

from which

$$F_2 = -\frac{\partial T}{\partial x} - \frac{\partial M_2}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting for M_1, M_2 and T from (4), (5) and (6) in (7) and (8) we get

$$F_1 = \frac{mEt^3}{12(m+1)} \cdot \frac{\partial^3 w}{\partial x \,.\, \partial y^2} + \frac{mEt^3}{12(m^2-1)}\left[m\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \,.\, \partial y^2}\right]$$

or

$$F_1 = \frac{m^2Et^3}{12(m^2-1)}\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \,.\, \partial y^2}\right] \quad . \quad . \quad . \quad . \quad (9)$$

Similarly

$$F_2 = \frac{m^2Et^3}{12(m^2-1)}\left[\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right] \quad . \quad . \quad . \quad . \quad (10)$$

Finally we can obtain an equation for w by resolving vertically:

$$\delta F_1 \,.\, \delta y + \delta F_2 \,.\, \delta x = p \,.\, \delta x \,.\, \delta y$$

or

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Substituting for F_1 and F_2 from (9) and (10) we get:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + p, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where, as before, $D = m^2Et^3/12(m^2-1)$.

This is the general equation for the deflection of a thin plate bent by forces perpendicular to its surface.

380. Stress-Couples and Shearing Forces at the Boundary.—In Fig. 414 let AB be a length δs of the boundary curve of the plate.

Let PG be the normal at P making an angle θ with the axis of x.

Let M_{1P}, M_{2P} and T_P denote the stress-couples per unit length about axes through P parallel to the axes of x and y.

Let M_0 and T_0 denote the couples, per unit length of boundary, applied by the external forces, about the tangent and normal respectively. Then we must have, in the same way as we resolve stresses,

$$T_0.\delta s + M_{1P}.\delta y \sin\theta + T_P.\delta x.\sin\theta - M_{2P}\delta x.\cos\theta - T_P\,\delta y\cos\theta = 0,$$

and

$$M_0\delta s - M_{1P}\delta y\cos\theta - T_P\,\delta x\cos\theta - M_{2P}\delta x\sin\theta - T_P\,\delta y\sin\theta = 0.$$

Putting $\delta x = \delta s\sin\theta$ and $\delta y = \delta s\cos\theta$, these give

$$T_0 = \tfrac{1}{2}(M_{2P} - M_{1P})\sin 2\theta + T_P\cos 2\theta \quad . \quad . \quad . \quad . \quad (13)$$

$$M_0 = M_{1P}\cos^2\theta + M_{2P}\sin^2\theta + T_P\sin 2\theta . . \quad . \quad . \quad (14)$$

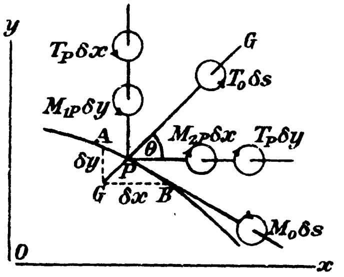

FIG. 414.

Again, let F_0 be the shearing force, per unit length of boundary, applied by the external forces, perpendicular to the plate. Then we must have

$$F_0\delta s = F_{1P}\delta y + F_{2P}.\delta x,$$

where F_{1P} and F_{2P} are the shearing forces per unit length on edges δy and δx, of an element at P bounded by δx, δy, δs.

Since $\delta y = \delta s\cos\theta$ and $\delta x = \delta s\sin\theta$, this equation becomes

$$F_0 = F_{1P}\cos\theta + F_{2P}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Equations (13), (14), (15) give us the conditions which must be satisfied at the boundary, according to the nature of the forces or couples applied to the boundary.

At a freely supported edge M_0, T_0, and w must vanish; $\int F_0 ds$, taken round the boundary, must equal the total load applied to the plate.

At a clamped edge $w = 0$ and $\frac{\partial w}{\partial \nu} = 0$, where $\delta\nu$ is an element of the normal. The second of these conditions gives

$$\frac{\partial w}{\partial x}.\frac{\partial \nu}{\partial x} + \frac{\partial w}{\partial y}.\frac{\partial \nu}{\partial y} = 0,$$

or

$$\cos\theta.\frac{\partial w}{\partial x} + \sin\theta.\frac{\partial w}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Up to the present time mathematicians have not been able to find solutions to many cases of practical interest.*

381. Elliptic Plate, Uniformly Distributed Load, Clamped Edge.—G. H. Bryan has given the following solution:

$$w = \frac{p}{8D} \cdot \frac{(x^2/a^2 + y^2/b^2 - 1)^2}{(3/a^4 + 3/b^4 + 2/a^2b^2)}, \quad . \quad . \quad . \quad . \quad (17)$$

where p is the pressure per unit area over a plate whose boundary is the ellipse $x^2/a^2 + y^2/b^2 = 1$. From this we find by (9) and (10)

$$F_1 = \frac{px/a^2.(1/b^2 + 3/a^2)}{3/a^4 + 3/b^4 + 2/a^2b^2}.$$

$$F_2 = \frac{py/b^2.(1/a^2 + 3/b^2)}{3/a^4 + 3/b^4 + 2/a^2b^2}.$$

Then, at a point (x, y) on the boundary, (15) gives

$$F_o = \frac{p}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}} \left[\frac{x \cos\theta}{a^2}\left(\frac{1}{b^2} + \frac{3}{a^2}\right) + \frac{y \sin\theta}{b^2}\left(\frac{1}{a^2} + \frac{3}{b^2}\right)\right]$$

$$\therefore F_o ds = \frac{p}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}} \left[\frac{xdy}{a^2}\left(\frac{1}{b^2} + \frac{3}{a^2}\right) + \frac{ydx}{b^2}\left(\frac{1}{a^2} + \frac{3}{b^2}\right)\right].$$

Hence we have

$$\int F_o ds = \frac{4p}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}} \left[\left(\frac{1}{b^2} + \frac{3}{a^2}\right)\int_0^b \frac{xdy}{a_2} + \left(\frac{1}{a^2} + \frac{3}{b^2}\right)\int_0^a \frac{ydx}{b^2}\right],$$

the integral $\int F_o ds$ being taken all round the boundary. Remembering that each of the integrals on the right-hand side represents one quarter of the area of the curve, that is, $\pi ab/4$, it is easily verified that the total shearing force round the boundary is equal to the total load on the plate.

It is also easily seen that w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ all vanish at every point on the boundary.

The stresses are found from (1), (2) and (3); on the surface, where $z = \pm t/2$,

$$p_x = \pm \frac{3p}{mt^2} \cdot \frac{\frac{x^2}{a^2}\left(\frac{1}{b^2} + \frac{3m}{a^2}\right) + \frac{y^2}{b^2}\left(\frac{m}{a^2} + \frac{3}{b^2}\right) - \left(\frac{1}{b^2} + \frac{m}{a^2}\right)}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}} \quad . \quad . \quad (18)$$

$$p_y = \pm \frac{3p}{mt^2} \cdot \frac{\frac{x^2}{a^2}\left(\frac{m}{b^2} + \frac{3}{a^2}\right) + \frac{y^2}{b^2}\left(\frac{1}{a^2} + \frac{3m}{b^2}\right) - \left(\frac{m}{b^2} + \frac{1}{a^2}\right)}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}} \quad . \quad . \quad (19)$$

$$q = \pm \frac{6(m-1)p}{mt^2} \cdot \frac{xy/a^2b^2}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

---

* An approximate method of treatment is given in Prescott's *Applied Elasticity* (Longmans, Green and Co., 1924).

It is easily verified that p_x and p_y have their greatest values at the centre, where

$$\left.\begin{aligned} p_x(\text{max.}) &= \pm\frac{3p}{mt^2}\left(\frac{1}{b^2}+\frac{m}{a^2}\right)\bigg/\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right) \\ p_y(\text{max.}) &= \pm\frac{3p}{mt^2}\left(\frac{1}{a^2}+\frac{m}{b^2}\right)\bigg/\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right) \end{aligned}\right\} \quad . \quad . \quad (21)$$

The stresses q are zero at the centre.

The solution is not known for an elliptic plate with the edge freely supported.

382. Elliptic Plate, Clamped Edge, Load px per Unit Area.—Bryan gives in this case

$$w = \frac{px}{24D}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2\bigg/\left(\frac{5}{a^4}+\frac{1}{b^4}+\frac{2}{a^2b^2}\right), \quad . \quad . \quad . \quad (22)$$

from which we can deduce expressions for the stresses. From this and the last article we can find the stresses in an elliptic plate with clamped edge and loading given by $p = p_0 + p'x + p''y$.

383. Rectangular Plate, Supported at the Edges.—If the sides of the plate are of lengths a and b, and the axes are taken from one corner of the plate, the displacement is given by

$$w = \frac{16}{\pi^6}\frac{p}{D}\sum_{r=1}^{r=\infty}\sum_{s=1}^{s=\infty}\frac{\sin\dfrac{r\pi x}{a}\sin\dfrac{s\pi y}{b}}{rs\left(\dfrac{r^2}{a^2}+\dfrac{s^2}{b^2}\right)^2}, \quad . \quad . \quad . \quad . \quad (23)$$

where p is the uniform load per unit area, whilst r and s denote odd integers. From this we can arrive at any degree of approximation that may be desired, using equations (1), (2), (3) to find the stresses.

384. Rectangular Plate under Thrusts in its Plane.—We shall consider the case of a flat rectangular plate subjected to uniform thrusts along each pair of opposite edges.

Let the origin be at one corner of the plate and let the equations of the sides be $x = 0$, $x = a$, $y = 0$, $y = b$, so that the lengths of the sides are a and b.

Let P_1 and P_2 be the thrusts on the plate, per unit length of edge-line, in directions parallel to the sides a and b, as shown in Fig. 415. As before, consider the equilibrium of an element of the plate, the sides being δx and δy. Let ϕ and $\phi + \delta\phi$ be the angles made by the faces AH and DE with the plane zy, so that $\phi = -\frac{\delta w}{\partial x}$ as on p. 490. Then, without sensible error, we can say that the normal force on each of the faces AH and DE is $P_1\delta y$; these two forces will have a resultant $P_1\delta y.\delta\phi$ perpendicular to the face AD of the element. Similarly, the thrust P_2 will produce a force $P_2\delta x.\delta\psi$ in the same direction, where $\psi = -\frac{\partial w}{\partial y}$ as above.

Then, resolving at right angles to AD, we have

$$\delta F_1.\delta y + \delta F_2.\delta x = P_1 \delta y.\delta \phi + P_2 \delta x \delta \psi,$$

which gives

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = P_1 \frac{\partial \phi}{\partial x} + P_2 \frac{\partial \psi}{\partial y} = - P_1 \frac{\partial^2 w}{\partial x^2} - P_2 \frac{\partial^2 w}{\partial y^2}.$$

Putting in the values of F_1 and F_2 from (9) and (10), this becomes

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 . \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + P_1 \frac{\partial^2 w}{\partial x^2} + P_2 \frac{\partial^2 w}{\partial y^2} = 0. \quad . \quad . \quad (24)$$

The conditions to be satisfied at the boundary are

$w = 0$, when $x = 0$ or a, for all values of y,
$w = 0$, ,, $y = 0$ or b, for all values of x.

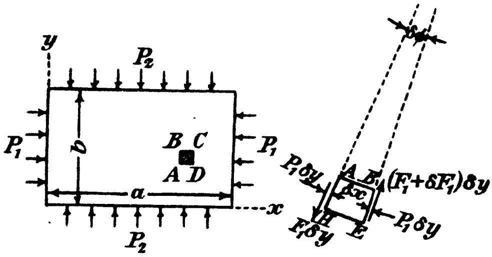

FIG. 415.

If the edges are freely supported we must also have $M_1 = 0$ when $x = 0$ or a, and $M_2 = 0$ when $y = 0$ or b, that is

$$\left.\begin{matrix} \dfrac{\partial^2 w}{\partial x^2} = 0 \\ \dfrac{\partial^2 w}{\partial y^2} = 0 \end{matrix}\right\} \left\{\begin{matrix} \text{when } x = 0 \text{ or } a, \text{ for all values of } y. \\ \text{when } y = 0 \text{ or } b, \text{ for all values of } x. \end{matrix}\right.$$

A solution* which satisfies all the conditions is

$$w = A \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where r and s are integers, provided

$$D\left(\frac{r^2\pi^2}{a^2} + \frac{s^2\pi^2}{b^2}\right)^2 = P_1 \frac{r^2\pi^2}{a^2} + P_2 \frac{s^2\pi^2}{b^2}. \quad . \quad . \quad . \quad . \quad (26)$$

Hence the plate can be held deflected by the applied thrusts, that is it becomes unstable after the manner of a strut, if equation (26) is satisfied.

If $P_1 = P_2 = P$, we have for the least load which can cause instability,*

$$\mathrm{P} = \pi^2 \mathrm{D}\left(\frac{1}{\mathrm{a}^2} + \frac{1}{\mathrm{b}^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

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* These results are due to G. H. Bryan (*Proc. London Math. Society*, 1891), who, however, obtained them by more rigorous analysis than the above.

If $P_2 = 0$, the least load is found by taking $s = 1$, and giving r an integral value such that a^2/b^2 lies between $r(r+1)$ and $r(r-1)$.*

If P_1 and P_2 are unequal, the plate will buckle into waves parallel to the side b if $P_2 < \frac{1}{2}P_1$, so that $r = 1$ and $s \neq 1$.*

If $P_2 = 0$ and a is very large compared with b, the crippling load is given by

$$\mathrm{P} = \frac{\pi^2\mathrm{D}}{\mathrm{b}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Thus the problem of a very long parallel strip subjected to thrusts in its plane, applied to the opposite edges, is analagous to that of a strut, D taking the place of EI. If the edges are clamped the critical load is given by $P = 4\pi^2D/b^2$.

It can also be shown* that a circular plate, of radius a, with clamped edge, subjected to a thrust P per unit length round the circumference, becomes unstable when

$$P = \lambda^2D/a^2,$$

λ being the least root of $J_1(\lambda) = 0$ (i.e. $\lambda = 3{\cdot}832$).

7. FLEXURE OF CURVED RODS, APPROXIMATE THEORY

385.—We have considered special cases of the bending of curved beams in the treatment of arches in Chapter XVII, whilst in Chapter XXV beams whose cross sections are comparable in dimensions with the radius of curvature of the axis are dealt with. In the present chapter we shall deal with the general problem of the bending and twisting of a beam whose central line is initially a plane curve, under any system of external forces, on the supposition that the largest dimension of a cross section is small compared with the least radius of curvature of the central line, and that the deformation is small. Before entering upon the mathematics of this problem it will be helpful to call to our minds the methods that were employed to solve problems of a general nature in dealing with straight beams. First we had equations of equilibrium in the form $dF/dx = w$, and $dM/dx = F$, (p. 116), which sufficed for finding the bending moments and shearing forces in simple cases. Next we had the equation which connects the bending moment with the distortion of the beam, in the form $M = EI/R$, (p. 159), and finally we had the purely geometrical relationship expressed by the equation $1/R = d^2y/dx^2$ (p. 217). Between the last two equations we connect the bending moment M directly with the shape of the strained axis of the beam, and we have to resort to this equation in the more general types of bending. Similar steps must be followed in the present instance, but, as we shall now have to consider bending in two planes, as well as torsion, the analysis will necessarily be more involved.

386. Nature of the Deformation.—The axis of the beam is supposed initially to be a plane curve, and one of the principal axes of every cross section is supposed to be in this plane in the unstrained state, so that the beam at first has no twist. In the most general case the axis of the beam may be bent in its original plane, it may be bent in a plane perpendicular to this and tangential to the axis, and it may be twisted; this covers the most general type of deformation possible if we exclude a stretching of the axis itself. In the unstrained state one set of corresponding principal axes of all cross sections are in the same plane, in the strained state this will not be the case.

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* These results are due to G. H. Bryan (*Proc. London Math. Society*, 1891), who, however, obtained them by more rigorous analysis than the above.

Thus in Fig. 416 let P_1Q_1 be an element of length δs of the axis of the deformed beam; let P_1A_1 and P_1B_1 be the principal axes of the cross section at P, and let Q_1A_2 and Q_1B_2 be the corresponding axes of the cross section at Q_1. Then in the unstrained state Q_1A_2 and Q_1B_2 are respectively parallel to P_1A_1 and P_1B_1.

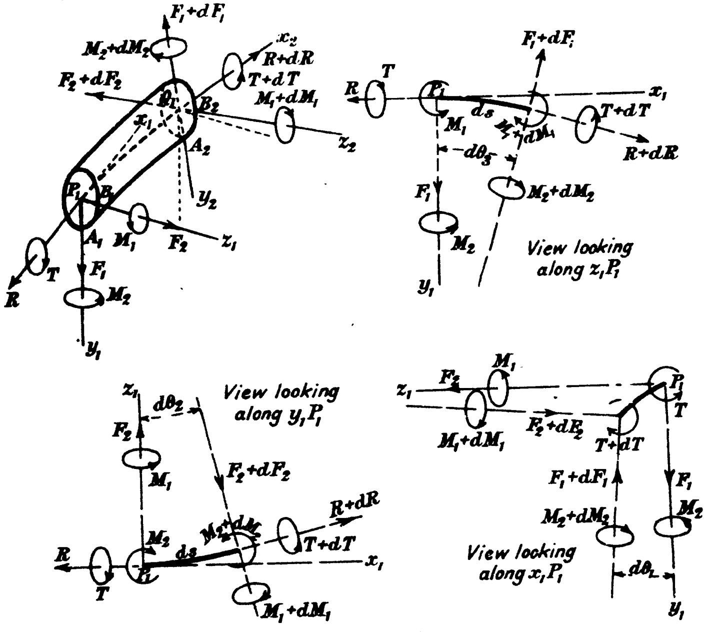

FIG. 416.

Let P_1 be the origin of a system of rectangular axes P_1x_1, P_1y_1, P_1z_1 in which P_1x_1 is the tangent at P_1 to P_1Q_1, whilst P_1y_1 and P_1z_1 lie along P_1A_1 and P_1B_1; also let Q_1 be the origin of a second similar system of axes Q_1x_2, Q_1y_2, Q_1z_2 where Q_1x_2 is the tangent at Q to P_1Q_1. Then the second system of axes can be obtained from the first by a translation δs along P_1Q_1, and small rotations $\delta\theta_1$, $\delta\theta_2$ and $\delta\theta_3$ about P_1x_1, P_1y_1 and P_1z_1.

Then, if ρ_1 and ρ_2 be the radii of curvature at P_1 in the planes xy and xz respectively,

$$\left\{\begin{aligned} &\delta\theta_1 = \phi.\delta s, \text{ where } \phi \text{ is the twist per unit length.} \\ &\delta\theta_2 = \frac{\delta s}{\rho_2} \\ &\delta\theta_3 = \frac{\delta s}{\rho_1} \end{aligned}\right. \qquad (1)$$

In the unstrained state there is no twist, ρ_1 is infinite, and we will denote the initial value of ρ_2 by ρ_0.

387. Equations of Equilibrium.—The total action between the portion of the rod to the left of P_1 on the element P_1Q_1 can be resolved into two shearing forces F_1 and F_2, and a tension R, along the axes of y, z, x respectively, and, in addition, bending moments M_1 and M_2 and a torque T about the same axes, as shown in Fig. 416.

Similarly the action of the portion to the right of Q_1 can be resolved into $F_1 + \delta F_1$, $F_2 + \delta F_2$, $R + \delta R$; $M_1 + \delta M_1$, $M_2 + \delta M_2$, and $T + \delta T$, as shown.

Let the external forces acting on P_1Q_1 be $X\delta s$, $Y\delta s$, and $Z\delta s$ in the direction P_1x_1, P_1y_1 and P_1z_1.

Then, resolving along the axis P_1x_1, we have:

$$-R + (R + \delta R) + (F_1 + \delta F_1)\delta\theta_3 + (F_2 + \delta F_2)\delta\theta_2 + X\delta s = 0.$$

Substituting for $\delta\theta_2$ and $\delta\theta_3$ from (1), dividing by δs, and proceeding to the limit when $\delta s = 0$, we get:

$$\frac{dR}{ds} + \frac{F_1}{\rho_1} + \frac{F_2}{\rho_2} + X = 0,$$

neglecting quantities of the second order such as $\delta F_1.\delta\theta_3$. Similarly, resolving along P_1y_1 and P_1z_1 we obtain two other equations of the same type. Thus we have the three equations:

$$\left.\begin{aligned}&\frac{dR}{ds} + \frac{F_1}{\rho_1} + \frac{F_2}{\rho_2} + X = 0\\&\frac{dF_1}{ds} - F_2\phi - \frac{R}{\rho_1} - Y = 0\\&\frac{dF_2}{ds} + F_1\phi - \frac{R}{\rho_2} - Z = 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, taking moments about P_1x_1, P_1y_1 and P_1z_1, we have:

$$(T + \delta T) - T + (M_2 + \delta M_2)\delta\theta_3 - (M_1 + \delta M_1)\delta\theta_2 = 0.$$
$$(M_2 + \delta M_2) - M_2 + (T + \delta T)\delta\theta_3 - (M_1 + \delta M_1)\delta\theta_1 - F_2\delta s = 0.$$
$$(M_1 + \delta M_1) - M_1 + (T + \delta T)\delta\theta_2 - (M_2 + \delta M_2)\delta\theta_1 - F_1\delta s = 0.$$

These equations reduce to

$$\left.\begin{aligned}&\frac{dT}{ds} + \frac{M_2}{\rho_1} - \frac{M_1}{\rho_2} = 0\\&\frac{dM_2}{ds} - \frac{T}{\rho_1} - M_1\phi - F_2 = 0\\&\frac{dM_1}{ds} + \frac{T}{\rho_2} - M_2\phi - F_1 = 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If, in addition to the forces $X\delta s$, $Y\delta s$, $Z\delta s$, the element P_1P_2 be acted on by external couples $L_1\delta s$, $L_2\delta s$ and $L_3\delta s$, about P_1x_1, P_1y_1, and P_1z_1, in directions y_1z_1, x_1z_1, x_2y_2 respectively, we must add L_3, L_2 and L_1 to the left-hand sides of the first, second, and third of equation (3) respectively.

Equations (2) and (3) are the six equations of equilibrium of the element P_1Q_1, and correspond with such equations as $dF/dx = w$ and $dM/dx = F$ in the simple theory of flexure.

388. Approximate Theory.—A generalization of the Bernoulli-Euler theory of flexure provides us with three relations* connecting the curvature and twist of the rod with its elastic properties:

$$-M_1 = \frac{EI_1}{\rho_1};\ -M_2 = EI_2\left(\frac{1}{\rho_2} - \frac{1}{\rho_0}\right);\ -T = CK\phi. \quad . \quad . \quad (4)$$

In these formulæ I_1 and I_2 are the second moments of area of the cross section about the axes P_1B_1 and P_1A_1 respectively, whilst CK is the torsional rigidity defined in Chapter XXX, and E is Young's Modulus. The nine equations (1), (2), (3) are sufficient to find the nine unknown quantities M_1, M_2, T; F_1, F_2, R; ρ_1, ρ_2, ϕ, but it is more convenient to study the deformation by means of displacements.

389. Geometrical Equations for the Strained Beam.—Let PQ be an element of the unstrained axis, the length of PQ being δs_0.

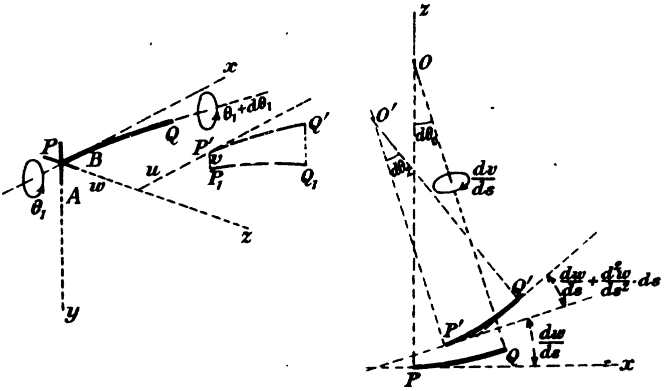

FIG. 417.

Let P, xyz be a system of rectangular axes such that Px is along the tangent at P to PQ, whilst Py and Pz are along the principal axes PA and PB of the cross section of the beam. The unstrained axis lies entirely in the plane xPz.

Let P_1Q_1 denote the position of PQ in the strained state, $P'Q'$ being the projection of P_1Q_1 on the plane xPz.

Let u, v, w denote the displacements of P referred to axes of x, y, z respectively.

Let $\delta\theta_0$ = the angle between the normals at P and Q.

Let $\delta\theta_2$ = the angle between the normals at P' and Q'.

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* For the justification of the use of these formulæ here see Love's *Mathematic Theory of Elasticity*, 3rd ed., p. 395.

Let ρ_o = the radius of curvature of PQ.
Let ρ_2 = the radius of curvature of $P'Q'$.
Then

$$\frac{1}{\rho_o} = \frac{\delta\theta_o}{\delta s_o}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Now the co-ordinates of P' referred to x, P, z are (u, w), so that the inclination of the tangent at P' to Px is $\frac{dw}{ds}$.

Similarly the inclination of the tangent at Q' to a line drawn at right angles to OQ is

$$\frac{dw}{ds} + \frac{d}{ds}\left(\frac{dw}{ds}\right)\delta s_o.$$

Therefore the angle between the tangents at P' and Q' is greater than the angle between the tangents at P and Q by an amount $\frac{d^2w}{ds^2} \cdot \delta s_o$. Thus the angle $P'O'Q'$ is

$$\delta\theta_2 = d\theta_o + \frac{d^2w}{ds^2}\delta s_o.$$

Also, to the first order of small quantities, the length of $P'Q'$ is $(\rho_o - w)\delta\theta_o$, i.e. it is

$$\delta s = \delta s_o\left(1 - \frac{w}{\rho_o}\right), \text{ by (5).}$$

Hence, to the first order of small quantities,

$$\frac{1}{\rho_2} = \frac{d\theta + \frac{d^2w}{ds^2}ds}{ds\left(1 - \frac{w}{\rho_o}\right)} = \frac{d\theta}{ds}\left(1 + \frac{w}{\rho_o}\right) + \frac{d^2w}{ds^2}\left(1 + \frac{w}{\rho_o}\right)$$

$$= \frac{1}{\rho_o} + \frac{w}{\rho_o^2} + \frac{d_2w}{ds^2}.$$

For small displacements, ρ_2 may be considered equal to the radius of curvature of the strained element P_1Q_1 in the direction of the principal axis PB, and we have

$$\frac{1}{\rho_2} = \frac{1}{\rho_o} + \frac{w}{\rho_o^2} + \frac{d_2w}{ds^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Next, consider the effects of the displacement v. On account of the v-displacements, the section at Q undergoes a rotation $\frac{dv}{ds}$ about the axis QB, the component of which about the tangent at P is

$$\frac{dv}{ds}\sin(d\theta_o), \text{ i.e. } \frac{dv}{ds} \cdot \frac{ds_o}{\rho_o}.$$

Thus: the v-displacements cause a twist per unit length $\frac{dv}{\rho_o \cdot ds}$.

Again, let θ_1 = the rotation of the cross section at P about the tangent at P to PQ, the corresponding rotation of the section at Q being $\theta_1 + d\theta_1$. Then there is a twist per unit length $\frac{d\theta_1}{ds}$. Adding this to the twist due to the v-displacements we get for the total twist per unit length

$$\phi = \frac{d\theta_1}{ds} - \frac{1}{\rho_o} \cdot \frac{dv}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But the rotation θ_1 about the tangent at P has a component $\theta_1 \sin (\delta\theta_o)$ about the normal at Q, i.e. there is curvature in the plane xPy of amount $\theta_1 \delta\theta_o = \theta_1 \cdot \frac{\delta s_o}{\rho_o}$, or a curvature per unit length $\frac{\theta_1}{\rho_o}$. If we add this to the curvature due to the v-displacement, we get

$$\frac{1}{\rho_1} = \frac{d_2v}{ds^2} + \frac{\theta_1}{\rho_o}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Finally, we have the condition that the length of the element PQ does not change. On account of the displacement w, we found above a change of length $-w\frac{ds_o}{\rho_o}$. On account of the u-displacements, the length increases by $\frac{du}{ds}$. Therefore the condition for no extension is

$$\frac{du}{ds} - \frac{w}{\rho_o} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We can now express the bending moments and torque in terms of u, v, w by means of the equations (4), p. 573.

Collecting results, we have, then:

$$\left.\begin{aligned} -\frac{M_1}{EI_1} &= \frac{1}{\rho_1} = \frac{d^2v}{ds^2} + \frac{\theta_1}{\rho_o} \\ -\frac{M_2}{EI_2} &= \frac{w}{\rho_o^2} + \frac{d^2w}{ds^2} \\ -\frac{T}{CK} &= \frac{d\theta_1}{ds} - \frac{1}{\rho_o} \cdot \frac{dv}{ds} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

390. Circular Beam Bent in its Plane.—When the axis of the beam is initially a circular arc of radius a in the xz plane, and the only external forces are in the plane of this arc, the above equations can be simplified. We shall have

$$\rho_o = a, \quad \theta_1 = 0, \quad \text{and } v = 0;$$

we can also write $s = a\theta$; and $d/ds = a^{-1} \cdot d/d\theta$, where θ is the angle between any radius and some convenient radius of reference.

(9) becomes

$$\frac{du}{d\theta} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

The first and last of the equations (10) show that M_1 and T vanish; ϕ vanishes also.

We then have

$$-M_2 = \frac{EI_2}{a^2}\left(\frac{d^2w}{d\theta^2} + \frac{du}{d\theta}\right) = \frac{EI_2}{a^2}\left(\frac{d^3u}{d\theta^3} + \frac{du}{d\theta}\right) \quad . \quad . \quad \text{(ii)}$$

If the external forces are X per unit length along the tangent and Z per unit length along the radius, the equations of equilibrium (2) and (3) give

$$\left.\begin{aligned} &\frac{dR}{d\theta} + F_2 + aX = 0, \text{ from first of (2)} \\ &\frac{dF_2}{d\theta} - R - aZ = 0, \text{ from third of (2)} \\ &\frac{dM_2}{d\theta} - aF_2 = 0, \quad \text{ from second of (3)} \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(iii)}$$

From the last of these and (ii) we have

$$F_2 = \frac{1}{a}\frac{dM_2}{d\theta} = -\frac{EI_2}{a^3}\left(\frac{d^4u}{d\theta^4} + \frac{d^2u}{d\theta^2}\right) \quad . \quad . \quad . \quad \text{(iv)}$$

From the second and (iv) we get

$$R = -aZ + \frac{dF_2}{d\theta} = -aZ - \frac{EI_2}{a^3}\left(\frac{d^5u}{d\theta^5} + \frac{d^3u}{d\theta^3}\right) \quad . \quad . \quad \text{(v)}$$

Substituting for R and F_2 in the first of (iii) we get

$$-a\frac{dZ}{d\theta} - \frac{EI_2}{a^3}\left(\frac{d^6u}{d\theta^6} + \frac{d^4u}{d\theta^4}\right) - \frac{EI_2}{a^3}\left(\frac{d^4u}{d\theta^4} + \frac{d^2u}{d\theta^2}\right) + aX = 0$$

or

$$\frac{d^6u}{d\theta^6} + 2\frac{d^4u}{d\theta^4} + \frac{d^2u}{d\theta^2} = \frac{a^4}{EI_2}\left(X - \frac{dZ}{d\theta}\right) . \quad . \quad . \quad . \quad \text{(vi)}$$

When u has been found, M_2, F_2 and R are obtained from (ii), (iv) and (v), and the stresses can then be determined.

391. Circular Beam Loaded at Right Angles to its Plane.—We shall now consider in detail the case of a beam whose axis is initially a circular arc; in a horizontal plane, subtending an angle of 180° at the centre of the circle. The ends of the beam are supposed built-in horizontally at A and B (Fig. 418), and single concentrated vertical load W is applied at the middle point C.

Now if the beam has a circular cross section, and the load be attached to a ring riding loosely on the beam, as shown at (*a*) Fig. 418, the line of action of W will always pass through the centroid of the section. Otherwise, when the beam deforms, the axis of the section at C will not remain vertical, and the load W will apply a torque at C; for instance, if the section be rectangular and W be attached at the top, as at (*b*), Fig. 418, W will apply at C a torque $W\theta_1 d/2$, if θ_1 be small and d denotes the depth of the section. Hence we shall in general suppose there is a torque $2T_0$ applied at C.

Let a be the radius of the semicircle, and let $AOP = \theta$, where P is any point on the arc. Then we can write $ds = ad\theta$.

When the load is applied, the beam will deflect to the position $AP'C'B$. At P take axes x, y, z as before. We have, considering the half AC:

$$F_1 = \frac{W}{2}, F_2 = \theta_1 \frac{W}{2}, R = \frac{W}{2}\frac{dv}{ds} = \frac{W}{2a}\cdot\frac{dv}{d\theta}. \quad . \quad . \text{ (i)}$$

At C' let the action on $C'A$ be represented by the twisting moment T_0, and a bending moment M_0 about the axis of z at C'. Then we can consider the half AC' acted on by $\frac{W}{2}$ vertical and the couples T_0 and M_0.

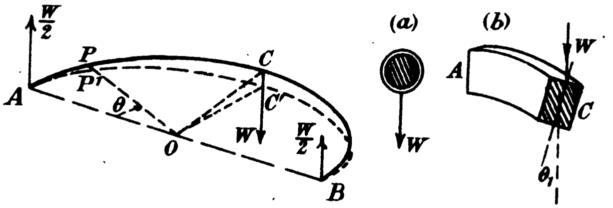

FIG. 418.

The equations of moments [(3) of § 386] give, putting $\rho_2 = a$ and $\rho_1 = \infty$, θ being then zero from (2),

$$\frac{dT}{ad\theta} - \frac{M_1}{a} = 0, \text{ i.e. } \frac{dT}{d\theta} - M_1 = 0 \quad . \quad . \quad . \quad . \text{ (ii)}$$

$$\frac{dM_2}{a\,.\,d\theta} - F_2 = 0, \text{ i.e. } \frac{dM_2}{d\theta} - a\theta_1\frac{W}{2} = 0 \quad . \quad . \quad . \text{ (iii)}$$

$$\frac{dM_1}{a\,.\,d\theta} + \frac{T}{a} - F_1 = 0, \text{ i.e. } \frac{dM_1}{d\theta} + T = \frac{aW}{2} \quad . \quad . \text{ (iv)}$$

From (ii) and (iv) we have:

$$\frac{d^2M_1}{d\theta^2} + M_1 = 0$$

$$\therefore\ M_1 = A\sin\theta + B\cos\theta, \quad . \quad . \quad . \quad . \text{ (v)}$$

where A and B are constants. When $\theta = \frac{\pi}{2}$ we must have $M_1 = M_0$, and therefore $A = M_0$. Hence, from (iv) and (v):

$$T = -M_0\cos\theta + B\sin\theta + \frac{aW}{2}.$$

When $\theta = \frac{\pi}{2}$, $T = T_0$; therefore $B = -\frac{aW}{2} + T_0$.

Hence

$$M_1 = M_o \sin\theta - \left(\frac{aW}{2} - T_o\right)\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v})$$

$$T = -M_o\cos\theta - \left(\frac{aW}{2} - T_o\right)\sin\theta + \frac{aW}{2}. \quad . \quad . \quad (\text{vi})$$

From the third of equations (10), § 389, we get:

$$T = -CK\left(\frac{d\theta_1}{ad\theta} - \frac{1}{a^2}\frac{d^2v}{d\theta}\right) = \frac{-CK}{a^2}\frac{d}{d\theta}(a\theta_1 - v),$$

since $\rho_o = a$; from this and (vi) we have

$$\frac{CK}{a^2}\cdot\frac{d}{d\theta}(v - a\theta_1) = -M_o\cos\theta - \left(\frac{aW}{2} - T_o\right)\sin\theta + \frac{aW}{2}.$$

Therefore

$$\frac{CK}{a^2}(v - a\theta_1) = -M_o\sin\theta + \left(\frac{aW}{2} - T_o\right)\cos\theta + \frac{aW\theta}{2} + A'.$$

Now since the end A is built in, we must have $v = 0$ and $\theta_1 = 0$, when $\theta = 0$. Hence

$$A' = -\left(\frac{aW}{2} - T_o\right).$$

Then

$$v - a\theta_1 = -\frac{a^2}{CK}\left\{M_o\sin\theta - \left(\frac{aW}{2} - T_o\right)(\cos\theta - 1) - \frac{aW\theta}{2}\right\} \quad (\text{vii})$$

Again, from the first of equations (10), § 389, we have

$$M_1 = -EI_1\left(\frac{\theta_1}{a} + \frac{d^2v}{a^2d\theta^2}\right) = -\frac{EI_1}{a^2}\left(a\theta_1 + \frac{d^2v}{d\theta^2}\right).$$

Hence, from (v)

$$a\theta_1 + \frac{d^2v}{d\theta^2} = -\frac{a^2}{EI_1}\left\{M_o\sin\theta - \left(\frac{aW}{2} - T_o\right)\cos\theta\right\} \quad . \quad (\text{viii})$$

Adding (vii) and (viii) we get

$$\frac{d^2v}{d\theta_2} + v = -a^2M_o\left(\frac{1}{EI_1} + \frac{1}{CK}\right)\sin\theta + a^2\left(\frac{aW}{2} - T_o\right)\left(\frac{1}{EI_1} - \frac{1}{CK}\right)\cos\theta + \frac{a^3W\theta}{2CK} - \frac{a^2}{CK}\left(\frac{aW}{2} - T_o\right).$$

The integral of this equation is:

$$v = A''\sin\theta + B''\cos\theta + \frac{a^3W\theta}{2CK} - \frac{a^2}{CK}\left(\frac{aW}{2} - T_o\right) - a^2\left(\frac{1}{EI_1} + \frac{1}{CK}\right) \left\{M_o\left(\frac{\sin\theta}{4} - \frac{\theta\cos\theta}{2}\right) - \left(\frac{aW}{2} - T_o\right)\left(\frac{\theta\sin\theta}{2} - \frac{\cos\theta}{4}\right)\right\} \quad . \quad (\text{ix})$$

From this we have:

$$\frac{dv}{d\theta} = A'' \cos\theta - B'' \sin\theta + \frac{a^3 W}{2CK} - a^2\left(\frac{1}{EI_1} + \frac{1}{CK}\right)$$
$$\left\{M_o\left(\frac{\theta\sin\theta}{2} - \frac{\cos\theta}{4}\right) - \left(\frac{aW}{2} - T_o\right)\left(\frac{3\sin\theta}{4} + \frac{\theta\cos\theta}{2}\right)\right\} \quad . \text{ (x)}$$

The conditions to be satisfied by v are $v = 0$ and $\frac{dv}{d\theta} = 0$, when $\theta = 0$.

These give, from (ix) and (x),

$$B'' = \frac{a^2}{CK}\left(\frac{aW}{2} - T_o\right) - \frac{a_2}{4}\left(\frac{1}{EI_1} + \frac{1}{CK}\right)\left(\frac{aW}{2} - T_o\right).$$
$$A'' = -\frac{a^3 W}{2CK} + \frac{a^2 M_o}{4}\left(\frac{1}{EI_1} + \frac{1}{CK}\right).$$

Substituting these values of A'' and B'' in (ix) we get:

$$v = \frac{a^3 W}{2CK}(\cos\theta - \sin\theta + \theta - 1) + \frac{a^3 W}{4}\left(\frac{1}{EI_1} + \frac{1}{CK}\right)\theta\sin\theta$$
$$- \frac{a^2 M_o}{2}\left(\frac{1}{EI_1} + \frac{1}{CK}\right)(\sin\theta - \theta\cos\theta).$$
$$+ a^2 T_o\left\{\frac{1 - \cos\theta}{CK} - \frac{1}{2}\left(\frac{1}{EI_1} + \frac{1}{CK}\right)\theta\sin\theta\right\} \quad . \quad . \text{ (xi)}$$

If the weight at C act through the centroid of the cross section, T_o will be zero by symmetry. We also have, on account of symmetry, $\frac{dv}{d\theta} = 0$ when $\theta = \frac{\pi}{2}$.

Hence from (x)

$$B'' = \frac{Wa^3}{2CK} - \frac{\pi a^3 M_o}{4}\left(\frac{1}{EI_1} + \frac{1}{CK}\right) - \frac{3a^2}{4}\left(T_o - \frac{Wa}{2}\right)\left(\frac{1}{EI} + \frac{1}{CK}\right).$$

Equating this to the value of B'' found above, we get

$$M_o = \frac{aW}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \text{ (xii)}$$

We then get from (v) and (vi) the following expressions for the bending moment and torque:

$$M_1 = \frac{aW}{\pi}\sin\theta - \frac{aW}{2}\cos\theta$$
$$T = -\frac{aW}{\pi}\cos\theta - \frac{aW}{2}\sin\theta + \frac{aW}{2} \quad . \quad . \quad . \text{ (xiii)}$$

From the above we can deduce the solution of the problem of a rod in the form of a circular quadrant, fixed at one end in a vertical plane, and carrying a vertical load W. The modifications to be made in the above equations are: $\frac{W}{2}$ must be replaced by W, whilst M_0 and T_0 must be made zero. We then obtain the equation

$$v = \frac{a^3W}{CK}(\cos\theta - \sin\theta + \theta - 1) + \frac{a^3W}{2}\left(\frac{1}{EI_1} + \frac{1}{CK}\right)\theta\sin\theta.$$

The bending moment and torque at any point are

$$M_1 = -aW\cos\theta$$
$$T = aW(1 - \sin\theta).$$

Some useful results of applying the above are given below; the proofs are left as exercises for the student.

1. The ends of a beam in the form of a circular arc subtending an angle 2ϕ at the centre are pushed together by equal and opposite forces P acting along the chord joining them. The displacements of the axis are given by

$$u = -\frac{a^3P}{EI}(\theta\cos\phi + \tfrac{1}{2}\theta\cos\theta);\ w = \frac{\partial u}{\partial\theta},$$

where a is the initial radius of the axis, and I is the moment of inertia of the cross section about an axis through the centroid perpendicular to the plane of bending.*

2. If the beam forms a complete circular ring, and is acted on by radial pressures P acting at opposite ends of a diameter, the displacement u is given by

$$u = -\frac{a^3P}{EI}\left[\frac{\theta}{\pi} - \tfrac{1}{2}(1 - \cos\theta - \tfrac{1}{2}\theta\sin\theta)\right],$$

where θ is measured from the diameter along which the forces P act.*

3. If a circular hoop rotate about a vertical diameter with angular velocity ω, this diameter is shortened by $\frac{w\omega^2a^5}{6gEI}$, w denoting the weight of the hoop per unit length.†

4. If a circular ring be subjected to an external radial pressure p per unit length, the ring is unstable if $p > 3EI/a^3$.

5. A piston ring, when free, has a nearly circular middle line, in which there is a gap of width c. The free form of the middle line is given by

$$r = a + \frac{c}{3\pi}(1 + \tfrac{1}{2}\theta\sin\theta),$$

$\theta = \pi$ corresponding with the radius through the centre of the gap. Then a uniform pressure, per unit length, equal to $EIc/3\pi a^2$, will compress the ring into circular form with radius a, the gap then being closed (Prescott).

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* H. Lamb, *London Math. Soc. Proc.*, Vol. 19 (1888).

† G. A. V. Peschka, *Zeitschr. f. Math. v. Phys.* (Schlömilch), Bd. 13 (1868). See also J. J. Guest, *Proc. R. Society* (Series A), Vol. 95, 1918.

6. A circular ring is subjected to equal radial thrusts P evenly spaced round the circumference at angular intervals 2α. The bending moments at the points of application of P, and midway between these points, are $\frac{P a}{2}\left(\frac{1}{\alpha} - \cot \alpha\right)$ and $\frac{P a}{2}\left(\operatorname{cosec} \alpha - \frac{1}{\alpha}\right)$ respectively (Prescott.)

8. STRESS-CONCENTRATIONS DUE TO SHARP CORNERS AND HOLES

392. In spite of the fact that it has been known for many years that sharp inwardly projecting corners in parts of machines and structures cause large local stress concentrations, failures still occur on account of insufficient attention being paid to this point. This is particularly true in all machines and in structures subjected to fluctuating stresses, it being estimated that 90 per cent. of failures in practice arise from fatigue. Here we can do no more than append a few notes which should be useful to designers and others, and refer the reader to other sources of information.* Methods of manufacture and heat treatment frequently lead to surface discontinuities which are virtually equivalent to "notches," with their consequent stress concentrations. The following figures, quoted from the paper by Gough and Clenshaw (see note below) will show how very important the surface finish of machine parts is:

	Endurance Limit (Rotating Bending) Tons/in².	
	Chrome-Vanadium Steel.	Silico-Manganese Steel.
(i) Surfaces completely machined and polished after hardening and tempering	±42	±46
(ii) Material machined and polished, then normal hardening and tempering	±31	±24
(iii) As (i) but grooved with $\frac{1}{9}$ mm. semicircular grooves to represent surface irregularities after hardening and tempering	±20	±21
(iv) As (ii) but decarburized before hardening and tempering	±20	±21
(v) Surfaces machined, decarburized, grooved, and finally hardened and tempered . .	±15	±11

Usually stress concentrations cannot be estimated exactly, but there are many instances in practice where these concentrations are due to bad design, and the following notes should enable designers largely to avoid these errors.

393. Tension Members with Sharp Corners.—(*a*) The simplest case to consider is a straight rod in pure tension, of either circular or rectangular cross section, such as Fig. 419. The maximum stress intensity decreases rapidly as the radius of the corner is increased. By means of photo-elastic experiments with transparent celluloid, Professor Coker obtained the following results for a flat plate*:

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* *The Relation of Fatigue to Modern Engine Design*, MacGregor, Burn and Bacon, Trans. N.E. Coast Inst. of Eng. and Shipbuilders, 1935. *Testing of Engineering Materials.* Gough and Clenshaw, Inst. Marine Eng., 1935; *Photo-Elasticity*, Coker and Filon (C.U. Press).

The maximum stress occurs at points A on the circular arc, just before it runs into the straight parts AC.

Radius R. (ins.)	0·25	0·125	0·0625
Maximum stress intensity	1480	1725	1890
Mean stress in narrow section . . .	1200	1200	1200

The maximum stress tends to infinity as R approaches zero. The direction of the maximum stress at A is tangential to the boundary curve.

Similar concentrations of stress occur when the specimen has a circular section.†

(*b*) *Practical Significance of Stress Concentration.*—Consider these figures in relation to a steel specimen, the radius of the corners being 0·0625. Suppose that the elastic limit is 18 tons/in.^{2}. Now, when the

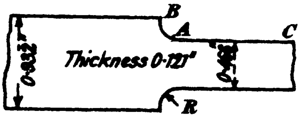

FIG. 419.

mean stress on the narrow section is, say, 16 tons/in.^{2}, the stress at A will be,

$$\frac{1890}{1200} \times 16 \text{ tons/in.}^2 = 25{\cdot}2 \text{ tons/in.}^2$$

That is to say, at this point A, the material will have very nearly, if not quite, reached the point of rupture, if the same conditions hold. But in reality the same conditions do not hold: in the experiment with celluloid the stresses were all less than the elastic limit of the material. In the steel specimen this will not be the case, as local yielding of the material will occur at an earlier stage; when the stress at A just reaches the elastic limit, the stress at a little distance from A will be only

$$\frac{1200}{1890} \times 18 \text{ tons/in.}^2 = 11{\cdot}4 \text{ tons/in.}^2.$$

When the mean stress becomes very little more than this, the material at A will yield, and a redistribution of stress will take place. The stress will tend to a more uniform distribution, so that the maximum stress will not reach such high values as indicated above. However, part of the material will have yielded and acquired permanent strain.

The practical conclusion is that, when a member is subjected to a steady tension there will not be so much danger from stress concentration as might at first be supposed, provided the material has good ductility. On the other hand, if the stress fluctuates periodically, there will be danger of a crack starting at the point of concentration, since the number of alternations required to cause fracture of the material at A will be far less than the number which would cause the fracture of the rod as a whole for a given value of the mean stress.

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* *Photo-Elasticity*, Coker and Filon, p. 564. † *Ibid.*, p. 567.

For example, Wohler, in one of his experiments on repeated tensile tests, found that with a stress range of 15·28 tons/in.^{2} the specimen required over 10 million repetitions to cause rupture. If the mean stress in our specimen had this value, the stress range at A would be 24 tons/in.^{2}. With the same steel Wohler found that with a stress range of 21 tons/in.^{2} rupture took place after 106,910 repetitions. Evidently in our case a crack would develop at A with even fewer repetitions; this will lead to still further concentration of stress at the end of the crack, which will ultimately spread through the material and cause a breakdown.

Similar remarks apply to all cases of stress concentration considered below.

394. Tension Members with Holes.—(*a*) *Circular Holes.*—When flat tension members are pierced by a circular hole the maximum tensile stress occurs at A (Fig. 420) and may reach a value three times the mean stress across AB, depending on the diameter of the hole in relation to the width of the specimen. This maximum stress can be calculated from the following formulæ.*

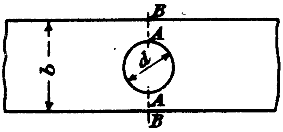

FIG. 420.

If p_m = the mean tensile stress across AB, the maximum tensile stress is

$$6c^3p_m/(2c^3 + 2c^2 + c + 1),$$

where $c = b/d$.

The circumferential tensile stress round the hole is given by

$$2c^3p_m(1 - 2\cos 2\theta)/(2c^3 + 2c^2 + c + 1),$$

θ being the angular co-ordinate measured from a horizontal line through the centre of the hole.

According to the first of these formulæ, as $c \to 1$, the maximum stress $\to p_m$. This, however, is not found to agree with optical investigation; for instance,† with a hole 5 ins. in diameter in a plate 5·2 ins. wide, the tensile stress at B was nearly zero, whilst that at A was approximately $2p_m$.

(*b*) *Elliptical Holes.*—If the hole is in the shape of an ellipse having axes $2a$, $2b$, as shown in Fig. 421, the tensile stress at the ends of the major axis is given by*

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* Coker and Filon, *loc. cit.*, p. 485.

† *Ibid.*, p. 486.

$$p_a = p\left(1 + \frac{2a}{b}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and the compressive stress at the ends of the minor axis is

$$p_b = p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where p denotes the uniformly distributed tensile stress at a large distance from the hole.

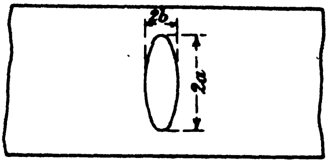

FIG. 421.

395. Notches in Tension Members.—When a tension member has semicircular notches cut in the sides, as in Fig. 422, the maximum tensile stress occurs at A and is given by†

$$p_{\max.} = 12c^3p_m/(6c^3 + 4c^2 + c + 1)$$

where

c = width of plate ÷ diameter of notches.
p_m = mean stress across AA.

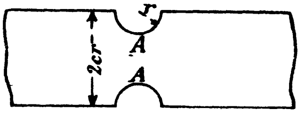

FIG. 422.

In the case of a V-notch, as shown in Fig. 423, if the bottom of the notch be cut with too small a radius the stress intensity may be very

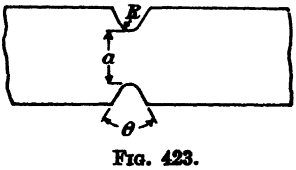

FIG. 423.

large. Professor Coker‡ found, with experiments on a celluloid plate 13 cm. wide, with $\theta = 45^\circ$, the following figures for the ratio

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[*] Coker and Filon, *loc. cit.*, p. 532 *et seq.*, give the analysis for an infinitely wide plate; with correction for effect of width of plate.

[†] *Ibid.*, p. 580.

[‡] *Ibid.*, p. 597.

$$\frac{p_{\text{max.}}}{p_m} = \frac{\text{Maximum stress at bottom of notch}}{\text{mean stress over width } a}$$

R (millmetres) . .	10	7·5	5	2·5	1·25	0·625
$p_{\text{max.}}/p_m$	2·2	2·5	2·7	3·5	4·78	6

Depth of notch = 1 cm.

396. Sharp Corners in Rods subjected to Torsion.—(*a*) *Abrupt Change of Diameter.*[*]—This is a case which is of frequent occurrence in practice, and if the radius ρ of the shoulder (Fig. 424) is too small, large stress concentrations may occur in the shoulder. The following figures should serve for all practical purposes.[†] The graphs below show the value of $q_{\text{max.}}/q_0$, where q_0 is the maximum shear stress calculated for the smaller part of the shaft by the standard formulæ.

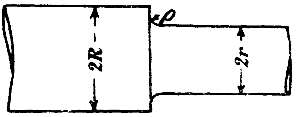

FIG. 424.

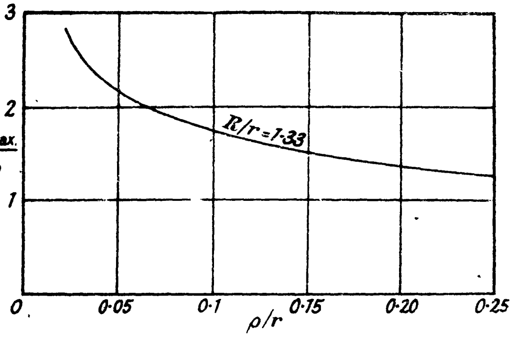

FIG. 425.

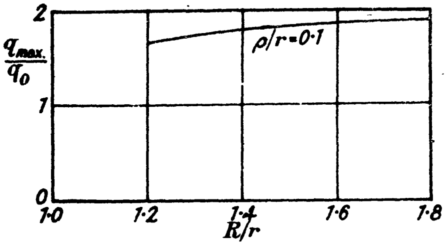

FIG. 426.

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[*] The general solution for circular rods of non-uniform section was given by Michell, *Proc. London Math. Soc.*, 1899.

[†] Timoshenko, *Applied Elasticity*, obtained these results by a graphical process given by Willers, *Zeitschr. f. Math. u. Phys.*, 1907.

For values of ρ/r less than 0·05, the maximum stress may be taken as proportional to $1/\sqrt[3]{\rho}$.

(*b*) *Keyways and Serrations.*—See pp. 111-112.

(*c*) *Small Semicircular Groove round the Shaft.*—The stress concentration for this case is given by Fig. 426,* in which R = the radius of the shaft, ρ = the radius of the cross section of the groove, the centre for ρ being on the surface of the shaft, and q_R denotes the shear stress at radius R calculated from the usual formula.

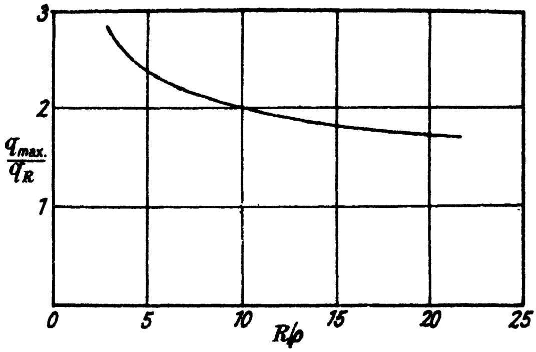

FIG. 427.

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* Timoshenko, *Applied Elasticity*, obtained these results by a graphical process given by Willers, *Zeitschr. f. Math. u. Phys.*, 1907.

APPENDIX II

TABLE OF ELASTIC CONSTANTS

The units are pounds and inches. E = Young's Modulus; C = Modulus of Rigidity.

1. Metals

Material.	Tension.			Compression.		Shear.	
	$E \div 10^6$	Elastic Limit.	Ultimate Strength.	Elastic Limit.	Ultimate Strength.	$C \div 10^6$	Ultimate Strength.
Aluminium, cast .	12·5	7,000	18,000	—	—	—	—
,, rolled .	—	12,500	27,000	—	—	—	13,000
,, wire. .	18·5	—	—	—	—	—	—
Copper, rolled . .	—	—	38,000	—	48,000	4·5 to 6·5	35,000
,, cast . . .	9·1	14,000	30,000	—	37,000	—	28,000
,, wire . . .	—	—	50,000	—	—	—	—
Brass . . *from*	12·0	—	18,000	—	11,000	5·5	18,000
,, wire . . .	—	—	80,000	—	—	—	—
Duralumin, rod . .	10·5	40,000	60,000	16,000 to 20,000	45,000 to 65,000	3·8	35,000
,, sheet .	—	30,000	55,000	—	—	—	—
,, wire . .	—	40,000	60,000	—	—	—	—
Iron, cast . *from*	14·0	—	13,400	—	80,000	5·5	18,000
,, ,, . . *to*	23·0	—	29,000	—	110,000	8·0	22,000
,, wrought *from*	27·0	23,000	36,000	21,000	36,000	11	37,000
,, ,, . *to*	29·0	40,000	65,000	24,000	60,000	13·5	41,000
,, wire. . . .	25·0	—	90,000	—	—	—	—
Steel, mild . *from*	29	40,000	60,000	40,000	60,000	12	47,000
,, ,, . . *to*	31	50,000	70,000	50,000	70,000	14·5	54,000

The strengths of special steels are too variable to be given. The particular specifications must be referred to.

Material.	$E \div 10^6$	Elastic Limit.	Ultimate Strength.	Elastic Limit.	Ultimate Strength.	$C \div 10^6$	Ultimate Strength.
Gun-metal . . .	—	17,000	31,000	—	—	—	—
Lead	—	—	3,300	—	—	—	—
Tin, cast	4·6	—	4,500	—	—	—	3,000
Zinc, cast . . .	—	—	7,400	—	—	—	9,000

2. Timber

Material.	Tension.			Compression.		Shear.	
	$E \div 10^6$ Parallel to Grain.	Elastic Limit.	Modulus of Rupture.	Strength Perpendicular to Grain.	Strength Parallel to Grain.	$C \div 10^6$	Ultimate Strength Parallel to Grain.
				Elastic Limit.	Max.		
Ash	1·4 to 1·65	7,700	12,000	1,300	6,000	0·14	1,750
Beech	1·5	7,400	12,000	—	6,000	—	1,700
Birch, American .	1·8 to 1·95	8,400	13,000	1,000	6,600	0·17	1,620
Cedar . . . *from*	1·0	4,200	6,000	400	4,000	—	800
,, ... *to*	1·7	6,000	10,000	700	5,300	—	1,100
Deal, Christiania .	1·6	—	—	—	—	—	—
Elm, English . .	1·3	5,500	—	—	—	—	—
,, Canadian . .	1·4	6,700	12,500	1,200	5,800	—	1,650
Hickory	1·9	8,900	16,000	1,800	7,300	—	1,800
Mahogany, Honduras	1·3	7,000	10,000	1,000	5,500	0·1	1,300
,, Spanish .	1·4	—	—	—	—	—	1,400
Oak, white commercial	1·4	6,700	12,000	1,300	6,000	--	1,760
Pine, white . . .	1·2	5,000	7,400	530	4,500	—	850
,, Norway . .	1·7	8,000	11,000	720	6,000	—	1,100
Padouk, dark . .	2·4	10,500	—	—	—	0·2	2,000
Spruce	1·4 to 1·81	5,500	7,900	500	5,500	0·08	500
Teak	2·4	—	7,000	—	—	—	—
Walnut, Brazil . .	1·3 to 1·5	5,800 to 8,000	11,000	1,000	6,100	0·125	1,500

ANSWERS TO EXAMPLES

EXAMPLES I. P. 31

1. 4·36 tons/in.^{2} **2.** 0·0076″; 0·00095. **3.** 3·33 tons/in.^{2}; 0·0248%. **5.** 0·00133″. **6.** 0·51″. **7.** 4·4″. **8.** 18·1 tons/in.^{2}; 34 tons/in.^{2}; 25%; 71·5%. **9.** 33·4 × 10^6 lbs./in.^{2} **10.** 0·0264″. **11.** 10,700 lbs./in.^{2}; 0·085″. **12.** 0·86″. **13.** Outer rods 214 lbs.; lower centre rod 572 lbs. **14.** Brass legs $\frac{2}{3}$ ton each; steel $\frac{11}{12}$ ton each. **15.** 6·7 tons. **16.** 9,460 lbs./in.^{2} steel tube; 6,100 lbs./in.^{2} copper; 0·218″; 31,000 lbs. **17.** 7,160 lbs./in.^{2} steel; 2,360 lbs./in.^{2} brass. **19.** 3·82 × 10^{-4} ins. and 1·27 × 10^{-4} ins. **20.** 493 lbs./in.^{2} **21.** 7·9966″. **22.** 0·035″. **23.** Ratio = 13·7. **24.** Steel 47,100 lbs./in.^{2}; brass 19,500 lbs./in.^{2}

EXAMPLES II. P. 50

2. 0·45″. **3.** 0·0217″. **4.** Top cross members 7·2 in.^{2}; lower ditto 4·33 in.^{2} Outside sloping members 10·4 in.^{2}; inside long, 1·44 in.^{2}; short continuation of the latter 2·9 in.^{2}; short members sloping from top outer corners, 2·16 in.^{2} Spread of base = 11·2″. **5.** *E*, 0·309″; *F*, 0·323″. **6.** 0·19″. **7.** 1,040 lbs. **8.** 6·10 tons/in.^{2} **9.** Load in member = 0·403 ton, tension.

EXAMPLES III. P. 61

1. 36·8 tons/in.^{2} **2.** 754 H.P. **3.** 12 revs./sec. **4.** 117 lbs. ft. **5.** $d = 2·42″$; thickness of cotter 0 605″: mean width 3·32″; distance of cotter hole from end of rod 1·25″; max. diam. of socket 5·08″; distance of cotter hole from end of socket 1·13″; outside diam. of socket 3·23″. **6.** 1,132 ft./sec.^{2} **7.** 0° 6′ 9″. **8.** 0·02 tons/in.^{2}

EXAMPLES IV. P. 70

1. 10·6 tons/ft. **2.** 14·1 tons/in.^{2}

EXAMPLES V. P. 84

1. $p_\theta = 3$, $q_\theta = 1·73$, tons/in.^{2} **2.** $p_\theta = 1·6$, $q_\theta = 1·2$ tons/in.^{2}, 3·75 tons compression. **3.** Max. stress 5·78 tons/in.^{2} on plane at 34° 44′ to cross section; max. shear stress 4·27 tons/in.^{2} on planes at 10° 16′ and 79° 44′ to cross section. **4.** 2, — 8, tons/in.^{2} **5.** 5·4 tons/in.^{2}, inclined 23° 37′ to horizontal. **6.** 6, — 4 tons/in.^{2}; strains 5·34 × 10^{-4} and 4·28 × 10^{-4}. **7.** 3·53, − 4·53, 4·03 tons/in.^{2} **8.** If p_1 and p_2 are the principal stresses, and θ denotes the angle between p_1 and the normal to the required plane, θ is given by

$$\theta = \frac{\pi}{4} + \frac{1}{2}\sin^{-1}\left(\frac{p_1 - p_2}{p_1 + p_2}\right)$$

when p_1 and p_2 both have the same sign. When p_1 and p_2 have opposite signs, the required plane is that for which the direct stress is zero, and θ is given by $\tan^2\theta = (-p_1/p_2)$. Resultant stress = 5·66 tons/in.^{2}, with $\theta = 54° 44′$. **9.** 0·5 and 3·46 tons/in.^{2} **10.** 1,030 lbs./in.^{2} **11.** (i) 2,030, (ii) 3,160 ft. lbs. per ton. **12.** ± 52·2 tons/in.^{2} **13.** 3·8, 2·2 tons/in.^{2}

EXAMPLES VII. P. 102

1. 1,470 lbs./in.^{2} **2.** 0·6 lbs./ft. **3.** 1,680 lbs./in.^{2} **4.** Rivet 4·73 tons/in.^{2}; boiler plates 4·42 tons/in.^{2}; cover plates 3·32 tons/in.^{2} **5.** 4·46. **6.** (i) 3,750 lbs./in.^{2}, (ii) 2,200 lbs./in.^{2} **7.** 1,875 lbs./in.^{2}; 1,500 lbs./in.^{2} **8.** 0·127%. **10.** (*a*) Copper 5,030 lbs./in.^{2}; steel 10,130 lbs./in.^{2} (*b*) Copper 10,670 lbs./in.^{2}, steel 22,830 lbs./in.^{2} **12.** 105·5 lbs./in.^{2} **13.** 19,800 lbs./in.^{2}

EXAMPLES VIII. P. 112

1. 2·38 tons/in.^{2}, 1°·04, 15·66″. **2.** 2·88 tons/in.^{2}. 5·05°. **3.** 14·4″, 8·64″. **4.** 4·12″; 2·22°. **5.** 7,170 lbs./in.^{2}; 4°·4. **6.** Wt. of hollow shaft = 0·644 of wt. of solid shaft. **8.** 4,880 lbs./in.^{2}, 0·59°. **9.** 450 lb. ins. **10.** 34,800 ft. lbs. **11.** 11·5 and — 1·98 tons/in.^{2}; $\theta = \frac{1}{2}\tan^{-1}2$. **12.** — 3·04, 1·65 tons/in.^{2} **13.** 146 ton. ins., 12·8°. **14.** 1·8, 10^{-3} ins.

EXAMPLES IX. P. 138

1. Max. B.M. = — 150 tons. ft. **2.** Max. B.M. = — 75 tons. ft. **3.** Max. B.M. = — 76 tons. ft. **4.** Max. B.M. = 100 tons. ft. **5.** Max. B.M. = 75 tons. ft. **6.** Max. B.M. = 5,000 lbs. ft. **7.** B.M. at centre = 1,320 lbs. ft. **8.** Max. B.M. = — 1,980 lbs. ft.; Max. S.F. = — 1,225 lbs. **9.** B.M. at ends = 0; B.M. at supports = — 1 ton. ft.; B.M. at centre = 2 tons. ft. **10.** 544,000 lbs. ft. **11.** B.M. at ground = 2,000 lbs. ft. **12.** B.M. at $A = 0$; at middle of AB = 90 tons. ft., at B = — 260 tons. ft. S.F. at A = — 9·5 tons, at left of B = 22·5 tons, at the right of B = — 18 tons. **13.** B.M. at $A = 0$, at B = — 44·6 tons. ft., at C = — 45 tons. ft., at D = — 9 tons. ft., at $E = 0$. S.F. at A = 3·18 tons; at B, 3·18 tons, at C, and D, — 3 tons; at E, 0. **16.** B.M. increases uniformly from zero at C to 6 tons. ft. at B, and keeps that value to A. **17.** B.M. decreases from zero at A to — 100 tons. ft. at C, and there increases suddenly to — 33·3 tons. ft., finally increasing uniformly to zero at B. **18.** B.M. at A = 159,000 lbs. ft.; 27·8 ft. above A. **19.** $\frac{1}{6}w_1l^2 + \frac{1}{3}w_2l^2$.

21. $M = \frac{10x}{3}\left(1 - \frac{x^2}{169}\right)$, where x is the distance in feet from the end at which the load is zero. **22.** B.M. at bearing nearest flywheel = 6·8 lbs. ft.

24. $M = \frac{w_0l^2}{\pi^2}\sin\frac{\pi x}{l}$.

EXAMPLES X. P. 154

1. In Fig. 144, taking $W_1 = 20$ tons, section required is 11′4″ from A; max. B.M. = 113 tons. ft. **5.** 187 tons. ft.; if w = equivalent uniformly distributed load, we must have $wl^2/8$ = 187 tons. ft., which gives for the total load on the girder 19·95 tons. **6.** Max. B.M. is $(100 - x)(0{\cdot}15x - 50)$ or $(14 - 0{\cdot}15x)x$ according as x is > or < 33′4″. In the first case it occurs when the 10 ton load is at the section; in the second case when the 5 ton load is at the section.

EXAMPLES XI. P. 182.

1. 173 tons. ins. **2.** 4·1 tons/in.^{2} **3.** 1,180 lbs./in.^{2} **4.** 4·25 tons/in.^{2}. **5.** 2 inches. **6.** 1·09 tons. **7.** 15,200 lbs./in.^{2} **8.** $9{\cdot}31/\sqrt{b}$ inches; $14{,}450 + 6{,}200\sqrt{b}$, lb. ins. **9.** (1) 2·42 in.^{2}; (2) 0·19 in.^{2}, but in this case, when the compressive stress in the concrete reaches 600 lbs./in.^{2}, the tensile stress will be 1,234 lbs./in.^{2}, so that the concrete would crack. Hence the larger reinforcement should be used; moment of resistance = 406,000 lbs. ins. **10.** Eight reinforcing bars are required; 2,320 lbs. **12.** 0·97 M tons/in.^{2}, if M be in tons. ins. **13.** The neutral axis is inclined at 58° to the horizontal; 8,000 lbs./in.^{2} **14.** 1·91 in.^{4} and 13·1 in.^{4} **16.** Depth = 28″, width = 11·2″, total section of reinforcement = 2·94 in.^{2} **17.** Ratio 1 to 9. **18.** $W^2l^3/8Ebd^3$; $f^2/18E$.

EXAMPLES XII. P. 198

1. 11·26 tons/in.^{2}, compression; 10·34 tons/in.^{2} tension. **2.** 1·84 tons, 0·84 tons/in.^{2} compression. **4.** Neutral axis is 3 ft. from AD, outside the section; max. compressive stresses 6·25 and 2·08 tons/ft.^{2} **5.** 0·288 in.^{2}. 4,625 lbs. **6.** 6,200 lbs./in.^{2}, compression; 5,120 lbs./in.^{2}, tension. **8.** 4·16 tons/in.^{2} compression; 3·40 tons/in.^{2} tension. **9.** 1,670 lbs./in.^{2}; 0·30″ towards B.

Examples XIII. P. 214

1. 0·50″; 1·4″. **2.** $p_{max} = 7·3$ tons/in.^{2}; $_{max} = 1·2$ tons/in.^{2} **3.** (i) 9·17, (ii) 9·78 and — 1·13 tons/in.^{2} **4.** (*a*) 8·2 tons/in.^{2}, (*b*) 1·14, (*c*) 7·54 and — 0·08 tons/in.^{2} **5.** The web takes 96 per cent. of the shearing force, and the flanges 87 per cent. of the B.M. **6.** 2·17 tons/in.^{2} **7.** 6·8 and — 0·6 tons/in.^{2} **8.** 0·505″; 2·29″. **9.** 95 lbs./in.^{2} **10.** 79 lbs./in.^{2} **12.** 57·6 tons total; 3″. **13.** 49·8 lbs./in.^{2} **16.** 3,220 and 336 lbs./in.^{2}

Examples XIV. P. 246

(Unless stated otherwise, $E = 30 \times 10^6$ lbs./in.^{2} for steel.)

2. 16·5 tons/in.^{2}, 1″; deflection = 0·111″, stress = 3·8 tons/in.^{2} **3.** 0·6″ under 12 ton load; 0·75″ under 8 ton load. **4.** 18 rods 1·15″ diam. **6.** 306 lbs., 3,000 lbs./in.^{2} **7.** 0·187 ton at the end of 10′ span, 1·469 tons at middle support, 0·344 ton at other end. **8.** 2·94″ vertical, 4·32″ horizontal; 380 lbs., 1·46″. **12.** (*a*) 0·1375″, (*b*) 0·050.″ **14.** $y = (8/E)\sqrt{lp^3B}\,.\{(l + x)\sqrt{l} - \sqrt{l - x}\}$, where B = the width of the section, p = the maximum stress, l = the length, and x is measured from the fixed end.

Examples XV. P. 256

1. + 21·15 and — 31·35 tons. ft.; 5·6 ft. from ends. **2.** B.M. at centre 23·7 tons. ft.; — 45·5 and — 42·1 tons ft. at ends; 4·15 ft. from left-hand end, 4·66 ft. from right-hand end. **6.** B.M. at supports = — 160 tons. ft.; at middle = 80 tons. ft.; $8·64 \times 10^6/EI$ lb. inch-units. **7.** 8·86 ft. from left-hand end; 0·0274″. **8.** — 53·3 and — 80 tons. ft. **9.** 20·57 ft. from right hand support; 0·0975″.

Examples XVI. P. 270

1. Ends 80 tons each; inner supports 220 tons each. **2.** (*a*) Max. B.M. = 538,400 lbs. ft.; B.M. at middle support = — 351,000 lbs. ft.; Max. S.F. = 42,200 lbs. (*b*) Max. B.M. = 395,600 lbs. ft.; B.M. at middle support = — 702,000 lbs. ft.; max. S.F. = 46,900 lbs. **3.** End pegs 0·0408 lb.; middle peg 0·1796 lb.; others 0·1306 lb. **5.** Max. B.M. = 68 tons. ft.; B.M. at inner supports = — 101 tons. ft.; points of inflexion 13·26 ft. from ends, and 25·5 ft. from ends. Max. S.F. = 20·5 tons. **6.** Bending moments at cross girders, from the left, are 0; 77,900; 53,200; — 74,000; 1,900; 26,600; 0; lbs. ft. **8.** 3,900 lbs.

Examples XVII. P. 281

2. Horizontal, 75 tons; vertical, 31·25, — 11·25, 28·75 tons. **4.** 186¾ tons. **7.** Decrease of tensile stress = 3 tons/in.^{2}; of compressive stress, 3·42 tons/in.^{2}

Examples XVIII. P. 323

(Unless otherwise stated, $E \doteq 30 \times 10^6$ lbs./in.^{2} for steel.)

1. 222 lbs. **2.** 1,930 lbs. **3.** 938 tons. **4.** 1,850 lbs. **5.** By the Euler formula, thickness = 0·0264″. **6.** 34·2 tons, using the result of Example 21. **7.** 1,720 lbs. **8.** 64,300 lbs.; yes, 32,000 lbs. **9.** 437 tons. **10.** The straight line formula gives smaller values of the load if $20 < l/k < 90$; 28·3 tons. **13.** 2·125 tons. **24.** 20 tons.

Examples XIX. P. 341

1. $b_0 = 1·58″$; $b_1 = 0·41″$. **2.** $b_0 = 2·16″$; $b_1 = 1·36″$; 15%. **3.** 1,390 lbs.

Examples XX. P. 353

1. 10·8 tons. **2.** Yes, 7·56 tons. **6.** 2·46″, say 2½″. **7.** Depth of section = 4″ gives a maximum bending stress of 11,400 lbs./in.^{2} **9.** M_B = — 5,420 lb. ins.; M_{max} = 5.510 lb. ins.; 5,320 lbs./in.^{2}

Examples XXI. P. 367

4. B.M. at A = 923 lb. ins.; B.M. at B and C = 336 lb. ins; thrusts in AB and AC = 7,030 lbs.; tension in BC = 4,944 lbs.

Examples XXII. P. 373

1. 7,000 lbs./in.″ **2.** 3,490. **3.** 6·65″; no. **4.** 354″; 10,970 lbs./in.^{2} **5.** 4·23 tons/in.^{2} **6.** 1,275 lbs./in.^{2}; 1,935 lbs./in.^{2}; 2,566, — 634 lbs./in.^{2} **7.** 10,630; — 5,980 lbs./in.^{2} **8.** 1,577 lbs.; 22,745 lbs. ft.; 13,435 lbs./in.^{2}; 6,985 lbs./in.^{2} **9.** 2·9″.

Examples XXIV. P. 396

1. 4,890 lbs./in.^{2}; 2·66″. **2.** 0·97″; say 1″, then 6·1 turns. **4.** 4·54″; 47°. **5.** 0·655″; 16,300 lbs./in.^{2} **6.** (i) 4·03″. (ii) 19,560 lbs./in.^{2} (iii) 80·6 lb. ins. **7.** 0·83″; 135″ assuming the plates to be straightened by the load. **8.** The periodic time $= 2\pi\sqrt{m/gs}$, where m = attached mass and s = the stiffness. 0·0938 sec. **9.** 58″. **11.** 6·8″. **12.** 4·57 lbs./in. **14.** 300 ins. lbs.

Examples XXV. P. 410

2. 3·28, — 4·58 tons/in.^{2} **3.** For a rectangular cross section of radial depth D and thickness B, when the centroid is at a distance $\rho_0 + \bar{y}$ from the centre of curvature, we obtain the following formula (see p. 400):

$$\int \frac{\rho_0\, y_0{}^2 B \,.\, dy_0}{\rho_0 + y_0} = B\rho_0{}^3 \log \frac{\rho_0 + \bar{y} + D/2}{\rho_0 + \bar{y} - D/2} + BD\rho_0(\bar{y} - \rho_0).$$

This gives, for the section in question, $h^2 = 0{\cdot}622$ in.^{2}; stresses 4,100, — 5,250 lbs./in.^{2} **4.** 1·74 tons. **5.** 1·18, — 0·515 tons/in.^{2}

Examples XXVIII. P. 462

1. (*a*) 13·2 tons/in.^{2} (*b*) 9·75 tons/in.^{2} **2.** 3·7″. **3.** 333 lbs./in.^{2}. **4.** 0·52″. **5.** 943·6 ins.^{3} **7.** 5·9958″; 9·47 tons/in.^{2} **8.** 23·912″. **9.** 0·00219″; 49·8°; 3 tons/in.^{2}, 7·13 tons/in.^{2}; change of bore = 0·00175″; of outside diameter 0·00446″. **10.** 3·32″; 1·99776″; new torque = 5·05 tons. ins. **11.** 0·00755″; 90° C. **13.** 46·3 tons/in.^{2}

Examples XXIX. P. 475

4. 0·942″. **5.** 4,000 r.p.m. **6.** 3·56; 21·2 tons/in.^{2}

Examples XXX. P. 488

1. $\theta = 0{\cdot}891T/C$; $q_1 = 0{\cdot}889T$. **2.** 1·61 ins.^{4} **3.** 0·00178 ins.^{4} **4.** 33·3 tons. ins. **5.** Square section is nearly 20% lighter.

Examples XXXI. P. 500

5. 50,000 lbs./in.^{2}

Examples XXXII. P. 510

4. 920 r.p.m. **5.** 2,340 r.p.m.; 33,600 lbs./in.^{2} **6.** 552 r.p.m. **7.** 473 r.p.m. **9.** 915 r.p.m. to 1,115 r.p.m.

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